Joint with:

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https://arxiv.org/abs/2012.06931 ($\approx$ José's talk)

https://arxiv.org/abs/2105.13948 ($\approx$ this talk)

$B_i(z) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & z & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$ $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$ braid (positive)

$\rightsquigarrow \underbrace{B_\beta(z_1 \ldots z_k)}_{\text{braid matrix}} = B_{i_1}(z_1) \cdots B_{i_k}(z_k)$

Assume $\delta(\beta) = w_0$

$\underbrace{X(\beta)}_{\text{braid variety}} = \{ z_1 \ldots z_k \mid B_\beta(z_1 \ldots z_k) w_0 \text{ upper-triangular} \}$ $w_0 = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$

Thm (folklore) (Kálmán) $\# X(\beta)$ over $\mathbb{F}_q$ = coefficient at top power of a in HOMFLY-PT polynomial of $\overline{\beta w_0}$

Idea of proof: $\beta = \ldots \overset{z_1}{\sigma_i} \overset{z_2}{\sigma_i} \ldots$

$\beta' = \ldots \sigma_i \ldots$ $\beta'' \ldots 1 \ldots$

José explained in his talk that $X(\beta) = \underbrace{\mathbb{C}^*}_{z_1 \neq 0} \times X(\beta') \sqcup \underbrace{\mathbb{C}}_{z_1 = 0} \times X(\beta'')$

A similar computation over $\mathbb{F}_q$ gives $\boxed{\# X(\beta) = (q-1) \# X(\beta') + q \, \# X(\beta'')}$

This is skein relation!

 $= (q-1)$ $+\, q$

This is skein relation!

If β is reduced ($=$ permutation),

$$X(\beta) = \emptyset \quad \beta \neq w_0, \quad X(w_0) = pt$$

Ex $X(\sigma^3)$: $\begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{w_0} = \begin{pmatrix} z_1+z_3+z_1z_2z_3 & * \\ & * \end{pmatrix}$

$$\underset{\mathbb{C}^3}{\{z_1+z_3+z_1z_2z_3 = 0\}} \simeq \underset{\mathbb{C}^2}{\{1+z_1z_2 \neq 0\}} = X(\sigma^3)$$

$z_1 + z_3(1+z_1z_2)$ $\qquad \# X(\sigma^3) = q^2 - q + 1$

Thm (Webster–Williamson, Mellit, Trinh) $H_T^*(X(\beta))$ = component in HOMFLY-PT homology of βw_0 of top a-degree.

Here $T = (\mathbb{C}^*)^{n-1}$ is a torus which acts on $X(\beta)$ and $H_T^*(X(\beta))$ is bigraded by homological degree & weight filtration.

Rmk If βw_0 closes to a knot then T acts freely and

$$H_T^*(X(\beta)) = H^*(X(\beta)/T).$$

Ex (Galashin–Lam) $\beta = T(m,n)$ = torus braid

$X(\beta)$ = positroid variety in $Gr(m,n)$ (see José's talk)

$H_T^*(X(\beta))$ = q,t - Catalan number

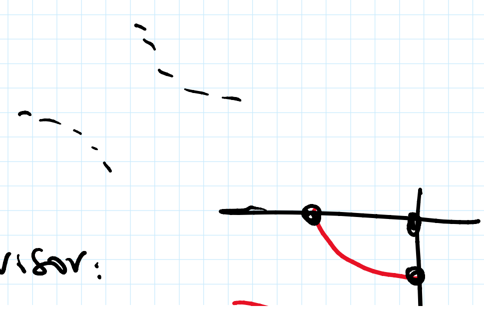
(by the work of Hogancamp and Mellit we know HOMFLY-PT homology)

Idea of proof of thm: compactification

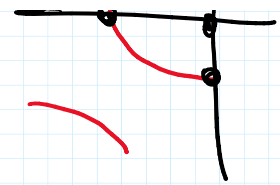
$$X(\sigma^3) \sim \{1+z_1z_2 \neq 0\} \subset \mathbb{C}^2$$

Compactify to $\mathbb{P}^1 \times \mathbb{P}^1$

Complement = normal crossing divisor:



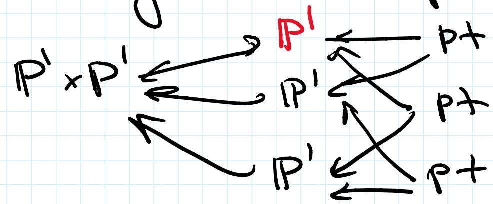
Complement = normal crossing divisor:



hyperbola + two $(\mathbb{P}^1)$'s at ∞

intersecting at 3 points transversally.

