How to know if a parabolic subgroup of an Artin group merges conjugacy classes

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Braid in representation theory and algebraic combinatorics
Presentation of the braid group with $n + 1$ strands

$$A_n = \left\langle \sigma_1, \ldots, \sigma_n \bigg| \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| > 1 \end{array} \right\rangle$$
Presentation of the braid group with $n + 1$ strands

\[ A_n = \left\langle \sigma_1, \ldots, \sigma_n \mid \begin{array}{l}
\sigma_i \sigma_j = \sigma_j \sigma_i, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
i = 1, \ldots, n-1
\end{array} \right. \text{ if } |i - j| > 1 \right\rangle \]
Presentation of the braid group with $n + 1$ strands

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\sigma_i \sigma_i+1 \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, \ldots, n-1
\end{align*} \right\rangle
\]
Artin groups
Artin groups

- $S$ finite set of generators.

\[ W_S = \left\langle S \mid s^2 = 1, \underbrace{stst \ldots}_{m_{s,t} \text{ elements}} = \underbrace{tsts \ldots}_{m_{s,t} \text{ elements}}, \forall s \in S, \forall s, t \in S, s \neq t, m_{s,t} \neq \infty \right\rangle. \]

Images from John Baez’ blog

\[ A_S = \langle S \mid \underbrace{stst \ldots}_{m_{s,t} \text{ elements}} = \underbrace{tsts \ldots}_{m_{s,t} \text{ elements}}, \forall s, t \in S, s \neq t, m_{s,t} \neq \infty \rangle. \]
Artin groups

- $S$ finite set of generators.

Coxeter group (finite)

$$W_S = \left\langle S \mid s^2 = 1, \begin{array}{c} \text{stst \ldots} = \text{tsts \ldots} \end{array}, \forall s \in S, \forall s, t \in S, s \neq t, m_{s,t} \neq \infty \right\rangle.$$ 

Artin group (of spherical type)

$$A_S = \langle S \mid \begin{array}{c} \text{stst \ldots} = \text{tsts \ldots} \end{array}, \forall s, t \in S, s \neq t, m_{s,t} \neq \infty \rangle.$$
Coxeter graphs

- $\mathcal{V} = S$
Coxeter graphs

- $\mathcal{V} = S$

- Si $m_{s,t} = 2$

\[ s \quad t \]
Coxeter graphs

- $\mathcal{V} = S$

- Si $m_{s,t} = 2$

- If $m_{s,t} \neq 2$
Coxeter graphs

- $\mathcal{V} = S$
- If $m_{s,t} = 2$

\[
\begin{array}{c}
\circ \quad s \quad \circ \\
\hline
\circ \quad m_{s,t} \quad \circ \\
\circ \quad t \quad \circ
\end{array}
\]

- If $m_{s,t} \neq 2$

\[
\begin{array}{c}
\circ \quad s \quad \circ \\
\hline
\circ \quad m_{s,t} \quad \circ \\
\circ \quad t \quad \circ
\end{array}
\]

Some families of Artin groups

- RAAGs: $m_{s,t} \in \{2, \infty\}$.
- Spherical (or finite type): finite Coxeter group.
- FC-type: All complete subgraphs without $\infty$ are spherical.
- 2-dimensional: $\frac{1}{m_{s,t}} + \frac{1}{m_{s,r}} + \frac{1}{m_{t,r}} \leq 1$, $\forall s, t, r \in S$. 
Parabolic subgroups

Standard parabolic subgroup $A_{S'}$ of $A_S$

It is the subgroup generated by a subset $S' \subseteq S$. 

Theorem [Van der Lek 1983]

$A_{S'}$ is again an Artin group.
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Parabolic subgroup $P$ de $A_S$

$$P = \alpha^{-1} A_{S'} \alpha,$$

where $A_{S'}$ is a standard parabolic subgroup and $\alpha \in A_S$. 
Complexes using parabolic subgroups

• Deligne complex [Charney & Davis 1995]
  ▶ It uses spherical parabolic subgroups.
  ▶ CAT(0) in some cases.
  ▶ It has been used to study classic problems:
    ▶ $K(\pi, 1)$ conjecture [Charney & Davis ’95, Paris ’14].
    ▶ Acylindrical hyperbolicity [Martin & Przytycki ’19, Charney & Morris-Wright, Vaskou].
    ▶ Tits Alternative [Martin & Przytycki ’19].

