How to know if a parabolic subgroup of an Artin group merges conjugacy classes

María Cumplido Cabello

15th of February, 2022

Braid in representation theory and algebraic combinatorics

Presentation of the braid group with $n + 1$ strands

$$A_n = \left\langle \sigma_1, \dots, \sigma_n \,\middle|\, \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{if } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, \dots, n-1 \end{array} \right\rangle$$

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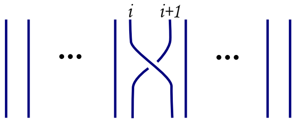

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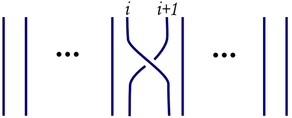

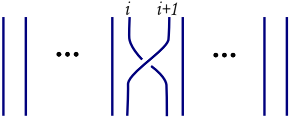

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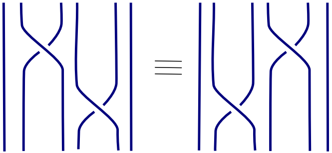

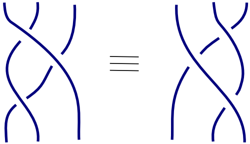

Artin groups

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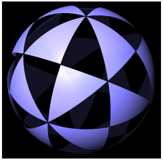

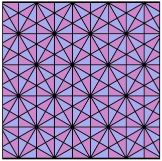

Images from John Baez' blog

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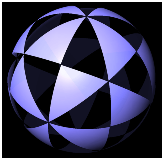

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Artin group (of spherical type)

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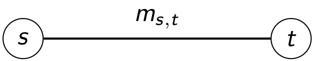

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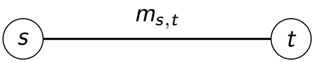


Some families of Artin groups

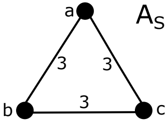
- RAAGs: $m_{s,t} \in \{2, \infty\}$.
- Spherical (or finite type): finite Coxeter group.
- FC-type: All complete subgraphs without ∞ are spherical.
- 2-dimensional: $\frac{1}{m_{s,t}} + \frac{1}{m_{s,r}} + \frac{1}{m_{t,r}} \leq 1, \forall s, t, r \in S$.

Parabolic subgroups

Standard parabolic subgroup $A_{S'}$ of A_S

It is the subgroup generated by a subset $S' \subseteq S$.

Parabolic subgroups



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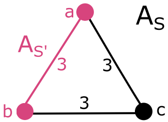

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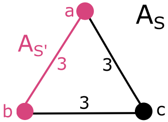

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Theorem [Van der Lek 1983]

$A_{S'}$ is again an Artin group.

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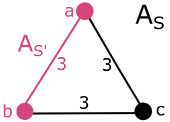

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Parabolic subgroup P de A_S

$$P = \alpha^{-1} A_{S'} \alpha,$$

where $A_{S'}$ is a standard parabolic subgroup and $\alpha \in A_S$.

Complexes using parabolic subgroups

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- **Deligne complex** [Charney & Davis 1995]
  - It uses spherical parabolic subgroups.
  - $CAT(0)$ in some cases.
  - It has been used to study classic problems:
    - $K(\pi, 1)$ conjecture [Charney & Davis '95, Paris '14].
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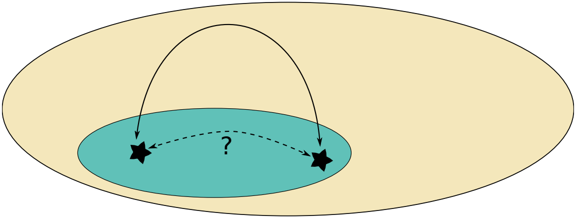
- **Artin complex** [C., Martin & Vaskou'20]

Conjugacy stability problem for parabolic sg $A_X < A_S$

If two elements of a parabolic subgroup A_X are conjugate in our Artin group A_S, are they conjugate "inside this parabolic subgroup" ?