We have a diagram of spaces:



By taking cohomology (equivariant), we get a complex computing $H^*_T(X(\beta))$

It agrees with definition of HOMFLY-PT homology.

In general, compactify to Bott-Samelson/brick variety

Thm (G., Hogancamp, Mellit, Nakagane)

Top HOMFLY-PT homology (βw_0) = Bottom HOMFLY-PT homology of $\beta \cdot w_0^{-1}$

highest a-degree — lowest a-degree

Cor $H^*_T(X(\beta))$ = Bottom HOMFLY-PT homology of $\beta \cdot w_0^{-1}$

is invariant under braid moves, conjugation + positive stabilization of $\beta \cdot w_0^{-1}$.

Thm (Casals, G., Gorsky, Simental)

The variety $X(\beta)$ is invariant under braid moves, conjugation + positive stabilization of $\beta \cdot w_0^{-1}$.

(up to $(\mathbb{C}^*)^{\cdots}$).

Ex $\beta = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_1^{-1} \cdot w_0^{(4)}$ braid on 4 strands

$X(\beta) = ??$

Non-positive but easily equivalent to a positive one

By thm, it is supposed to be controlled by a ...

non positive but easily equivalent to a positive one

By thm, it is supposed to be controlled by a non-positive braid

$$\sigma_2\sigma_1\underbrace{\sigma_3\sigma_2\sigma_3^{-1}}_{\text{braid move}}\sigma_1^{-1} = \sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1} = \sigma_2\cancel{\sigma_1}\cancel{\sigma_2^{-1}}\cancel{\sigma_2}\cancel{\sigma_1^{-1}} = 1$$

$(\cdot\, w_0^{(4)})$ $\quad$ $(\cdot\, w_0^{(4)})$ $\quad$ positive stabilization $\quad$ $(\cdot\, w_0^{(3)})$ $\quad$ $(\cdot\, w_0^{(2)})$

$X(w_0^{(4)}) = pt \Rightarrow X(\beta) = (\mathbb{C}^*)^2$

$\boxed{\text{Warning: Really need negative crossings for this!}}$

To explain the proof, we need a detour to Legendrian link invariants.

$(\mathbb{R}^3,\ \alpha = dz - y\,dx)$ $\quad$ $L \subset \mathbb{R}^3$ Legendrian if $\alpha|_L = 0$.

contact form

Given a Legendrian link, Chekanov defined a differential graded algebra $\mathcal{A}_L$ which is a Legendrian link invariant.

(now known as Chekanov-Eliashberg dga)

Ex γ = positive braid

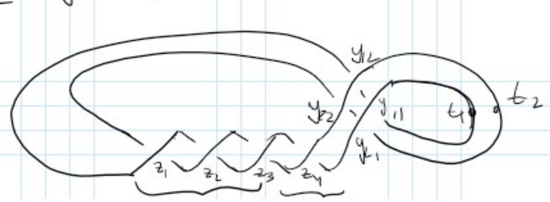
Casals-Ng: there is a Legendrian link with (x,y) projection

degree 0 $\quad$ degree 1

Generators of dga: all crossings in this picture

Differential counts certain disks.

