Dual Braids and the Braid Arrangement

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Overview

This talk focuses on the relationship between two classifying spaces for the braid groups:

- the one coming from the **braid arrangement**, and
- the one coming from its **dual Garside presentation.**

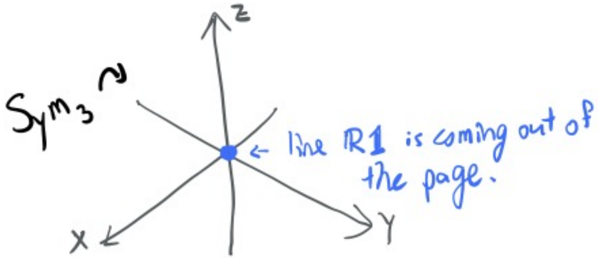
The first is extremely natural and the second has an orthoscheme metric which is (conjecturally) CAT(0). My goal is to advertise how to use polynomials to draw a connection between these two.

- From the GGT side, the polynomials are a surprise.
- For complex analysts, the dual braid complex is a surprise.

All new results are joint work wtih Michael Dougherty.

Symmetric Groups

The symmetric group SYM_d naturally acts on $\mathbb{R}^d$ (or $\mathbb{C}^d$) by permuting coordinates.



Since the line $\mathbb{R}\mathbf{1}$ (or $\mathbb{C}\mathbf{1}$) is pointwise fixed under this action, this action descends to $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ (or $\mathbb{C}^d/\mathbb{C}\mathbf{1}$).

Braid Arrangement

This $\textsc{Sym}_d$ action is not free, but the points with non-trivial stabilizers are a union of hyperplanes.

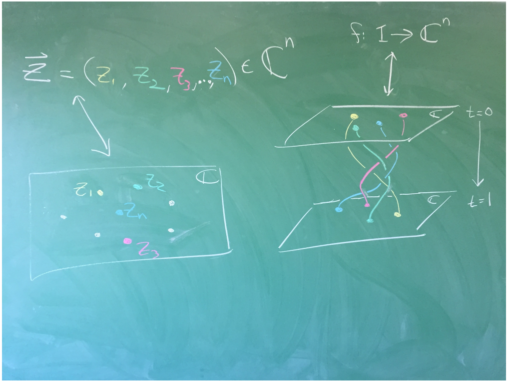
Definition

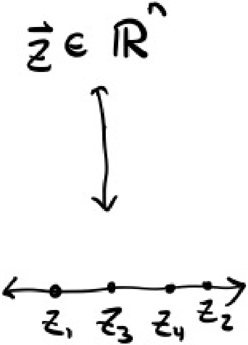
Let $H_{ij} = \{\mathbf{z} \mid z_i = z_j\}$ and let $\mathcal{H} = \bigcup_{i,j \in [d]} H_{ij}$ be the union of these hyperplanes. This is the **braid arrangement** and there are versions of it in $\mathbb{R}^d$, $\mathbb{R}^d/\mathbb{R}\mathbf{1}$, $\mathbb{C}^d$ and $\mathbb{C}^d/\mathbb{C}\mathbf{1}$.

We're most interested in the space $Y = \textsc{Sym}_d \backslash (\mathbb{C}^d - \mathcal{H})/\mathbb{C}\mathbf{1}$, an open manifold of dimension $2n$ with $n = d - 1$. Its natural metric is locally euclidean, but not compact and not complete.

Configuration Space Trick

A single point in $\mathbb{C}^d$ can be viewed as d labeled points in $\mathbb{C}$.





Complex Case: Braids

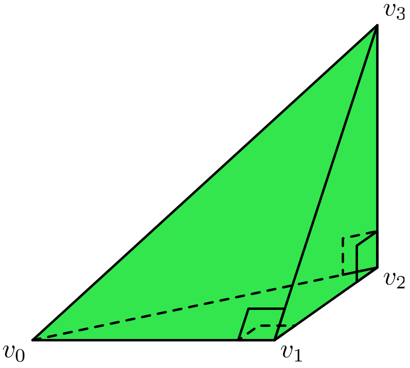
Theorem

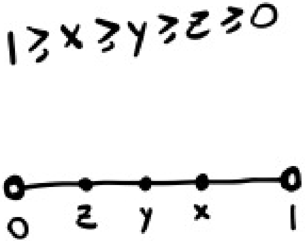
$\pi_1(\mathbb{C}^d - \mathcal{H}) = \text{PB}\textsc{raid}_d$ *and* $\pi_1(\textsc{Sym}_d \backslash (\mathbb{C}^d - \mathcal{H})/\mathbb{C}\mathbf{1}) = \text{B}\textsc{raid}_d$.