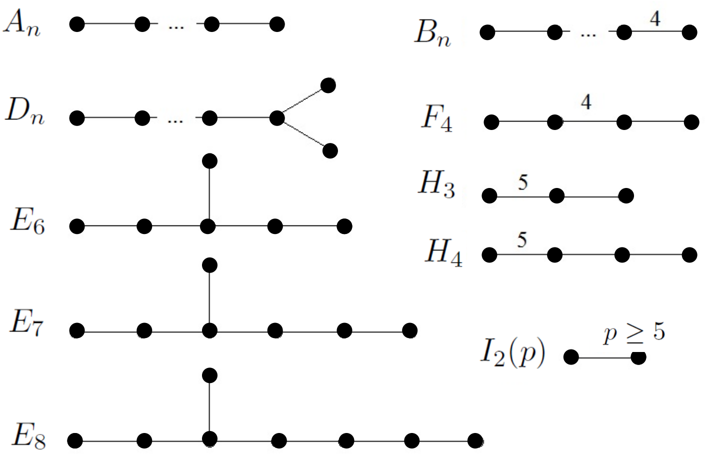
$$\exists\ c \in A_S,\ c^{-1}ac = b \Longrightarrow \exists c' \in A_X,\ {c'}^{-1}ac' = b?$$

If the answer is yes, we say that A_X is **conjugacy stable**.



This is always true for braids [González-Meneses 2014], but it is not true in general.

Irreducible Coxeter graphs (of finite type)



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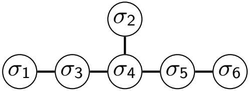

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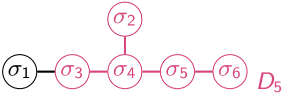

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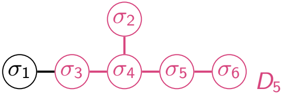

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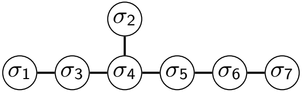

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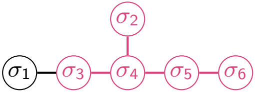

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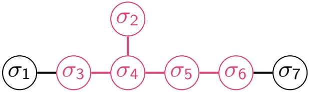

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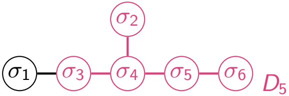

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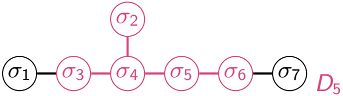

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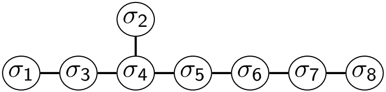

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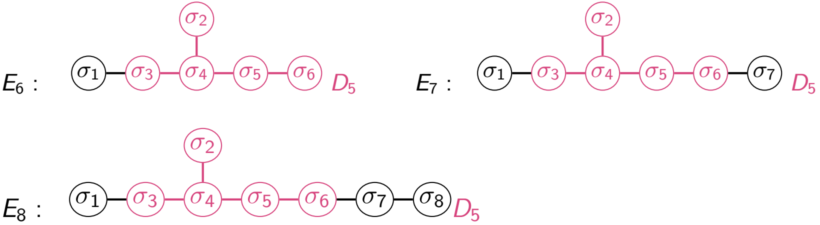

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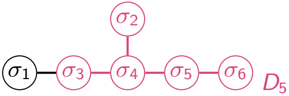

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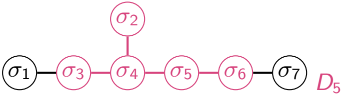

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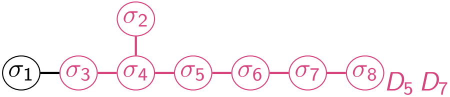

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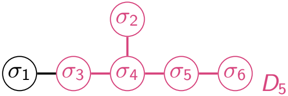

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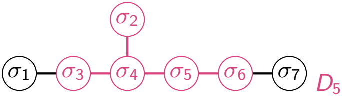

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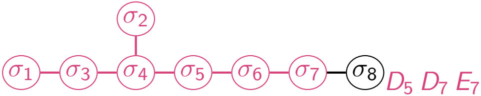