• Complex of irreducible parabolic subgroups [C., Gebhardt, González-Meneses & Wiest ’19]
  ▶ It has been created for spherical Artin groups.
  ▶ It is totally analogous to the curve complex for the braid case.
  ▶ Generalized to FC-type Artin groups [Morris-Wright ’20].
  ▶ Its properties are being studied.

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- **Artin complex** [C., Martin & Vaskou’20]
Conjugacy stability problem for parabolic sg $A_X < A_S$

If two elements of a parabolic subgroup $A_X$ are conjugate in our Artin group $A_S$, are they conjugate “inside this parabolic subgroup”?

$$\exists \, c \in A_S, \, c^{-1}ac = b \implies \exists c' \in A_X, \, c'^{-1}ac' = b?$$

If the answer is yes, we say that $A_X$ is **conjugacy stable**.

This is always true for braids [González-Meneses 2014], but it is not true in general.
Irreducible Coxeter graphs (of finite type)
Theorem (Calvez, Cisneros, C., 2020)

Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:
Theorem (Calvez, Cisneros, C., 2020)

Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:

\[ E_6 : \quad \sigma_1 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6 \]
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$$D_5$$
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$E_6$:

\[ \sigma_1 \to \sigma_3 \to \sigma_4 \to \sigma_5 \to \sigma_6 \]

$D_5$

$E_7$:

\[ \sigma_1 \to \sigma_3 \to \sigma_4 \to \sigma_5 \to \sigma_6 \to \sigma_7 \]
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\[ E_6 : \quad \sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 D_5 \quad \text{and} \quad E_7 : \quad \sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 D_5 \]
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Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:

$E_6$:

\[ \sigma_1 \rightarrow \sigma_3 \rightarrow \sigma_4 \rightarrow \sigma_5 \rightarrow \sigma_6 \rightarrow D_5 \]

$E_7$:

\[ \sigma_1 \rightarrow \sigma_3 \rightarrow \sigma_4 \rightarrow \sigma_5 \rightarrow \sigma_6 \rightarrow \sigma_7 \rightarrow D_5 \]

$E_8$:

\[ \sigma_1 \rightarrow \sigma_3 \rightarrow \sigma_4 \rightarrow \sigma_5 \rightarrow \sigma_6 \rightarrow \sigma_7 \rightarrow \sigma_8 \]
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$E_7$:

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$E_8$:

\[ \sigma_1 \rightarrow \sigma_3 \rightarrow \sigma_4 \rightarrow \sigma_5 \rightarrow \sigma_6 \rightarrow \sigma_7 \rightarrow \sigma_8 \rightarrow D_5 \]
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Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:

- $E_6$: $\sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_2 D_5$
- $E_7$: $\sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_2 \sigma_7 D_5$
- $E_8$: $\sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 D_5 D_7$
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Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:

$E_6$:

\[
\begin{array}{c}
\sigma_1 \, \sigma_3 \, \sigma_4 \, \sigma_5 \, \sigma_6 \, D_5
\end{array}
\]

$E_7$:

\[
\begin{array}{c}
\sigma_1 \, \sigma_3 \, \sigma_4 \, \sigma_5 \, \sigma_6 \, \sigma_7 \, D_5
\end{array}
\]

$E_8$:

\[
\begin{array}{c}
\sigma_1 \, \sigma_3 \, \sigma_4 \, \sigma_5 \, \sigma_6 \, \sigma_7 \, \sigma_8 \, D_5 \, D_7 \, E_7
\end{array}
\]
Theorem (Calvez, Cisneros, C., 2020)

Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:

- $E_6$: $\sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 D_5$
- $E_7$: $\sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 D_5$
- $E_8$: $\sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 D_5 D_7 E_7$
- $H_4$: $\sigma_1 5 \sigma_2 \sigma_3 \sigma_4 H_3$
Theorem (Calvez, Cisneros, C., 2020)

Let $A_S$ be an Artin–Tits group of spherical type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following cases:

$$E_6 : \sigma_1 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6 \quad D_5$$  
$$E_7 : \sigma_1 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6 \quad D_5$$

$$E_8 : \sigma_1 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6 - \sigma_7 \quad D_5 \quad D_7 \quad E_7$$

$$H_4 : \sigma_1 - 5 \quad \sigma_2 - \sigma_3 - \sigma_4 \quad H_3$$  
$$\sigma_1 - \sigma_3 \quad \sigma_2k \quad D_{2k}$$
Theorem (C., Martin, Vaskou, preprint 2020)

Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:
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Theorem (C., Martin, Vaskou, preprint 2020)

Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

\[
\begin{array}{c}
\sigma_1 \overset{\text{odd}}{\longrightarrow} \sigma_2 \\
\sigma_1 \overset{\text{even}}{\longrightarrow} \sigma_2
\end{array}
\]
Theorem (C., Martin, Vaskou, preprint 2020)

Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

$\begin{align*}
\sigma_1 \text{ odd } & \quad \sigma_2 \\
\sigma_1 \text{ even } & \quad \sigma_2
\end{align*}$

conjugate by the generator of its centre:

$\begin{align*}
\sigma_2 \text{ odd } & \quad \sigma_1
\end{align*}$
Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

- $\sigma_1$ odd conjugate by the generator of its centre:
- $\sigma_2$ odd
- $\sigma_1$ even conjugate by the generator of its centre:
- $\sigma_2$ even
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Let \( A_S \) be an Artin–Tits group of large type and \( A_X \) a proper irreducible standard parabolic subgroup. \( A_X \) is conjugacy stable in \( A_S \) except for the following case:

\[ \sigma_1 \sigma_2 \]

conjugate by the generator of its centre:

\[ \sigma_2 \sigma_1 \]

\[ \sigma_1 \text{ even} \sigma_2 \]

conjugate by the generator of its centre:

\[ \sigma_1 \text{ even} \sigma_2 \]

\[ \sigma_1 \text{ odd} \sigma_2 \text{ odd} \sigma_3 \text{ odd} \sigma_4 \text{ even} \sigma_5 \text{ odd} \sigma_6 \]
Theorem (C., Martin, Vaskou, preprint 2020)

Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

- $\sigma_1 \text{odd} \quad \sigma_2 \text{odd}$ conjugate by the generator of its centre: $\sigma_2 \text{odd} \quad \sigma_1 \text{odd}$
- $\sigma_1 \text{even} \quad \sigma_2 \text{odd}$ conjugate by the generator of its centre: $\sigma_1 \text{even} \quad \sigma_2 \text{even}$
- $\sigma_1 \text{odd} \quad \sigma_2 \text{odd} \quad \sigma_3 \text{odd} \quad \sigma_4 \text{even} \quad \sigma_5 \text{odd} \quad \sigma_6 \text{odd}$

Coxeter graph of $A_S$

Coxeter graph of $A_S$

odd-labelled path

no odd-labelled path
Theorem (C., Martin, Vaskou, preprint 2020)

Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

- $\sigma_1$ odd, $\sigma_2$ odd: conjugate by the generator of its centre: $\sigma_2$ odd, $\sigma_1$ odd
- $\sigma_1$ even, $\sigma_2$ odd: conjugate by the generator of its centre: $\sigma_1$ even, $\sigma_2$ even
- $\sigma_1$ odd, $\sigma_2$ odd, $\sigma_3$ odd, $\sigma_4$ even, $\sigma_5$ odd, $\sigma_6$ odd
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Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

- $\sigma_1 \sigma_2$ odd conjugate by the generator of its centre: $\sigma_2 \sigma_1$ odd
- $\sigma_1$ even $\sigma_2$ even conjugate by the generator of its centre: $\sigma_1$ even $\sigma_2$
- $\sigma_1$ odd $\sigma_2$ odd $\sigma_3$ odd $\sigma_4$ even $\sigma_5$ odd $\sigma_6$
Theorem (C., Martin, Vaskou, preprint 2020)

Let $A_S$ be an Artin–Tits group of large type and $A_X$ a proper irreducible standard parabolic subgroup. $A_X$ is conjugacy stable in $A_S$ except for the following case:

Conjugate by the generator of its centre:

- $\sigma_1^{\text{odd}} \sigma_2^{\text{odd}}$
- $\sigma_1^{\text{even}} \sigma_2^{\text{odd}}$
- $\sigma_1^{\text{odd}} \sigma_2^{\text{even}} \sigma_3^{\text{odd}} \sigma_4^{\text{even}} \sigma_5^{\text{odd}} \sigma_6^{\text{odd}}$
When are two standard parabolic subgroups conjugate?

