Braids @ ICERM 2022

Categorifying Burau representations & fusion categories

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The Burau representation of $\mathbb{B}(I_2(n))$

- Rank two : $I_2(n) :=$ 

↳ $\mathbb{B}(I_2(3))$, $\mathbb{B}(I_2(4))$, $\mathbb{B}(I_2(5))$

$\cong Br_3$

The Burau representation of $\mathbb{B}(I_2(n))$

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$\hookrightarrow$ $\mathbb{B}(I_2(3))$, $\mathbb{B}(I_2(4))$, $\mathbb{B}(I_2(5))$

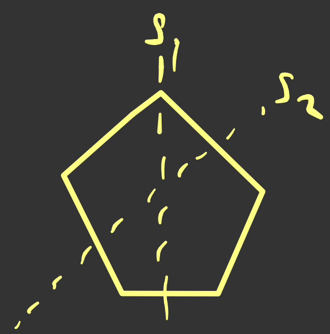
$\cong Br_3$

$\downarrow \qquad \downarrow \qquad \downarrow$

$\mathbb{W}(I_2(3))$ $\quad$ $\mathbb{W}(I_2(4))$ $\quad$ $\mathbb{W}(I_2(5))$

$n = 3$

$n = 4$



$n = 5$

☆ $W(I_2(n)) \cong D_n$

The Burau representation of $\mathbb{B}(I_2(n))$

– Rank two : $I_2(n) :=$ 

$\hookrightarrow$ $\mathbb{B}(I_2(3))$, $\mathbb{B}(I_2(4))$, $\mathbb{B}(I_2(5))$ $\xrightarrow{\quad\rho\quad}$ $GL_2(\mathbb{R}[q^{\pm 1}])$ Burau representation

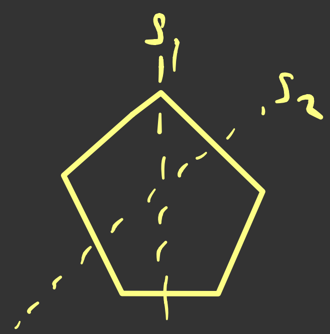
$\cong Br_3$

$$\begin{array}{ccccc}
\mathbb{B}(I_2(3)) & \mathbb{B}(I_2(4)) & \mathbb{B}(I_2(5)) & \xrightarrow{\rho} & GL_2(\mathbb{R}[q^{\pm 1}]) \\
\downarrow & \downarrow & \downarrow & & \downarrow q=-1 \\
W(I_2(3)) & W(I_2(4)) & W(I_2(5)) & \longrightarrow & GL_2(\mathbb{R})
\end{array}$$

Geometric representation.