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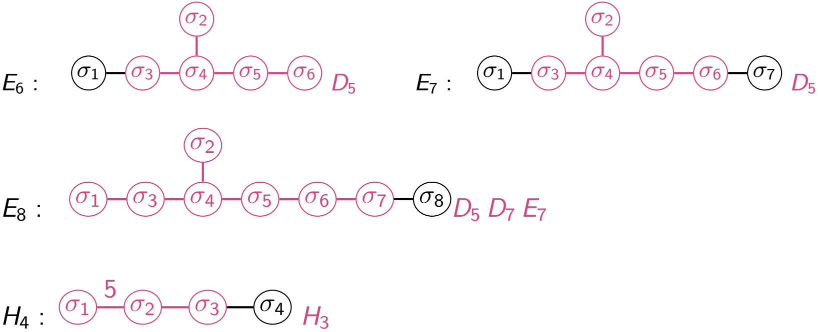

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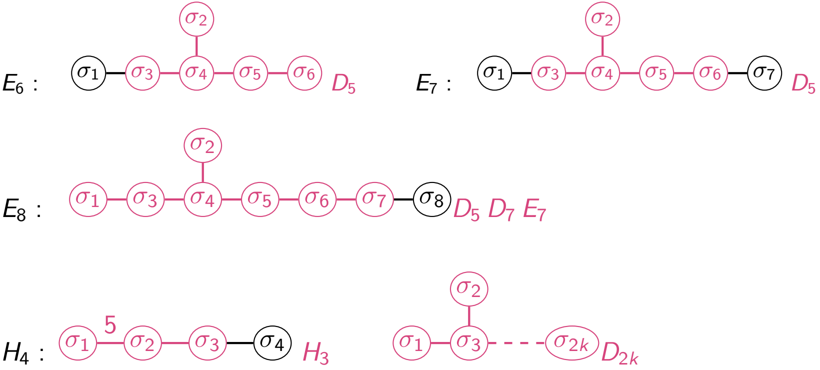

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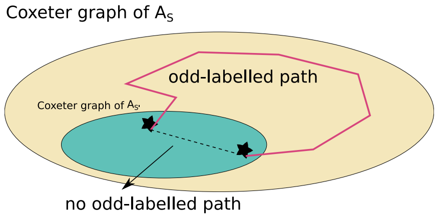

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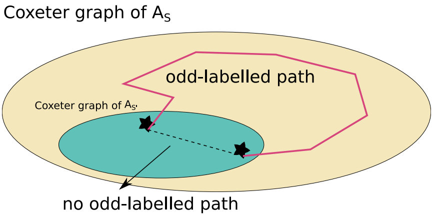

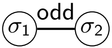

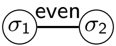

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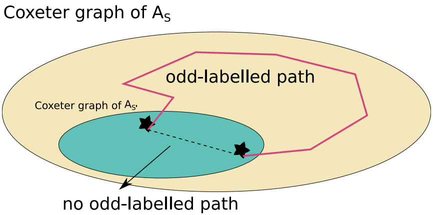

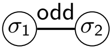

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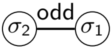

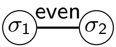

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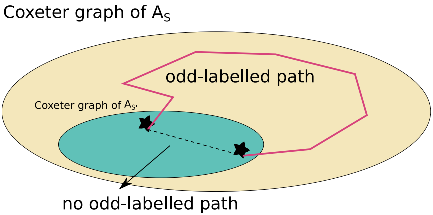

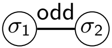

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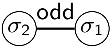

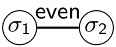

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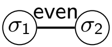

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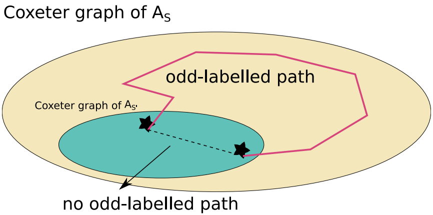

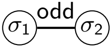

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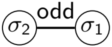

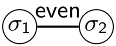

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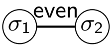