- If $A_X$ and $A_Y$ are different and $A_X$ is not spherical, they are never conjugate.
- If $A_X$ and $A_Y$ are spherical...

If $A_Z$ is spherical, we call $\Delta_Z$ the generator of the centre $Z(A_Z)$. The conjugation by $\Delta_Z$ is trivial except for the cases $A_m, E_6, D_n, I_2(n), n$ odd:

\[
\begin{align*}
A_n : & \quad (\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5) \rightarrow (\sigma_5 - \sigma_4 - \sigma_3 - \sigma_2 - \sigma_1) \\
E_6 : & \quad (\sigma_1 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6) \rightarrow (\sigma_6 - \sigma_5 - \sigma_4 - \sigma_3 - \sigma_1) \\
D_n, n \text{ odd:} & \quad (\sigma_1 - \sigma_3 - \sigma_4 - \sigma_5) \rightarrow (\sigma_2 - \sigma_3 - \sigma_4 - \sigma_5)
\end{align*}
\]

**Theorem (Paris '97)**

Two irreducible standard parabolic subgroups are conjugate if we can obtain one from the other by conjugating by these four types of $\Delta_Z$. 
An example of two conjugate parabolic subgroups
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?

$Z = D_5$

$\Delta_z$
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?

$Z = D_5$
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?

\[ Z = A_5 \]
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?

$Z = A_5$

$\Delta_Z$
An example of two conjugate parabolic subgroups

Are the red and blue parabolic subgroups conjugate?
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$Z = E_6$
Godelle’s conjectures for every Artin group $A_S$

1. Any element conjugating two parabolic subgroups decomposes as a product of minimal elements doing the same conjugations as the $\Delta_Z$’s of (Paris 97).
Godelle’s conjectures for every Artin group $A_S$

1. Any element conjugating two parabolic subgroups decomposes as a product of minimal elements doing the same conjugations as the $\Delta_Z$'s of (Paris ’97). That is, the conjugacy element depends on the “conjugacy paths” that we can combinatorially find in the Coxeter graph.
Godelle’s conjectures for every Artin group $A_S$

1. Any element conjugating two parabolic subgroups decomposes as a product of minimal elements doing the same conjugations as the $\Delta_Z$’s of (Paris 97). That is, the conjugacy element depends on the “conjugacy paths” that we can combinatorially find in the Coxeter graph.

2. If $P \subset A_X$ are two parabolic subgroups of some Artin group $A_S$, $P$ is also a parabolic subgroup of $A_X$.

These conjectures have been shown for spherical Artin groups [Paris '97], FC-type Artin groups [Godelle '03] and two-dimensional Artin groups [Godelle '07] and some Euclidean Artin groups [Haettel '21].
Parabolic closure

Conjecture 3
For every Artin group $A_S$ and any element $g \in A_S$, there is a minimal (with respect to the inclusion) parabolic subgroup containing $g$. We denote this parabolic subgroup $P_g$ and we call it **parabolic closure** of $g$.

We know this conjecture is true for spherical Artin groups [C., Gebhardt, González-Meneses, Wiest ’19], some FC-type cases [Morris-Wright ’20], large-type [C., Martin, Vaskou], some two-dimensional [Blufstein], some Euclidean [Haettel].
Conjecture 3
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Lemma (C. 2021, preprint).
Let $a, b, c \in A_S$. If every element in $A_S$ has a parabolic closure, then

$$c^{-1}ac = b \Rightarrow c^{-1}P_a c = P_b.$$
We (almost) have an algorithm!
We (almost) have an algorithm!
We (almost) have an algorithm!
We (almost) have an algorithm!
We (almost) have an algorithm!

\[ A_X \] (not conjugacy stable)

\[ A_{Y_a} \]

\[ A_{Y_b} \]
We (almost) have an algorithm!
We (almost) have an algorithm!

- $A_{X}$ (not conjugacy stable)
- Does not permute connected components

$A_{Y_{a}}$: permutes connected components

$A_{Y_{b}}$: permutes connected components
We (almost) have an algorithm!
We (almost) have an algorithm!
We (almost) have an algorithm!

\[ A_s \]

\[ A_X \] (not conjugacy stable)

\[ A_{Y_a} \]

\[ A_{Y_b} \]

does not exchange
and

\[ 1 \]

\[ 2 \]
We have an algorithm!
We write this algorithm!