$n=3$

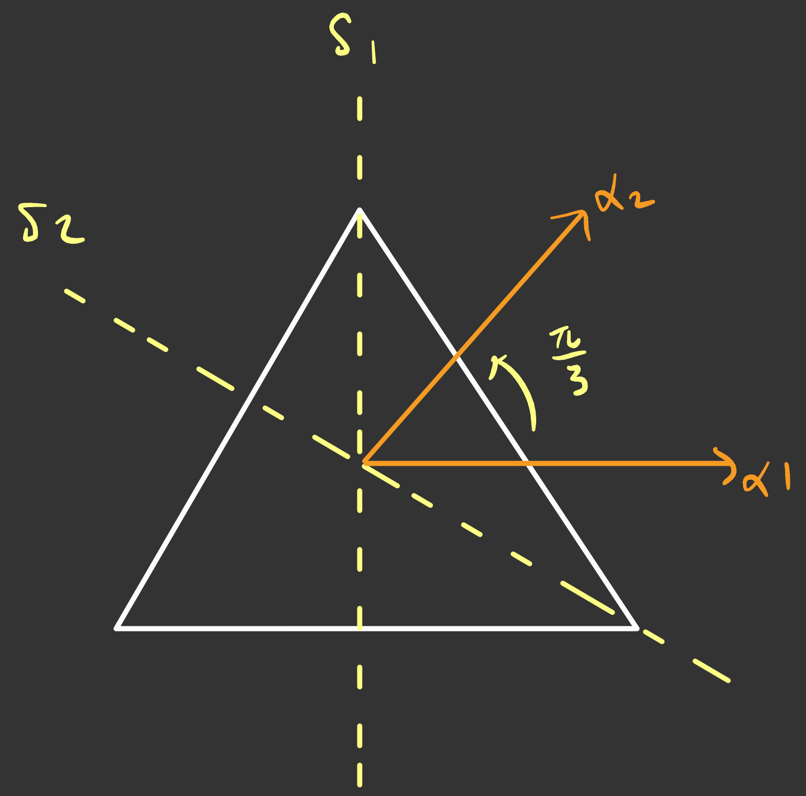
$n=4$

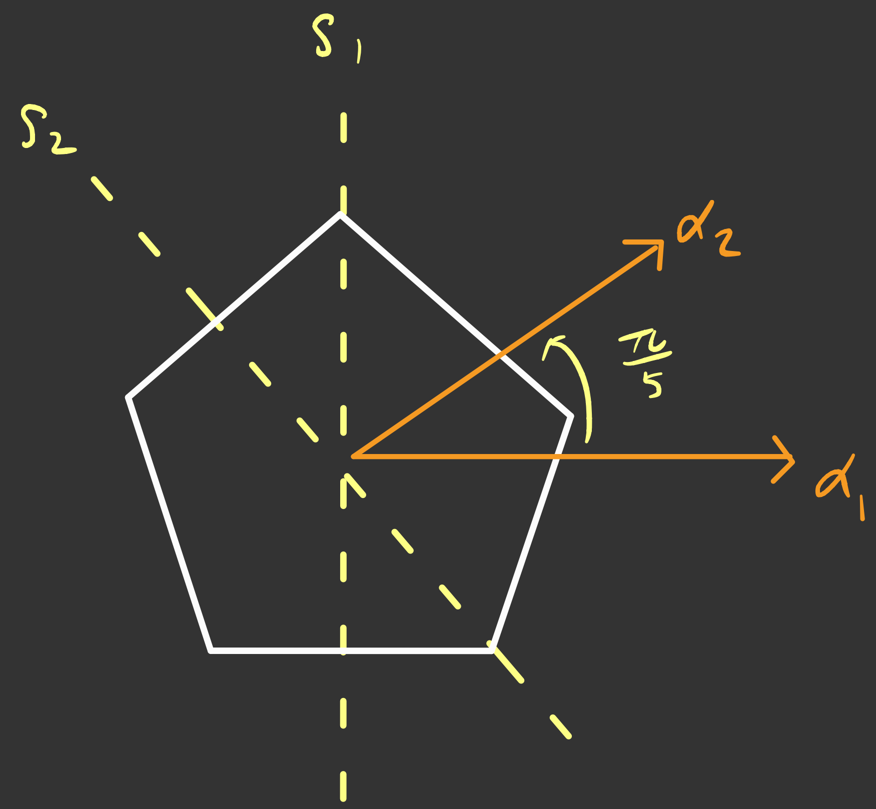


$n=5$

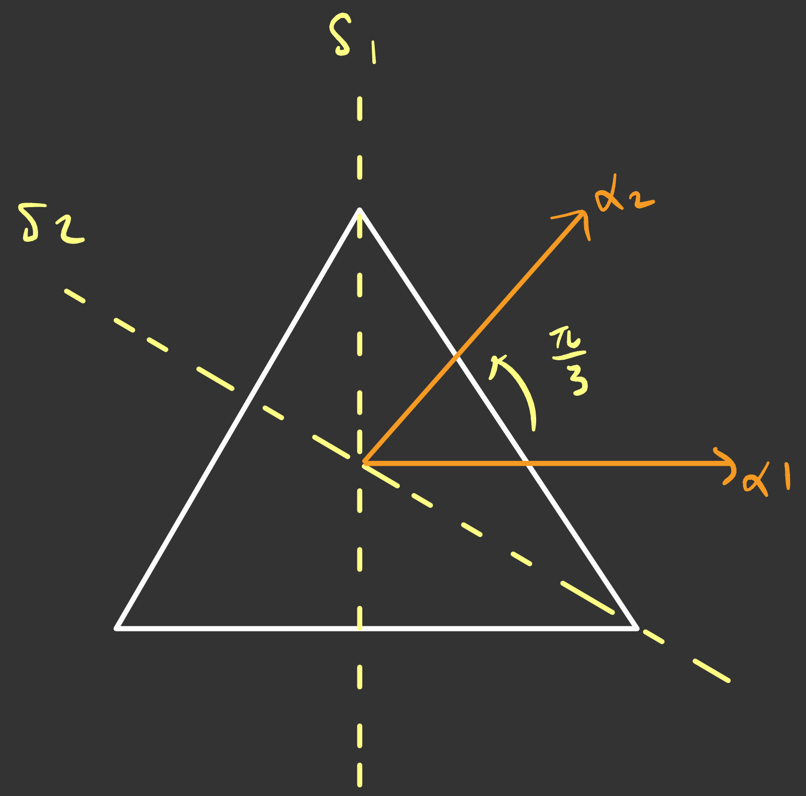
☆ $W(I_2(n)) \cong D_n$

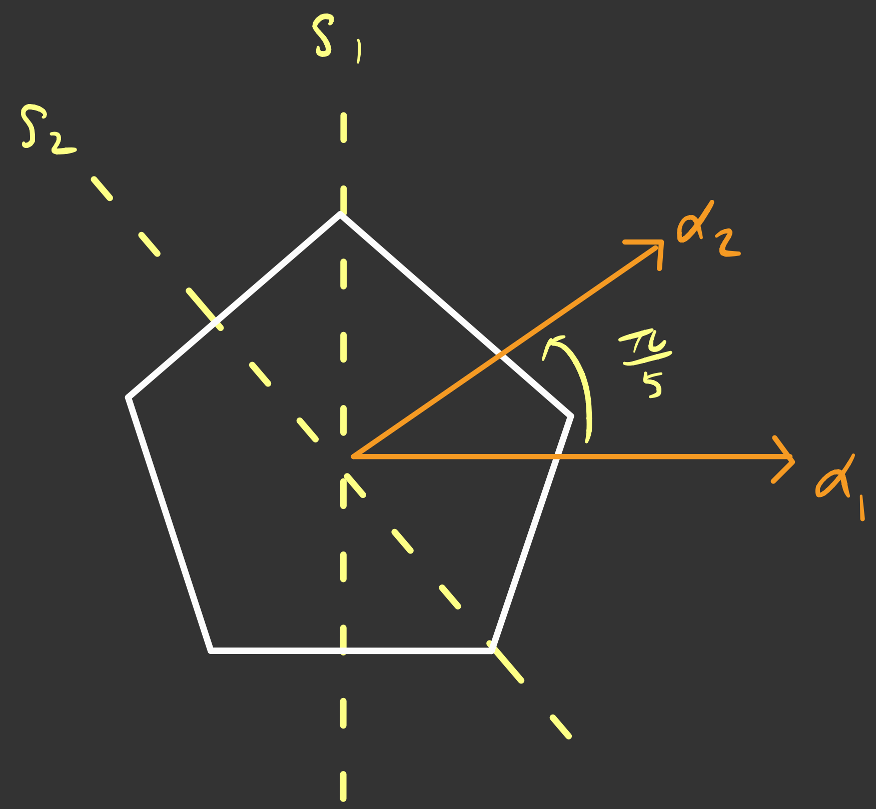
The Geometric representation of $W(I_2(n))$