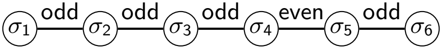

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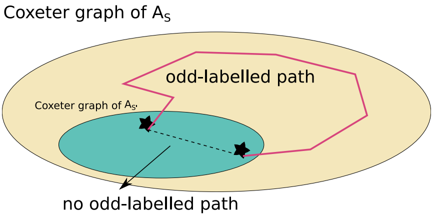

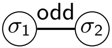

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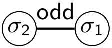

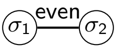

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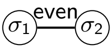

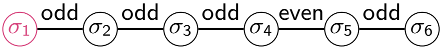

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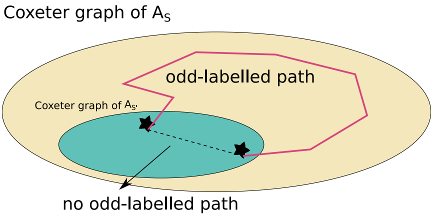

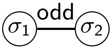

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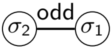

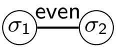

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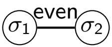

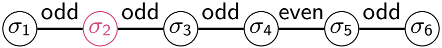

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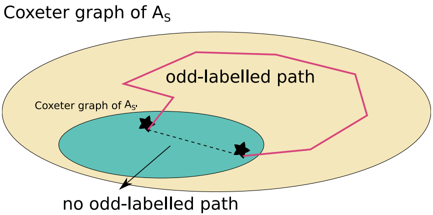

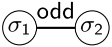

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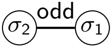

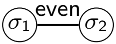

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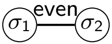

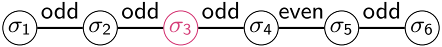

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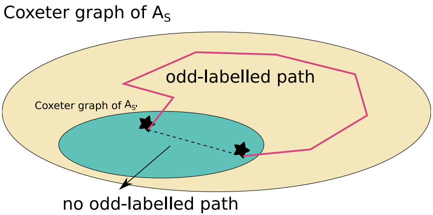

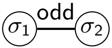

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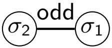

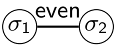

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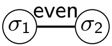
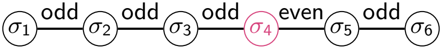


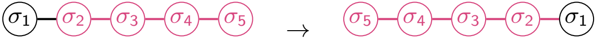


When are two standard parabolic subgroups conjugate?

- If A_X and A_Y are different and A_X is not spherical, they are never conjugate.
- If A_X and A_Y are spherical...

If A_Z is spherical, we call Δ_Z the generator of the centre $Z(A_Z)$. The conjugation by Δ_Z is trivial except for the cases $A_m, E_6, D_n, I_2(n), n$ odd:

A_n :



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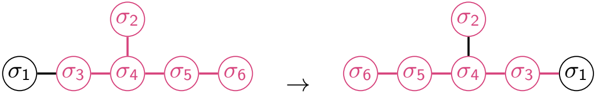
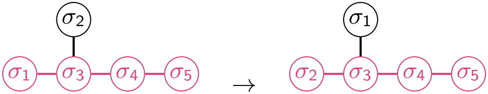


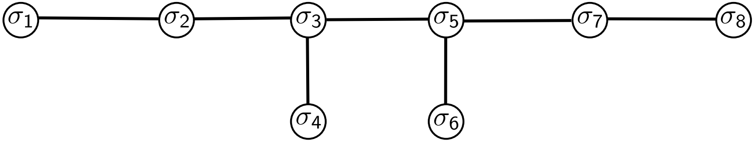
D_n, n odd:



Theorem (Paris '97)

Two irreducible standard parabolic subgroups are conjugate if we can obtain one from the other by conjugating by these four types of Δ_Z.