$\textsc{Sym}_d\backslash$ means points are unlabeled.
$-\mathcal{H}$ means points are distinct.
$/\mathbb{C}\mathbf{1}$ means point configurations up to rigid translation.

Real Case: Orthoschemes

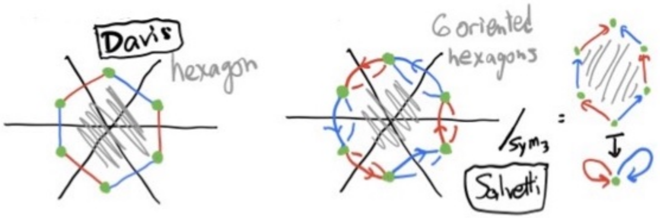
The corresponding theorem over the reals is that $\text{SYM}_d \backslash (\mathbb{R}^d - \mathcal{H})/\mathbb{R}\mathbf{1})$ is contractible. If we restrict the unlabeled points to an open unit interval, the result is the interior of an orthoscheme.





From ℝ to ℂ, from Davis to Salvetti

The cell structure dual to the real braid arrangement is a permutahedron and this is its Davis complex. The standard presentation of the braid group is derived from a slight complexification of the real arrangement and the standard classifying space (called the Salvetti complex) is built out of an oriented version of the Davis complex.



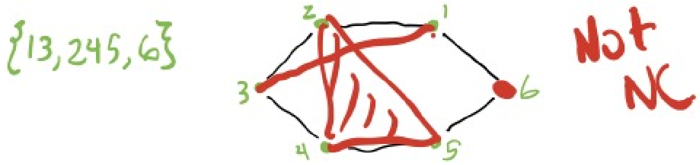
$$\text{BRAID}_3 = \langle a, b \mid aba = bab \rangle$$

Noncrossing Partitions

The standard presentation of the braid group starts with the points/strands in a row. The dual presentation starts with the points/strands arranged as the vertices of a convex polygon.

Definition (Noncrossing Partitions)

A partition π of the vertices of a convex d-gon is *noncrossing* if the convex hulls of the blocks are pairwise disjoint. The set of all noncrossing partitions, ordered by refinement, forms a lattice called NC_d.



Dual Braids

Definition (Dual Garside Structure)

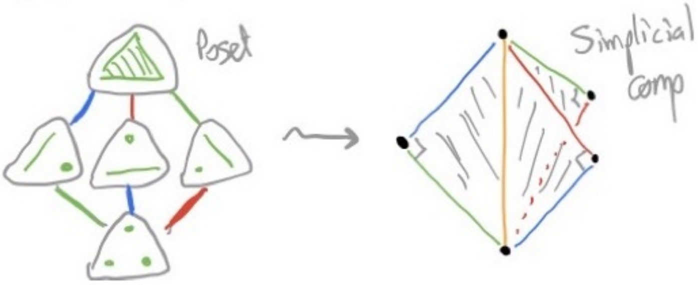
Noncrossing partitions lead of *dual simple braids*. The dual simple braids are part of a dual Garside structured. The Garside element is the d-cycle which rotates the vertices of the d-gon. The atoms are the set of $\binom{d}{2}$ half-twists.

Remark

The classifying space derived from this Garside structure is the *dual braid complex*. The geometric realization of NC_d is a strong fundamental domain from the BRAID_d action on the universal cover of the dual braid complex.