**INPUT**: An Artin group $A_S$ satisfying our 3 conjectures and a irreducible parabolic subgroup $A_X$.

**OUTPUT**: $A_X$ is conj. stable in $A_S$ or $A_X$ is not conj. stable in $A_S$.

**For** every pair $(A_Y, A_Z)$ of standard parabolic subgroups in $A_X$:

- **If** there are $D$ exceptions, return $A_X$ is not conj. stable in $A_S$;

- **If** they are conjugate in $A_S$:
  - **If** they are not conjugate in $A_X$, return $A_X$ is not conj. stable in $A_S$;
  - **If** they conjugate in $A_X$ but we cannot do the same permutation of components as in $A_S$, return $A_X$ is not conj. stable in $A_S$;

return $A_X$ is conj. stable in $A_S$;
We can now solve the conjugacy stability problem for new families of Artin groups...

**FC-type**
- Conjectures 1 and 2 [Godelle '03] ✓
- Partial results for Conjecture 3 [CGGW '19, Morris-Wright '20, C. '21].
- We can use the algorithm to know whether a spherical parabolic subgroup is conjugacy stable or not.

**2-dimensional**
- Conjectures 1 and 2 [Godelle '07] ✓
- Conjecture 3 for large [C., Martin, Vaskou, '20] and (2,2)-free [Blufstein, '21].

**Euclidean**
- All conjectures for types Ā and Ĉ [Haettel '21].
Algorithm 2: Algorithm to check the $D_k$, $k > 2$, exceptions described in the proof of Theorem 14.

Input: The Coxeter graph $Γ_S$ of an Artin group $A_S$ and three subgraphs $Γ_X ⊆ Γ_S$, $Γ_Y ⊆ Γ_X$ such that $A_X$ and $A_S$ satisfy the hypotheses of Theorem 14 and $Γ_Y$ is a connected component of $Γ_Y$ of type $D_k$.

Output: 1 (if we know that $A_X$ is not conjugacy stable) or 0.

Label the elements $x_1, x_2, ..., x_k$ of $Y$ as in Figure 1.

for $t ∈ Adj((x_k)) ∩ (S \setminus X)$ do
  if the connected component of $Γ_{Y∪\{t\}}$, containing $Y'$ and $t$, is of type $D_{2m+1}$, for some $m$ then
    for $t' ∈ Adj((x_k)) ∩ X$ do
      if the connected component of $Γ_{Y∪\{t\}}$, containing $Y'$ and $t'$, is of type $D_{2m+1}$, for some $m'$ then
        return 1;
      return 0;
  return 0;
return 0.

Algorithm 4: Algorithm that tell us if a parabolic subgroup is conjugacy stable or not.

Input: The Coxeter graph $Γ_S$ of an Artin group $A_S$ and a $Γ_X ⊆ Γ_S$ such that $A_X$ and $A_S$ satisfy the hypotheses of Theorem 14.

Output: “$A_X$ is conjugacy stable” or “$A_X$ is not conjugacy stable”.

for $(X_1, X_2) ⊆ (X, X)$ such that $|X_1| = |X_2|$ do
  if $Γ_X$ is of type $D_k$ then
    if $k > 2$ then
      run algorithm 2;
      if algorithm 2 returns 1 then
        return “$A_X$ is not conjugacy stable”;
    if $k = 2$ then
      run algorithm 3;
      if algorithm 3 returns 1 then
        return “$A_X$ is not conjugacy stable”;
  if $|X_1| = X_3$ then
    $D := \{(X_1, X_2, ..., X_m)\}$;
  else
    $D := \{\}$;
  for $(Y_1, Y_2, ..., Y_m) ∈ D$ do
    $Y := Y_1 ∪ Y_2 ∪ ... ∪ Y_m$;
    for $t ∈ X \setminus Adj(Y)$ do
      if the connected component $Γ_Y$ of $Γ_Y∪\{t\}$ containing $t$ is twistable then
        $Z = Δ_Y Y Δ_Y$;
        $T = (Δ_Y Y Δ_Y, Δ_Y Y Δ_Y, ..., Δ_Y Y Δ_Y, Δ_Y Y Δ_Y)$;
        if $T ∉ C$ then
          $C := C ∪ \{T\}$;
        if $Z = X_2$ then
          return “$A_X$ is not conjugacy stable”;