The Geometric representation of $W(I_2(n))$

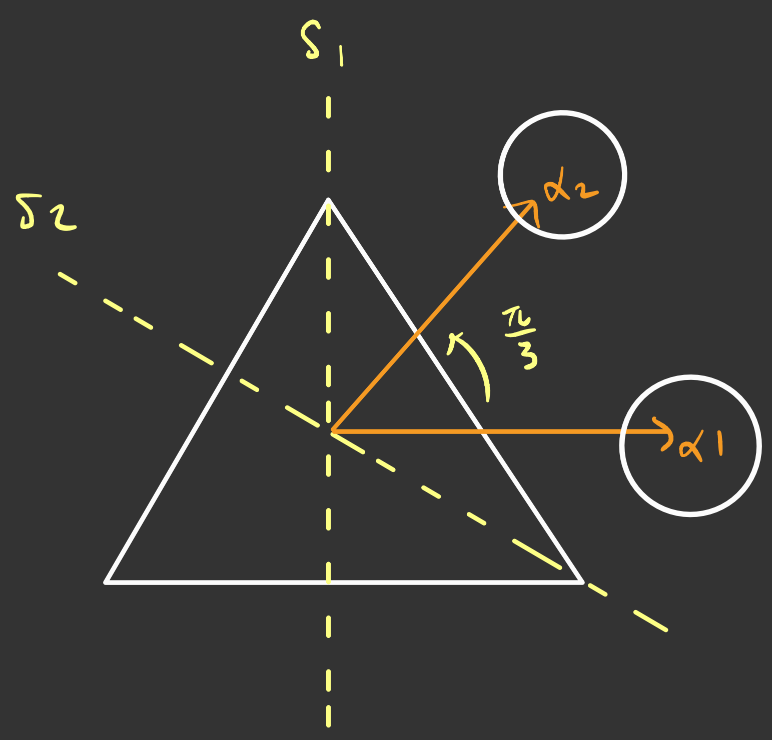




$$2\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i = j \\ 2\cos\frac{\pi}{n}, & i \neq j \end{cases}$$

$$s_i \cdot v = v - 2\langle \alpha_i, v \rangle \alpha_i$$

Categorifying i when $n=3$

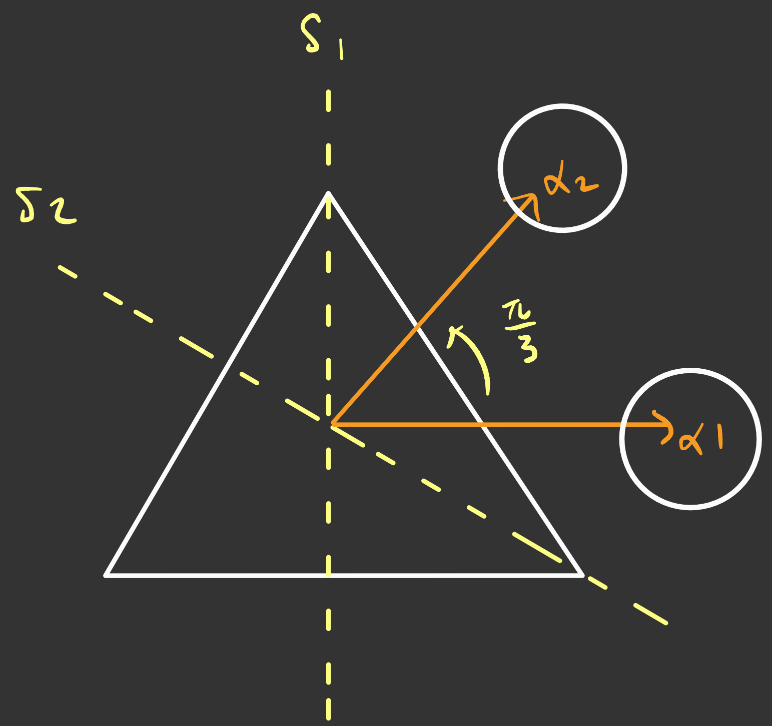


$$\boxed{2\cos\frac{\pi}{3} = 1 \in \mathbb{Z}}$$

$$2\langle\alpha_i,\alpha_j\rangle = \begin{cases} 2, & i=j \\ 1, & i\neq j \end{cases}$$