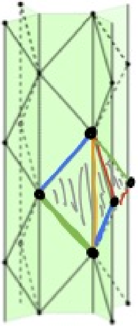
Dual Presentation: BRAID_3

The noncrossing partition lattice, its geometric realization and the dual presentation:



$\text{BRAID}_3 = \langle a, b, c, d \mid ab = bc = ca = d \rangle$

Dual Braid Complex: $d = 3$



Part of the (univ. cover of the) dual braid complex with $d = 3$.

Dual braids and orthoschemes

Tom Brady and I have conjectured that the dual braid complex with the orthoscheme metric is locally $\mathrm{CAT}(0)$ and this has been shown when the number of strands is very small.

Remark

The present work is part of a new approach to proving this theorem for all of the braid groups, but placing the dual braid complex in a natural broader context.

So, what's the connection?

Remark

The space $Y = \textsc{Sym}_d \backslash (\mathbb{C}^d - \mathcal{H})/\mathbb{C}\mathbf{1}$ and the dual braid complex are both classifying spaces for the group $\textsc{Braid}_d$, so they are homotopy equivalent.

But...is there a natural embedding of the dual braid complex into the space Y? so that the one deformation retracts to the other? Where does the orthoscheme metric come from? Where does the cell structure come from?

Remark

In 2017, Daan Krammer gave me a very strong hint (unpublished). The solution is to focus on point configurations as polynomials. In what follows, let $n = d - 1$.

Metric and Cell Structure

Definition (Point Configurations and Polynomials)

Let $Y = \text{Sym}_d \backslash (\mathbb{C}^d - \mathcal{H})/\mathbb{C}\mathbf{1}$ is the same as the space of monic degree d complex polynomials with distinct (unlabeled) roots up to (precomposition by) translation.

Theorem (Dougherty-M (in preparation))

*There is a natural **bounded** metric on Y so that its metric completion X is a compact space that supports a natural piecewise Euclidean cell structure.*

Remark

In X there are $d^{d-2} \cdot n!$ top-dimensional cells indexed by a maximal chain in NC_d and a permutation in Sym_n. Each such cell is the direct product of two d-dimensional orthoschemes.

Low dimensions

Example ($d = 2$)

When $d = 2$, the cell structure on X is an annulus built out of $2^0 \cdot 1! = 1$ rectangle (a 2-polytope).

Example ($d = 3$)

When $d = 3$, the cell structure on X is the union of $3^1 \cdot 2! = 6$ 4-polytopes.

Example ($d = 4$)

When $d = 4$, the cell structure on X is the union of $4^2 \cdot 3! = 96$ 6-polytopes.

Conjecture

X is $\mathrm{CAT}(0)$ iff the dual braid complex is $\mathrm{CAT}(0)$.

Roots, Critical Points and Critical Values

Let $p(z)$ be a degree-d complex polynomial. The roots and critical points of $p(z)$ are in the domain. The critical values of $p(z)$ are in the range.

Lemma

For a polynomial $p(z)$, the following are equivalent:

- *p has no repeated roots,*
- *the roots and critical points are disjoint sets, and*
- *the critical values of p are nonzero.*

Dual Braids and the Braid Arrangement

Theorem (Dual Braid Subspace)

The polynomials with all critical values on the unit circle form a subspace of X that is homeomorphic to the dual braid complex. And if you give X a metric defined by how its critical values move in the range, the dual braid subcomplex has its orthoscheme metric. Moreover, X deformation retracts to this dual braid subcomplex.

Theorem

The subspace of polynomials where all critical values lie in a fixed interval of the positive real axis is contractible, and as a metric object it is the orthoscheme realization of NC_d.

The scale depends on the choice of interval.

