From Artin monoids to Artin groups

Joint with

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Notation

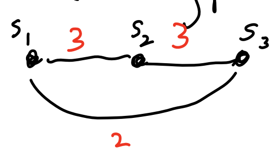
Γ = graph with vertices $\{s_1, \dots, s_n\}$ and edges labelled by $s_i \overset{m_{ij}}{\text{———}} s_j$, $m_{ij} \in \{2, 3, 4, \dots\}$

$$A_\Gamma = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \rangle$$

$\Downarrow$

W_Γ = associated Coxeter group formed by adding relators $s_i^2 = 1 \ \forall i$

Eg: Braid gp on 4 strands



(we include edges labelled 2)

A_Γ^+ = monoid generated by the same presentation

Two classes of Artin groups:

A_Γ spherical type $\iff$ W_Γ finite
A_Γ infinite type $\iff$ W_Γ infinite

For spherical type Artin groups, every element can be written in the form $a\Delta^{-n}$, $a \in A_\Gamma^+$, Δ = Garside element.

$\Downarrow$

Many questions about spherical-type A_Γ can be reduced to questions about A_Γ^+.

For infinite type, A_Γ^+ is still has nice combinatorial properties, but passing from A_Γ^+ to A_Γ is much more difficult!

$\Downarrow$

Many basic questions about infinite-type A_Γ remain open!

For all Γ, A_Γ^+ has

- a well-defined length function
$$\ell : A_\Gamma^+ \to \mathbb{Z}$$
- a partial order $a < b$ if $ac = b$ for some $c \in A_\Gamma^+$
- nice normal forms
- solvable word problem

Q: Can we use the Artin monoid to help answer questions about the Artin group?

Geometric viewpoint:

Coxeter groups can be realized as complex reflection groups

$W_\Gamma \circlearrowright \mathbb{C}^n$ generators (and conjugates) act as reflections in a hyperplane H_r

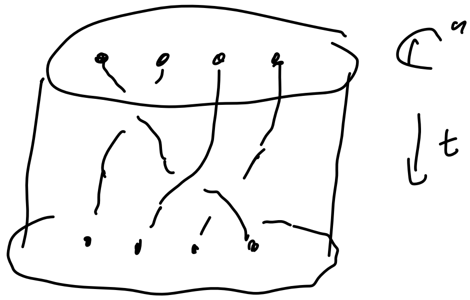
$W_\Gamma \overset{\text{free}}{\circlearrowright} \mathcal{H}_\Gamma = \mathbb{C}^n - \bigcup_r H_r$ (replace $\mathbb{C}^n$ by Tits cone in ∞-type case)

Thm (van der Lek) $A_\Gamma = \pi_1(\mathcal{H}_\Gamma / W_\Gamma)$

Eg: $W_\Gamma = \Sigma_n =$ symmetric gp $\circlearrowright \mathbb{C}^n$

$\mathcal{H}_\Gamma / \Sigma_n =$ configuration space of n distinct points in $\mathbb{C}$

$\pi_1(\mathcal{H}_\Gamma / \Sigma_n) = B_n$ braid group



In fact, this configuration space is a $K(\pi,1)$-space for B_n!

$K(\pi,1)$-Conjecture: For any Artin group, $\mathcal{H}_\Gamma / W_\Gamma$ is a $K(A_\Gamma, 1)$.
($\Leftrightarrow$ $\widetilde{\mathcal{H}_\Gamma}$ is contractible)

Deligne ('72): Conjecture holds for all spherical-type A_Γ

C-Davis ('95): Conjecture holds for for some infinity-type A_Γ
(FC-type & 2-dim'l)

Proof: Construct a simplicial complex $\mathcal{D}_\Gamma$ ("Deligne complex") and prove

(1) $\mathcal{D}_\Gamma \simeq \widetilde{\mathcal{H}_\Gamma}$ for all Γ

$$K(\pi,1)\text{-conj} \Longleftrightarrow \mathcal{D}_\Gamma \simeq pt$$

(2) $\mathcal{D}_\Gamma \simeq pt$ $\xrightarrow{\text{Deligne}}$ use combinatorial methods

$\xrightarrow{\text{C-Davis}}$ put a CAT(0) metric on $\mathcal{D}_\Gamma$

Current work with Boyd, Morris-Wright, Rees

Can we use properties of A_Γ^+ to help understand $\mathcal{D}_\Gamma$?