$$s_i \cdot v = -2\langle\alpha_i, v\rangle\alpha_i + v$$

Categorifying ; when $n=3$ (Khovanov - Huerfano, Rouquier - Zimmermann)



$$\boxed{2\cos\frac{\pi}{3} = 1 \in \mathbb{Z}}$$

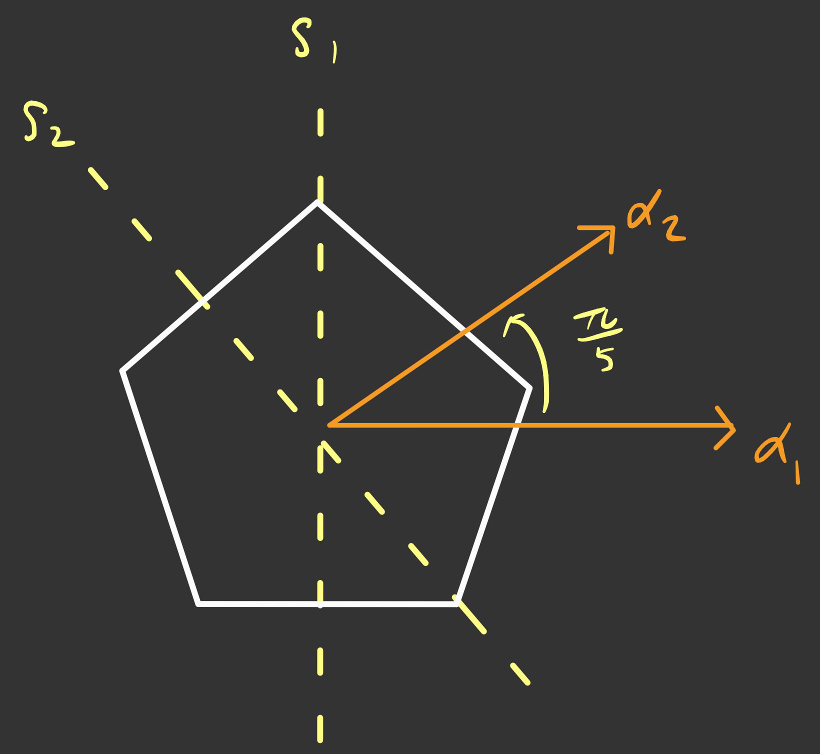
Construct a f.d. algebra A

$\rightsquigarrow \quad P_1, P_2 \in {}_A\mathrm{mod}$

$$2\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i=j \\ 1, & i\neq j \end{cases} \quad \rightsquigarrow \quad {}_iP \otimes_A P_j \cong \begin{cases} \mathbb{R}^2, & i=j \\ \mathbb{R}, & i\neq j \end{cases}$$

$$s_i \cdot v = -2\langle \alpha_i, v \rangle \alpha_i + v \quad \rightsquigarrow \quad \sigma_i \cdot V^{\bullet} = \mathrm{cone}\left(P_i \otimes {}_iP \otimes V^{\bullet} \to V^{\bullet}\right)$$

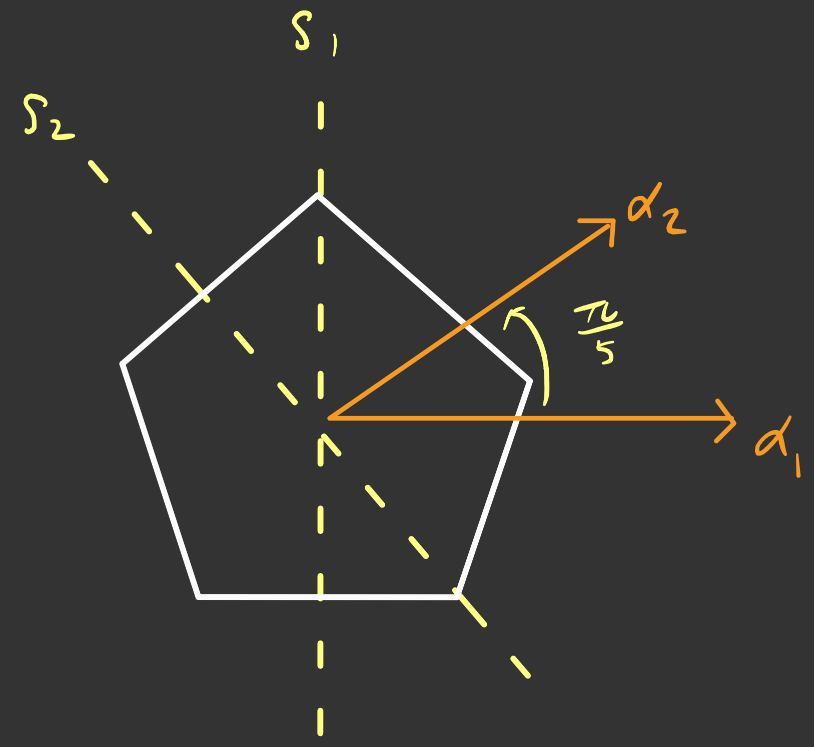
Categorifying ; when $n = 5$



$$2\cos\frac{\pi}{5} = \delta = \text{the golden ratio.}$$

$$2\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & i = j \\ \delta & i \neq j \end{cases} \rightsquigarrow {}_iP \otimes_A P_j \cong \begin{cases} \mathbb{1} \oplus \mathbb{1} & i = j \\ \boxed{?} & i \neq j \end{cases}$$