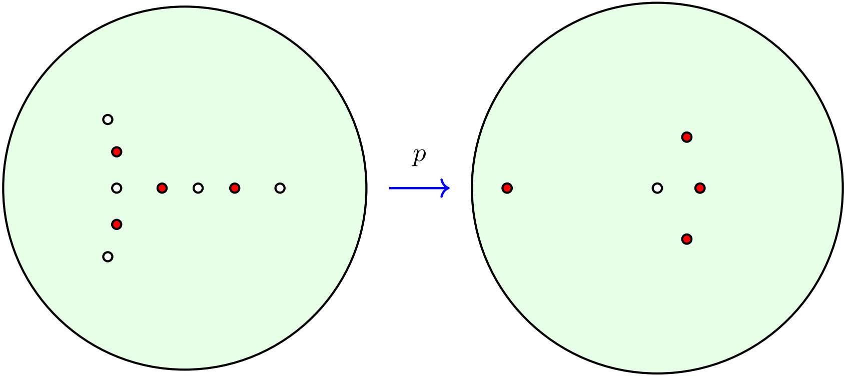
Individual Polynomials

Remark

Before putting an orthoscheme cross orthoscheme metric on X, we first assign a genus 0 surface with a CAT(0) rectangle cell structure to each polynoimal labeling a point in X.

In this final section I focus on a single example, to illustrate the construction.

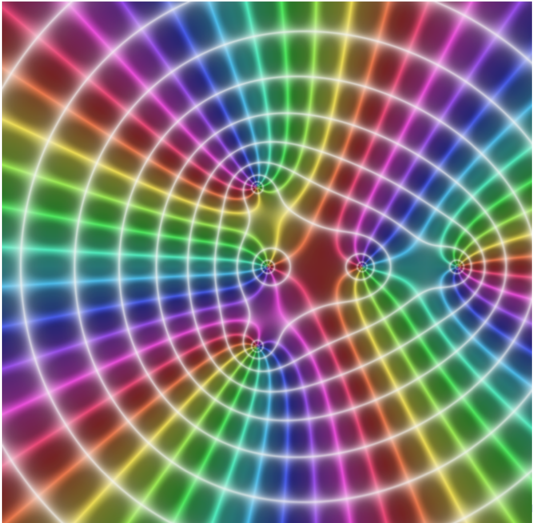
An Example: Roots 1



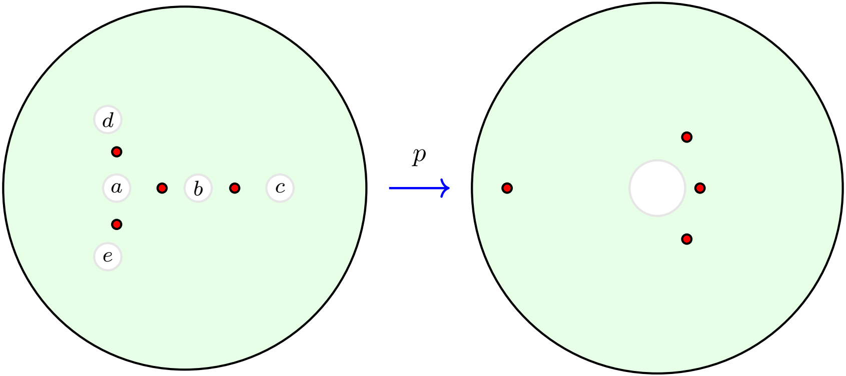
$$p(z) = 3z^5 - 15z^4 + 20z^3 - 30z^2 + 45z$$
$$p'(z) = 15(z^2+1)(z-1)(z-3)$$

Mathematica

Here is a tiled diagram produced by Mathematica.