$\mathcal{D}_\Gamma$ = geometric realization of the poset
$\{aA_R \mid a \in A_\Gamma,\ A_R = \text{spherical-type parabolic}\}$

A_R = subgp of A_Γ generated by $R \subseteq S$

We introduce an analogue for the monoid A_Γ^+:

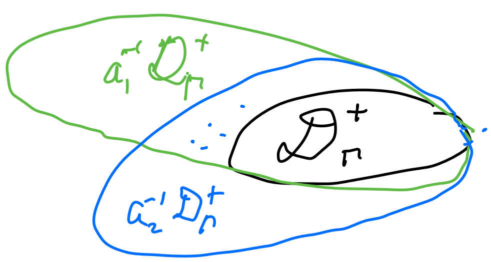
$\mathcal{D}_\Gamma^+$ = geometric realization of the poset
$\{aA_R^+ \mid a \in A_\Gamma^+,\ A_R^+ = \text{monoid of a spherical-type parabolic}\}$

<u>Thm</u> (Boyd-C-Morris-Wright) For any Artin group A_Γ, $\mathcal{D}_\Gamma^+$ is contractible and the natural map $\mathcal{D}_\Gamma^+ \to \mathcal{D}_\Gamma$ is a locally isometric embedding.

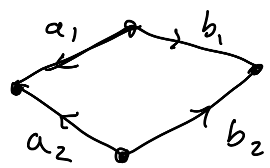
(The proof uses the combinatorial structure of A_Γ^+.)

$\mathcal{D}_\Gamma$ is covered by translates of $\mathcal{D}_\Gamma^+$
Can we use this to prove that $\mathcal{D}_\Gamma$ is contractible in some other cases?

Say $a \in A_\Gamma^+$, then
$$a\mathcal{D}_\Gamma^+ \subset \mathcal{D}_\Gamma^+ \quad \text{and} \quad \mathcal{D}_\Gamma^+ \subset a^{-1}\mathcal{D}_\Gamma^+$$
Now say $a_1, a_2 \in A_\Gamma^+$



What is $a_1^{-1}\mathcal{D}_\Gamma^+ \cap a_2^{-1}\mathcal{D}_\Gamma^+$. Is it just $\mathcal{D}_\Gamma^+$?
Is it contractible? Say
$$a_1^{-1}b_1 = a_2^{-1}b_2 \quad \text{for some } b_1, b_2 \in A_\Gamma^+$$



When can this happen?

More generally, need to understand expressions of elements $g \in A_\Gamma$ as alternating words $g = \dots a_2^{-1}b_2a_1^{-1}b_1$.

This leads to the study of the monoid Cayley graph

$\mathrm{Cay}(\underset{\text{group}}{G}, \underset{\text{gen set}}{S})$ = graph with vertices G and edges $g_1 \xrightarrow{s} g_2 = g_1 s$, $s \in S$

Monoid Cayley graph

$$\mathrm{Cay}^+(A_\Gamma) := \mathrm{Cay}(A_\Gamma, A_\Gamma^+)$$

Geodesic from 1 to g in $\mathrm{Cay}^+(A_\Gamma)$ is shortest word of the form

$$g = a_1 a_2^{-1} a_3 a_4^{-1} \cdots \qquad a_i \in A_\Gamma^+$$

Eg: Say A_Γ is spherical type. Then Every g can be decomposed as $g = a\Delta^{-n}$ $\quad 1 \xrightarrow{a} a \xleftarrow{\Delta^n} g \quad$ so $\mathrm{diam}(\mathrm{Cay}^+(A_\Gamma)) = 2$

Questions

① Is $\mathrm{diam}(\mathrm{Cay}^+(A_\Gamma)) = \infty$ for all infinite-type A_Γ?

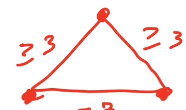
② Can we find geodesic paths in $\mathrm{Cay}^+(A_\Gamma)$? (Dehornoy, Holt, Rees, ...)

③ What properties of $\mathrm{Cay}^+(A_\Gamma)$ would allow us to prove that $\mathcal{D}_\Gamma^+$ is contractible?

Thm (Boyd - C - Morris-Wright - Rees)

① The following graphs Γ have infinite diameter $Cay^+(A_\Gamma)$.

2 vertices with $m_{ij} = \infty$



triangle with edge labels ≥ 3

② If $A_R \subseteq A_\Gamma$ is a standard parabolic subgp, then $Cay^+(A_R)$ geodesically embeds in $Cay^+(A_\Gamma)$. In particular

$$\mathrm{diam}(Cay^+(A_R)) = \infty \Rightarrow \mathrm{diam}(Cay^+(A_\Gamma)) = \infty$$

Proof

① Use work of Holt-Rees on "large-type" Artin groups to do the triangle case.

② Uses work of C-Paris and Godelle-Paris to construct a retraction

$$Cay^+(A_R) \underset{r}{\overset{\longrightarrow}{\longleftarrow}} Cay^+(A_\Gamma)$$