Categorifying when $n=5$

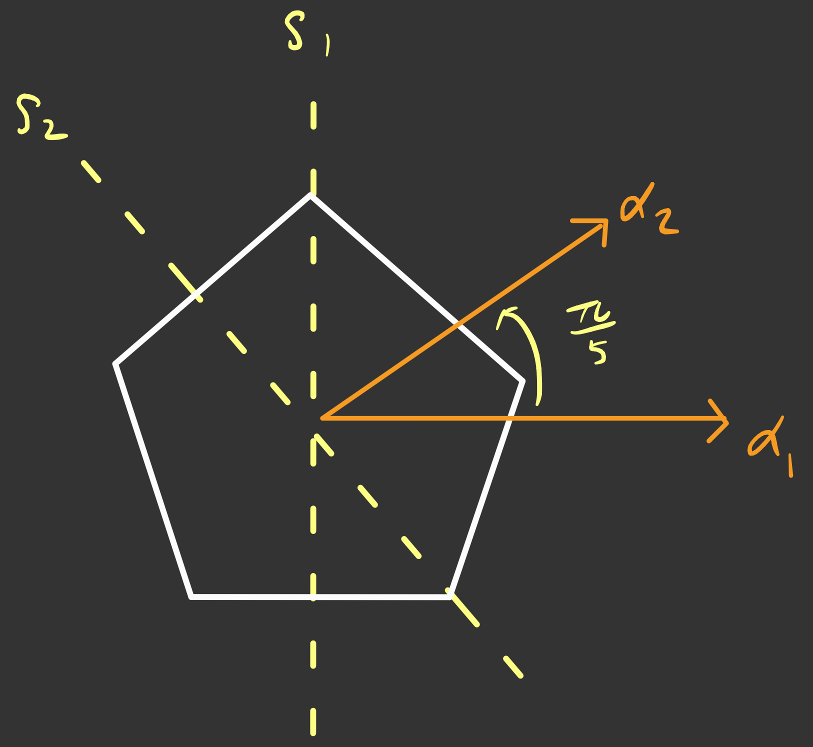


$2\cos\frac{\pi}{5} = \delta =$ the golden ratio.

$$2\langle\alpha_i, \alpha_j\rangle = \begin{cases} 2 & i=j \\ \delta & i\neq j \end{cases} \rightsquigarrow {}_iP \otimes_A P_j \cong \begin{cases} \mathbb{1}\oplus\mathbb{1} & i=j \\ \boxed{?} & i\neq j \end{cases}$$

We build **algebras** in category $\mathcal{C}$ that **categorifies** $\mathbb{Z}[\delta]$!

Categorifying when $n=5$



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Construct an algebra A in Fib

$\rightsquigarrow \quad P_1 , P_2 \in {}_A\mathrm{mod}$

$$2\langle \alpha_i , \alpha_j \rangle = \begin{cases} 2 & i=j \\ \delta & i \neq j \end{cases} \rightsquigarrow {}_iP \otimes_A P_j \cong \begin{cases} \mathbb{1} \oplus \mathbb{1} & i=j \\ \boxed{\tau} & i \neq j \end{cases}$$

$$s_i \cdot v = -2\langle \alpha_i , v \rangle \alpha_i + v \rightsquigarrow \sigma_i \cdot V^\bullet = \mathrm{cone}\left(P_i \otimes {}_iP \otimes V^\bullet \to V^\bullet \right)$$

$$\delta^2 = 1 + \delta \rightsquigarrow \tau \otimes \tau \cong \mathbb{1} \oplus \tau .$$

Categorifying: general n

Fact: For each $n \geq 3$, there is a fusion category $\mathcal{C}_n$ such that $K_0(\mathcal{C}_n) \twoheadrightarrow \mathbb{Z}[2\cos\frac{\pi}{n}]$.

$\overline{\mathrm{Rep}}(U_{e^{i\frac{\pi}{n}}}(\mathfrak{sl}_2)) \simeq TLJ_n$

Categorifying ; general n

Fact: For each $n \geq 3$, there is a fusion category $\mathcal{C}_n$ such that $K_0(\mathcal{C}_n) \twoheadrightarrow \mathbb{Z}[2\cos\frac{\pi}{n}]$.

Thm. [H.] To each $n \geq 3$, we associate an algebra $\mathbb{A}$ in $\mathcal{C}_n$, which induces a (faithful) categorical action of

$$\mathbb{B}(I_2(n)) \curvearrowright \mathrm{Kom}^b({}_{\mathbb{A}}\mathrm{mod}).$$

This categorifies the Burau representation of $\mathbb{B}(I_2(n))$.

Thm. [Nielsen - Thurston]:

Let S be a compact orientable surface. Every mapping class element $\varphi \in MCG(S)$ is either:

① periodic ② reducible ③ pseudo-Anosov.

categorical analogue!

Thm. [H.]

Let $\Gamma = I_2(n)$. Every element $g \in \mathbb{B}(I_2(n)) \curvearrowright \mathrm{Kom}(\mathcal{A}^{\mathrm{mod}})$ is either

① periodic ② reducible ③ pseudo-Anosov

THANK YOU !

Question ?