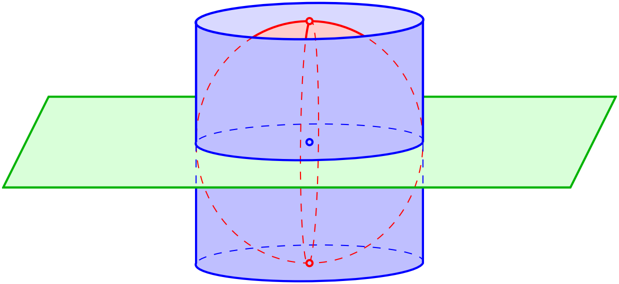


An Example: Roots 2



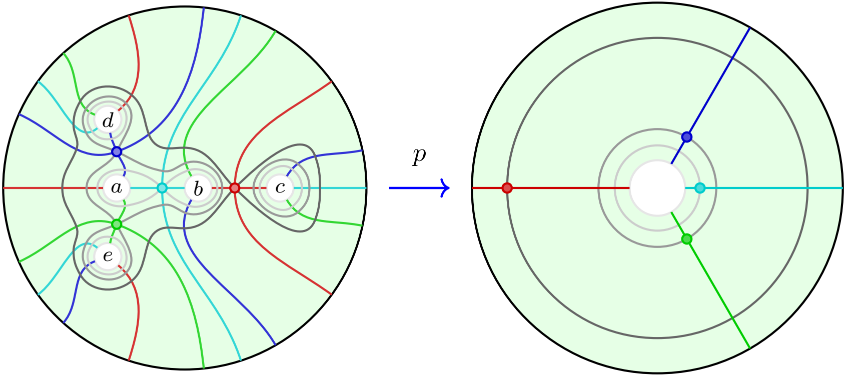
cleaning up:
label roots and pull back annulus

Technical Aside 1: Annular Metric



We give $\mathbb{C}_0$ the bounded metric of the open cylinder $\mathbb{T} \times \mathbb{I}^{\text{int}}$ (via stereographic projection followed by radial projection).
Rays and circles in $\mathbb{C}_0$ = longitudes and latitudes in $\mathbb{S}^2$
= horizontal circles and vertical lines in $\mathbb{T} \times \mathbb{I}$.

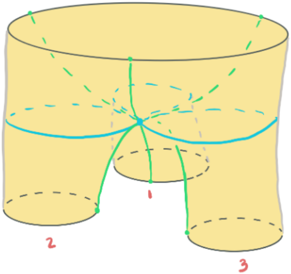
An Example: Square Tiling

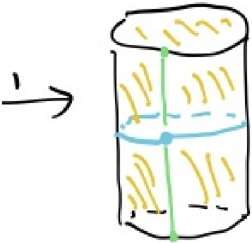


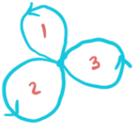
pull back both latitudes and longitudes

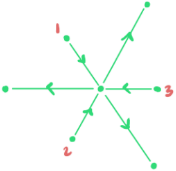
$\Longrightarrow$ square tiling of a disk with n holes

Technical Aside 2: Multipedal pants

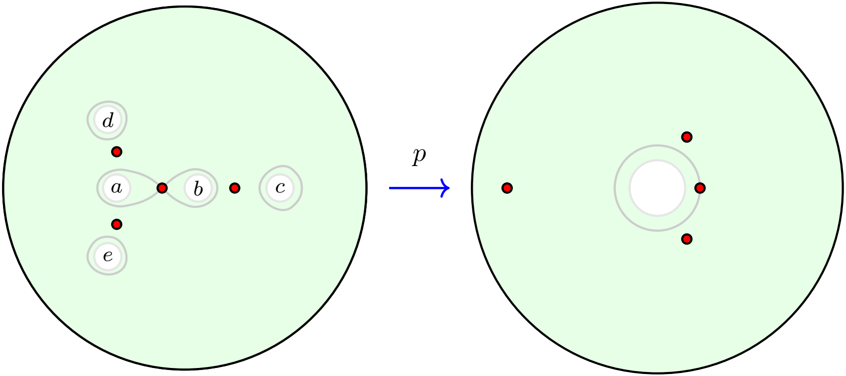








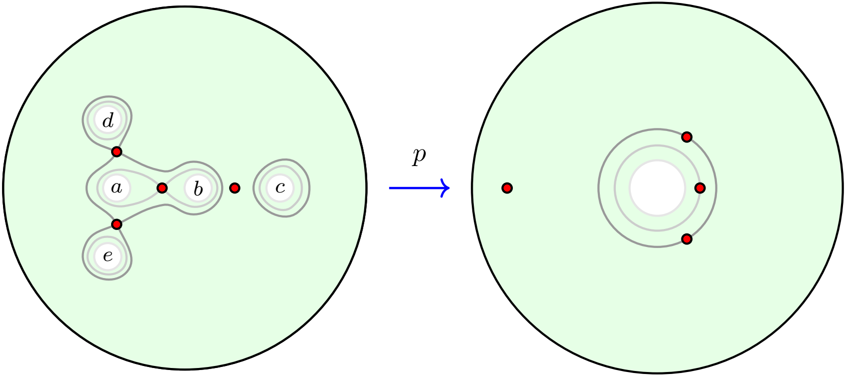
An Example: Latitudes 1



pull back circles (“latitudes”)