An example of two conjugate parabolic subgroups



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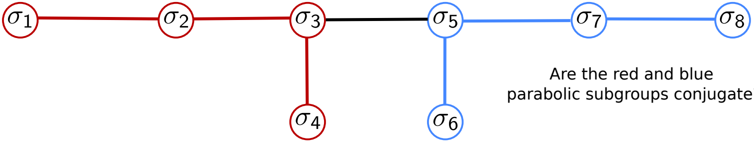

Are the red and blue
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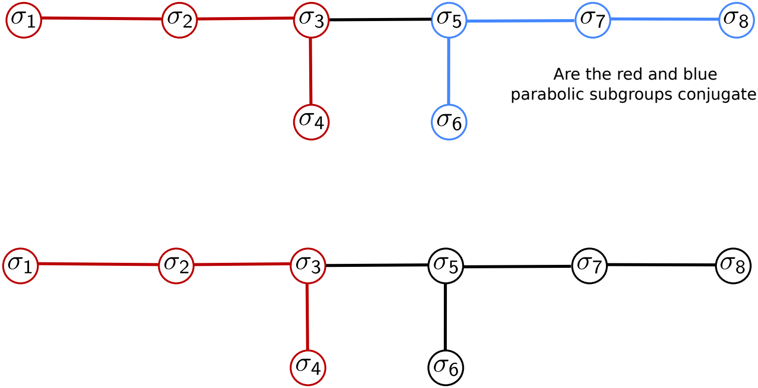

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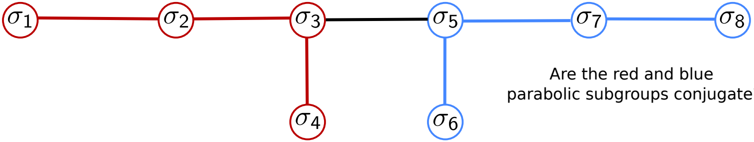

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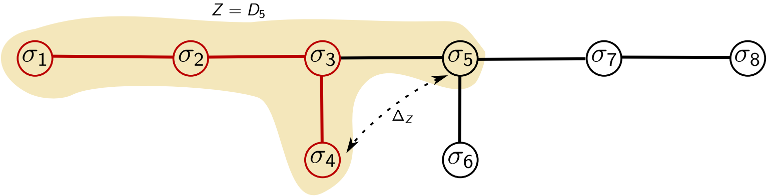

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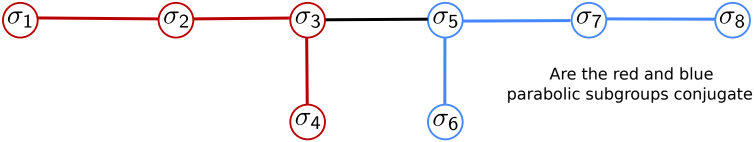

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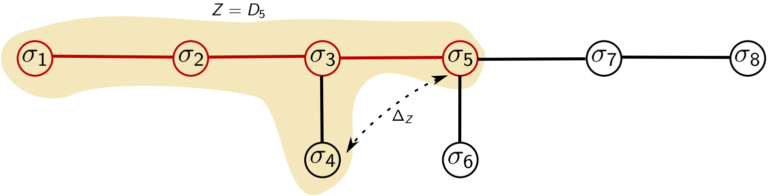

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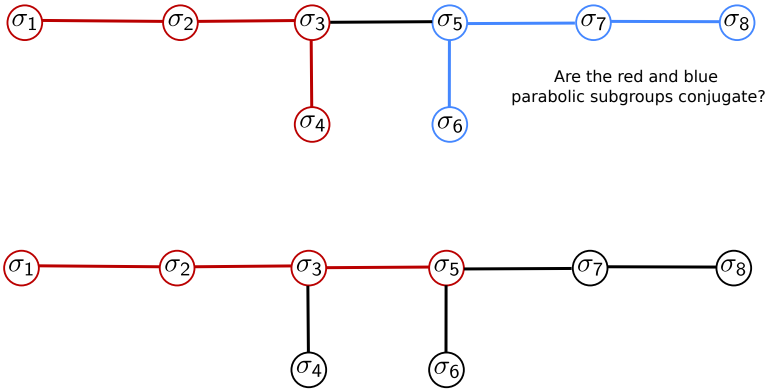

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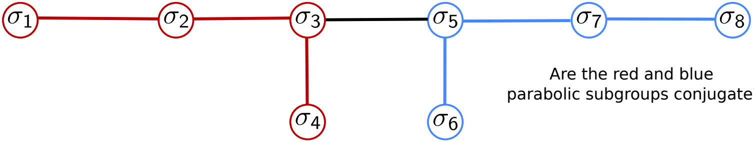