Census L-space knots are braid positive, except for one that is not

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Humboldt-Universität zu Berlin

Kazhdan-Laumon Categories and the Symplectic Fourier Transform

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Department of Mathematics
Massachusetts Institute of Technology

February 16, 2022

Overview

- Symplectic Fourier transforms: an interesting action of the braid group in the context of geometric representation theory
- It was used by Kazhdan-Laumon to construct a “glued category” of perverse sheaves on the basic affine space, with the hope of a new construction of representations of $G(\mathbb{F}_q)$
- Braid theory may help us solve this problem and carry out their original construction

The Hecke algebra and its regular representation

- G: semisimple algebraic group over k (algebraically closed). W: its Weyl group. B_W: generalized braid group.

Definition

For any $q \in \mathbb{C}$, the Hecke algebra $\mathcal{H}_q$ is the quotient of $\mathbb{C}[B_W]$ by the relations $(s_i + q^{-1})(s_i - 1) = 0$.

- Deformation of the Weyl group W. Finite-dimensional of dimension $|W|$.
- Regular representation: left action of $\mathcal{H}_q$ on itself.

Categorifying the regular representation

- G/U (which is $\mathbb{A}^2 \setminus \{(0,0)\}$ for SL_2).
- $\mathrm{Perv}_B(G/U) \iff \mathcal{O}_0$. Simple objects $\{L_w\}_{w \in W}$.
- $\dim K^0(\mathrm{Perv}_B(G/U)) = |W|$.

Proposition

- The braid group B_W "acts" on $\mathrm{Perv}_B(G/U)$ via endofunctors defined geometrically called *symplectic Fourier transforms* – which we will explain in a minute.
- This gives an action of $\mathbb{C}[B_W]$ on $K^0(\mathrm{Perv}_B(G/U))$. It factors through $\mathcal{H}_q$, and further, $K^0(\mathrm{Perv}_B(G/U))$ is isomorphic to the regular representation of $\mathcal{H}_q$.

Symplectic Fourier transforms on $\mathbb{A}^2_{\mathbb{F}_q}$

- Symplectic form: $\langle , \rangle : \mathbb{A}^2_{\overline{\mathbb{F}_q}} \to \overline{\mathbb{F}_q}$, $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_2 - x_2 y_1$.
- Let $\mathcal{F}$ be a perverse sheaf on $\mathbb{A}^2_{\mathbb{F}_q}$.
- Grothendieck's sheaf-function correspondence: from $[\mathcal{F}] \in K^0$, get element of $\prod_{n \geq 1} \mathbb{C}[\mathbb{A}^2_{\mathbb{F}_{q^n}}]$.
- Can define the symplectic Fourier transform via six functors & Artin-Schreier sheaves, but let's focus on functions.

Definition

Let $\psi : \mathbb{F}_{q^n} \to \mathbb{C}^*$ be an additive character. The symplectic Fourier transform on $\mathbb{A}^2_{\mathbb{F}_{q^n}}$ is an endomorphism of $\mathbb{C}[\mathbb{A}^2_{\mathbb{F}_{q^n}}]$ given by

$$\mathrm{FT}_\psi(f)(x) = \sum_{y \in \mathbb{A}^2_{\mathbb{F}_{q^n}}} f(y) \psi(\langle x, y \rangle) \tag{1}$$

Symplectic Fourier transforms on G/U

- Simple reflection $s \implies$ parabolic subgroup $P_s \subset G \implies Q_s = [P_s, P_s]$.
- $G/U \to G/Q_s$ fibration with $\mathbb{A}^2 \setminus \{(0,0)\}$ fibers. "Closure" $V \to G/Q_s$ with $\mathbb{A}^2$ fibers; $G/U \to G/Q_s$.
- Symplectic Fourier transform corresponding to s on perverse sheaves: "fiberwise" Fourier transform:
  $G/U \hookrightarrow V$, apply $\mathbb{A}^2$ Fourier transform, restrict to G/U.

Observations

The $\mathbb{A}^2$ Fourier transform from the previous slide gives a W-action: $s^2 = 1$. This Fourier transform gives a braid action, but we do *not* have $s^2 = 1$: this is because $\mathbb{A}^2 \setminus \{(0,0)\} \neq \mathbb{A}^2$. On $\mathrm{Perv}_B(G/U)$, it satisfies the Hecke relation $(s-1)(s+q^{-1}) = 0$.

What about on $\mathrm{Perv}(G/U)$?

- We defined the symplectic Fourier transform for all perverse sheaves, not just B-equivariant ones.
- E.g. for $\mathbb{A}^2$, Fourier transform on "B-equivariant sheaves" generates a 3-dimensional space spanned by $\chi_{(0,0)}, \chi_{x-\mathrm{axis}}, \chi_{\mathbb{A}^2}$.
- On $\mathrm{Perv}(G/U)$, the relation $(s-1)(s+q^{-1}) = 0$ does not always hold.

Theorem (Polishchuk, 1998).

It instead satisfies the cubic relation $(s^2-1)(s+q^{-1}) = 0$.

Some motivation: gluing perverse sheaves

- The reason I care about studying this symplectic Fourier transform: Kazhdan-Laumon's construction of gluing perverse sheaves.
- In 1988, Kazhdan-Laumon used this symplectic Fourier transform to construct a "glued category" from $|W|$-many copies of $\mathrm{Perv}(G/U)$.
- They conjectured that their category had finite cohomological dimension. They showed that if it did, then this category could be used to produce a new beautiful geometric construction of representations of $G(\mathbb{F}_q)$.