$\Longrightarrow$ lemniscates

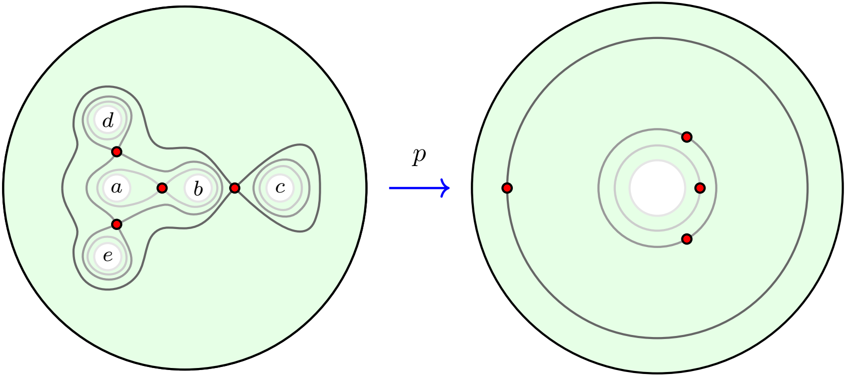
An Example: Latitudes 2



pull back circles (“latitudes”)

$\Longrightarrow$ lemniscates

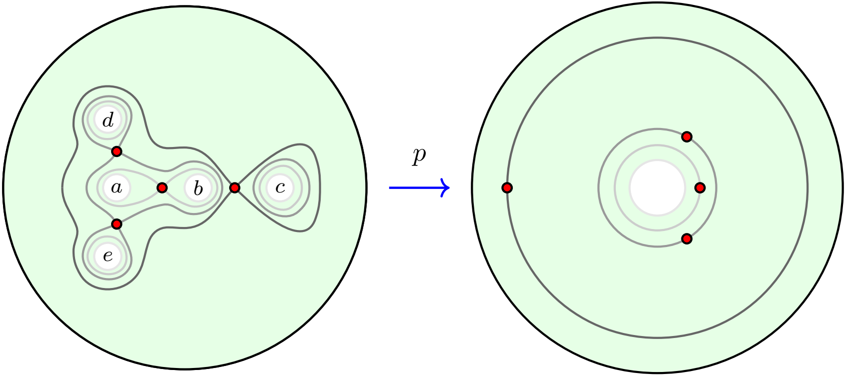
An Example: Latitudes 3



pull back circles (“latitudes”)

$\Longrightarrow$ lemniscates

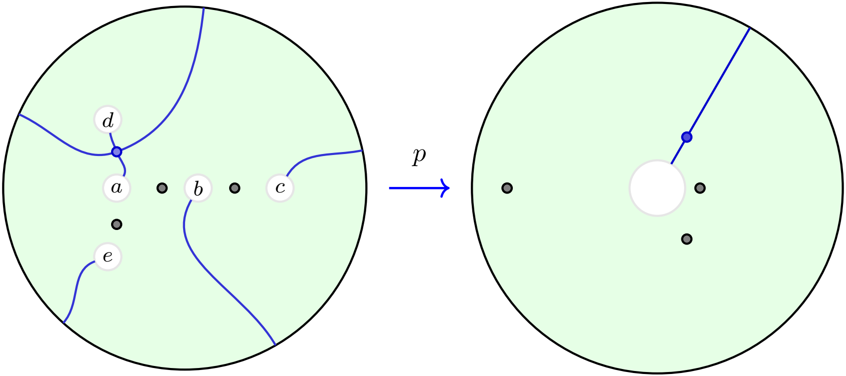
An Example: Chain of Partitions



determines a chain in the partition lattice:

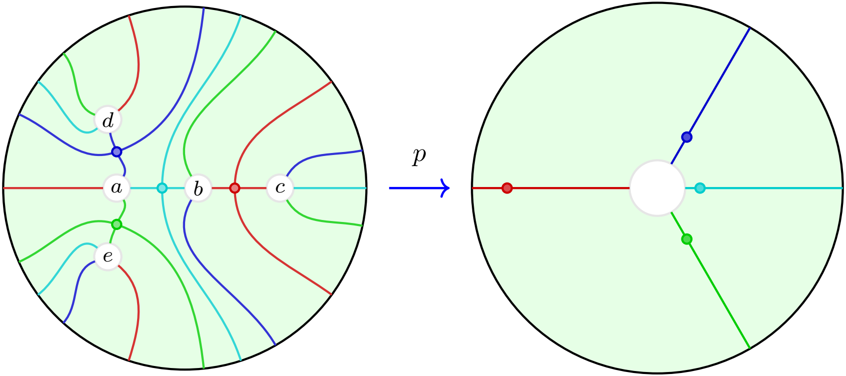
$\{a, b, c, d, e\} \subset \{ab, c, d, e\} \subset \{abde, c\} \subset \{abcde\}$

An Example: Longitude 1



pull back line segments (“longitudes”)
similar to basketballs (Martin-Savitt-Singer ’07)
and primitive majors (W. Thurston et al ’19)

An Example: Longitude 2



pull back line segments (“longitudes”)
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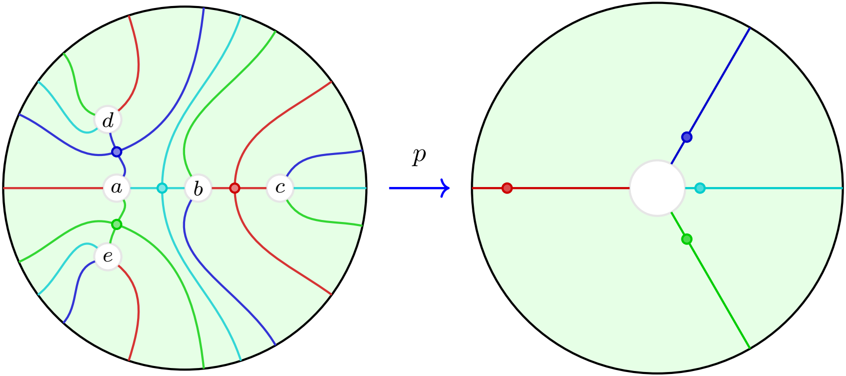
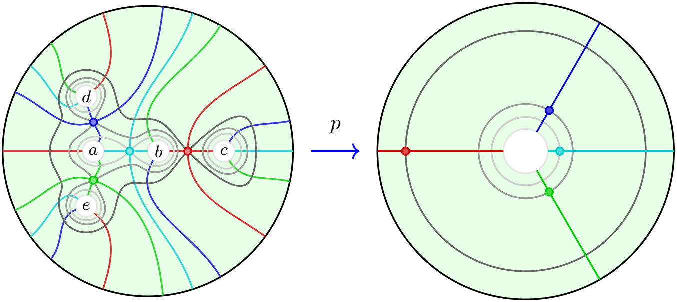
An Example: Factorization of a *d*-cycle



determines a cyclic word along the boundary
$\Longrightarrow$ factorization of an d-cycle permutation

From One Polynomial to the Space of Polynomials



Key Idea: use the geometry and combinatorics of (the surfaces assigned to) individual polynomials to add a bounded metric to the space X and then break it up into cells indexed by noncrossing paritition chains and permutations.

Top-dimensional Cells

Definition

A *truly generic polynomial* has nonzero critival values of multiplicity one, with distinct augments and magnitudes avoiding the negative real axis.

These are points in the interior of the $d^{d-2}n!$ top-dimensional cells in the critical value complex.

Thank You

References (all with Michael):

- Critical Points, Critical Values, and a Determinant Identity for Complex Polynomials *Proceedings AMS* 2020 (`arXiv:1908.10477`)
- Geometric Combinatorics of Polynomials I: The Case of a Single Polynomial *J.Algebra, to appear* (`arXiv:2104.07609`)
- Geometric Combinatorics of Polynomials II: The Critical Value Complex (in preparation)

(there's also a new survey article on my papers page based on this talk)