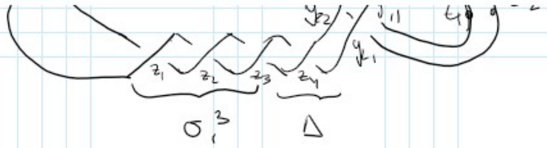
Ex $\gamma = \sigma_1^3$



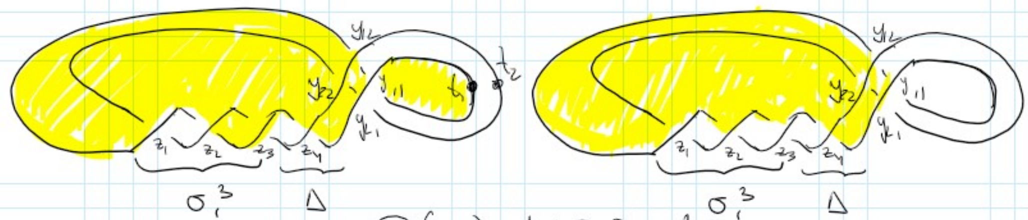
$\partial(z_i) = 0 \quad \deg(z_i) = 0$

$\partial(t_i) = 0 \quad \deg(t_i) = 0$

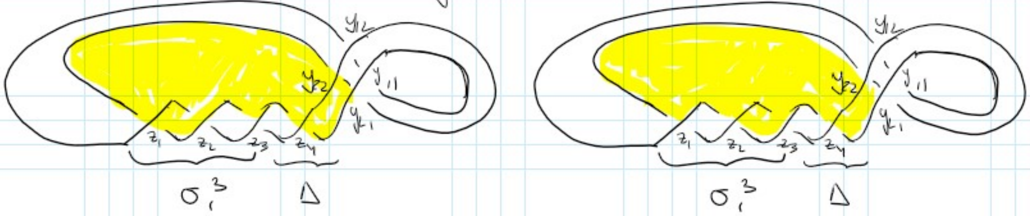
$\deg(y_{ij}) = 1$

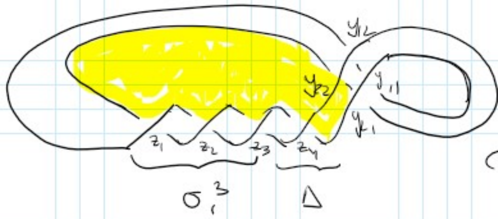


$\deg(y_{ij}) = 1$



$\partial(y_{11}) = 1 + z_2 z_3 - t_1$





$\partial(y_{21}) = z_3 + z_1 + z_1 z_2 z_3$

Thm (Kálmán, Casals-Ng) $X(\gamma) = \operatorname{Spec} H^0(\mathcal{A}_L)$

$L =$  $=$ augmentation variety of $\mathcal{A}_L$

In fact, $\partial(y_{ij})$ = defining relations for $X(\gamma)$.

- We use Chekanov-Eliashberg dga $\mathcal{A}_L$ to define braid varieties for <u>non-positive braids equivalent to positive ones</u> $(*)$
- Use them to verify invariance under Reidemeister 2 $\sigma_i \sigma_i^{-1} = 1$, Reidemeister 3 and positive stabilization using explicit geometric models
- For experts: for non-positive braids $\mathcal{A}_L$ has generators of negative degree, but $(*)$ implies that $\underline{H^{<0}(\mathcal{A}_L) = 0}$.

$$\left.\deg(y_{ij}) = 1\ \right]\ \text{equations on } \mathbb{C}^N$$

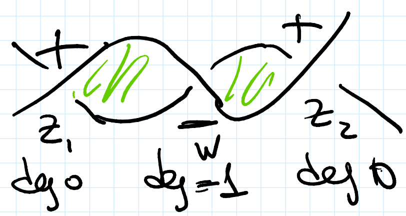
$$\left.\deg(z_i) = 0\ \right]\ \mathbb{C}^N$$

$$\left.\text{neg. crossing} \Rightarrow \deg -1\ \right]\ \text{vector fields on } \mathbb{C}^N$$

$d^2 = 0 \Rightarrow$ vector fields commute & preserve the equations

If $H^{<0}(\mathcal{A}_L) = 0$ then vector fields integrate to a free action of $\mathbb{C}^m$

<u>R2</u> Add a variable z_i, but free action of $\mathbb{C}$ kills it.



$\partial(z_1) = w + \ldots$

$\partial(z_2) = w + \ldots$

Vector field for w $\quad \frac{\partial}{\partial z_1} \pm \frac{\partial}{\partial z_2}$