Theorem (Bezrukavnikov and Polishchuk, 2001).

This category does not have finite cohomological dimension.

All hope is not lost!

- Kazhdan and Laumon's conjecture of finite cohomological dimension was too strong, but it wasn't strictly necessary to carry out their goal of constructing $G(\mathbb{F}_q)$-representations.
- In his 2001 paper "Gluing of perverse sheaves on the basic affine space", Polishchuk gave certain criteria for how we might salvage Kazhdan & Laumon's original idea to construct representations of $G(\mathbb{F}_q)$ from their glued category.
- He defined a notion of a "good representation" of $\mathbb{C}[B_W]$, and showed that if $K^0(\mathrm{Perv}(G/U))$ is such a representation, then the proposed construction can be made to work.

Toward Polishchuk's criteria

- The relation $(s^2 - 1)(s + q^{-1})$ satisfied by the symplectic Fourier transform means $K^0(\mathrm{Perv}(G/U))$ is not just a representation of $\mathbb{C}[B_W]$ but of the "cubic Hecke algebra" $\mathcal{H}_q^c$.

Theorem (M-F).

If this action of $\mathcal{H}_q^c$ factors through a finite-dimensional quotient, then $K^0(\mathrm{Perv}(G/U))$ is a "good representation", and so Polishchuk's criteria are satisfied.

- Unfortunately, $\mathcal{H}_q^c$ is not always finite-dimensional (but it is in small-rank cases like Type A_n for $n \leq 4$), so this doesn't complete the story.

Other relations

- Polishchuk found the cubic relation $(s^2 - 1)(s + q^{-1})$, but conjectured that other relations may also exist.

Theorem (M-F).

For any two simple reflections s_i, s_j, the relation

$$(s_i + q^{-1})(s_i - 1)s_j^2(s_i^2 - 1) = 0 \tag{2}$$

is satisfied by the action of the symplectic Fourier transform on $K^0(\mathrm{Perv}(G/U))$, and this relation is nontrivial in $\mathcal{H}_q^c$.

A question for braid theorists

- If we consider the algebra $\mathcal{H}_q^c/(r)$, where r is our new relation, we can then ask whether this algebra is finite-dimensional.
- This algebra (but with generic parameters: $(s-a)(s-b)(s-c)$, rather than our $a=1, b=-1, c=-q^{-1}$) was studied in detail by Marin in 2018. He conjectured in his paper that it *is* finite-dimensional, but this is still an open problem.
- So this open problem in geometric representation theory is connected to this open problem in braid theory which can be phrased purely in terms of understanding the structure of this mysterious quotient of the braid algebra.

Thanks!

A hooking conjecture on circle graphs motivated by Khovanov homology

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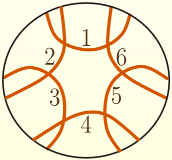
Universidad de Sevilla

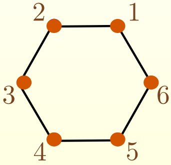
Braids in Representation Theory and Algebraic Combinatorics

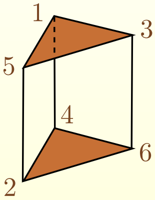
February 16, 2022 - Providence

The conjecture

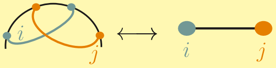
Circle with chords $\mathcal{C}$ → Circle graph $G_{\mathcal{C}}$ → Independent simpl. complex $X_{\mathcal{C}}$







chord $\longleftrightarrow$ vertex

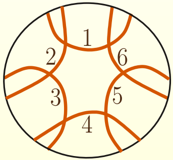


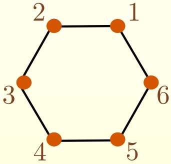
$$X_{\mathcal{C}} = \left\{ \begin{array}{c} \emptyset, \\ 1, 2, 3, 4, 5, 6, \\ 13, 14, 15, 24, 25, 26, 35, 36, 46, \\ 135, 246 \end{array} \right\}$$

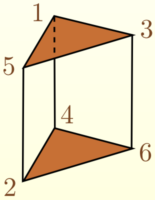
$\sigma \in X_{\mathcal{C}} \Leftrightarrow \sigma \equiv$ independent subsets of vertices of $G_{\mathcal{C}}$

The conjecture

Circle with chords $\mathcal{C}$ → Circle graph $G_\mathcal{C}$ → Independent simpl. complex $X_\mathcal{C}$







Conjecture:

$X_\mathcal{C}$ is homotopy equivalent to a wedge of spheres.

Motivation: extreme Khovanov homology

L

$H^{0,3}(\;) = \mathbb{Z}$

$H^{3,7}(\;) = \mathbb{Z}/2$

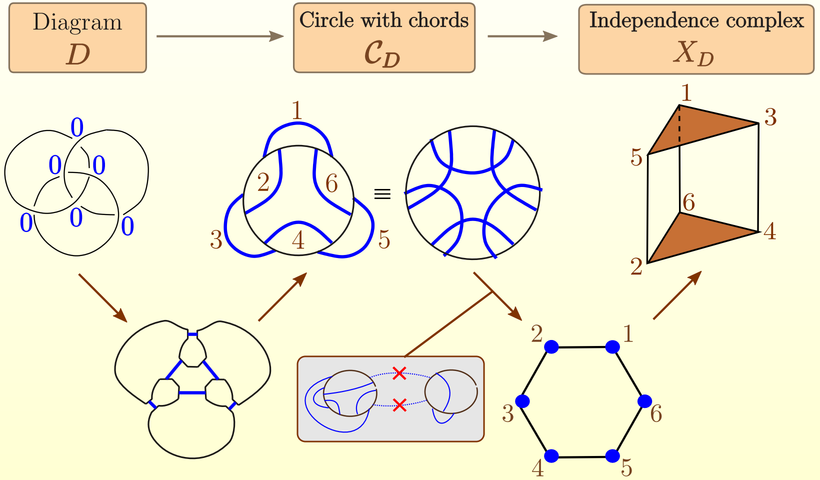
$H^{i,j}(\;)$ — i →, j ↕

$j \backslash i$	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}/2$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

$$J(L) = \sum_{i,j} (-1)^i \, q^j \, \mathrm{rk}(H^{i,j}(L))$$

$$J(\;) = q + q^3 + q^5 - q^9$$

Motivation: extreme Khovanov homology



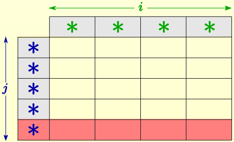
Motivation: extreme Khovanov homology

Diagram D → Circle with chords $\mathcal{C}_D$ → Independence complex X_D

Theorem [González-Meneses, Manchón, S.'17]:

The extreme Khovanov complex of D
is a copy of
the cohomology complex of X_D.

$$\boxed{H^{*,j_{min}}(D) \underset{\text{shift}}{=} H^*(X_D)}$$



Theorem [Cantero, S., 2019]:

There is a homotopy equivalence

$$\underset{\text{LS}}{\mathcal{X}^{j_{min}}}(D) \simeq_h \Sigma^{1-n}\Sigma^{\infty} \underset{\text{GMS}}{X_D}.$$

Conjecture holds for...

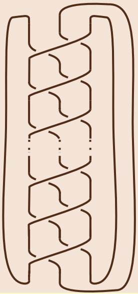
Conjecture: G circle graph $\Rightarrow X_G \sim$ wedge of spheres.

- L_n path $\Rightarrow X_{L_n} \sim \begin{cases} * & n = 3k \\ S^k & n = 3k+1,\ 3k+2 \end{cases}$
- T tree $\Rightarrow X_T \sim \begin{cases} * \\ \text{sphere} \end{cases}$
- C_n cycle $\Rightarrow X_{C_n} \sim \begin{cases} S^{k-1} & n = 3k \pm 1 \\ S^{k-1} \vee S^{k-1} & n = 3k \end{cases}$

$$G_{T(3,q)} = C_{2q}$$

$$\downarrow$$

$$H^{i,j_{min}}(T(3,q)) = \begin{cases} \mathbb{Z} & \text{if } 2q = 3k \pm 1, \quad i = 2q - k \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } 2q = 3k, \quad i = 2q - k \\ 0 & \text{otherwise} \end{cases}$$

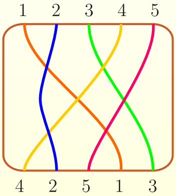


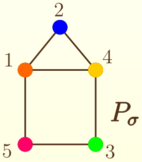
Conjecture holds for...

<u>Conjecture</u>: G circle graph $\Rightarrow X_G \sim$ wedge of spheres.

- P_σ permutation graph ✓

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

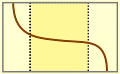
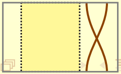




<u>Theorem</u> [Przytycki, S. '18]:

Given $\mathcal{X} = S^{k_1} \vee S^{k_2} \vee \cdots \vee S^{k_n}, \quad \exists \sigma \;/\; X_{P_\sigma} \sim \mathcal{X}$.

$$\mathcal{X} = S^3 \vee S^2 \vee S^1 \; = \; \sum \left(\sum \left(\sum (S^0) \vee S^0 \right) \vee S^0 \right)$$

- $\vee S^0 \longleftrightarrow$ 
- $\sum \longleftrightarrow$ 

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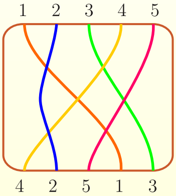
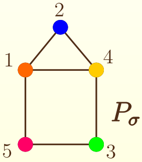
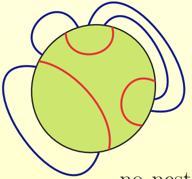
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$



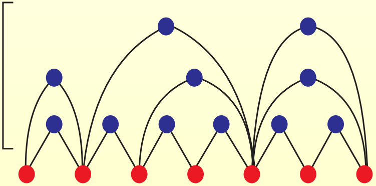


- Non-nested graph ✓



no nested chords in

adjacent 2



Conjecture holds for...

Conjecture: G circle graph $\Rightarrow X_G \sim$ wedge of spheres.

Theorem [Przytycki, S. '22]:

Let $\beta \in \mathbb{B}_n$, $n \leq 4$. Then, $X_{\hat{\beta}} \sim$ wedge of spheres.

More precisely, $X_{\hat{\beta}}$ has the homotopy type of a wedge of at most 4 spheres of the form $S^k \vee S^k \vee S^k \vee S^l$, for some $k, l \in \mathbb{Z}$.

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Thank you!