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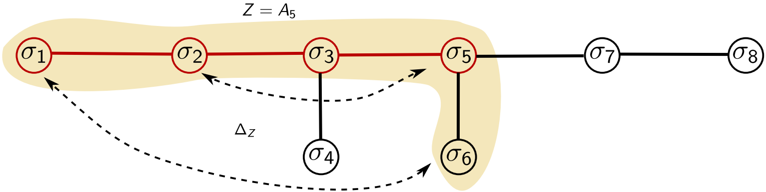

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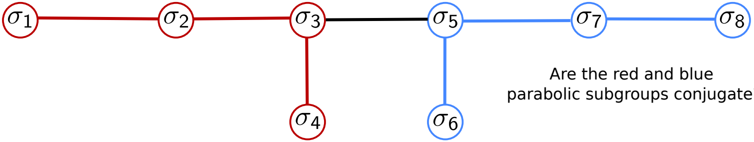

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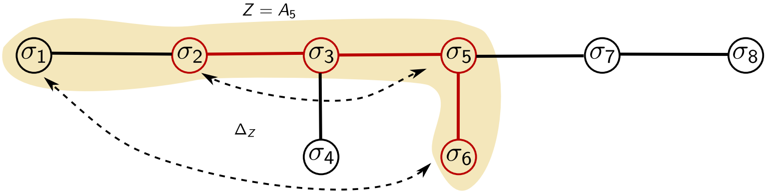

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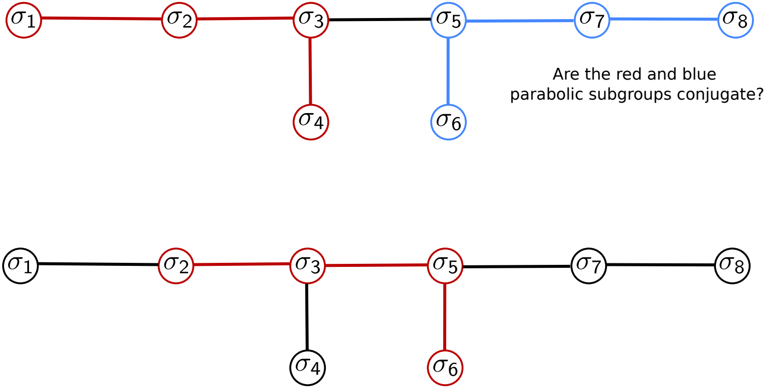

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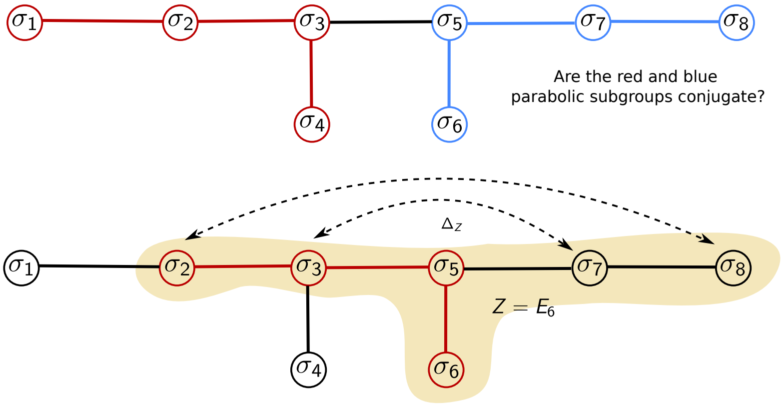

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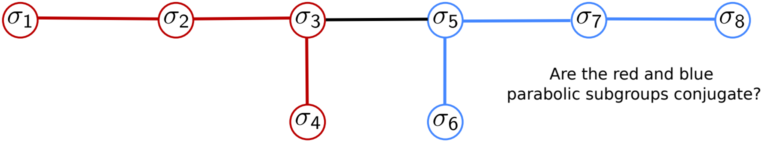

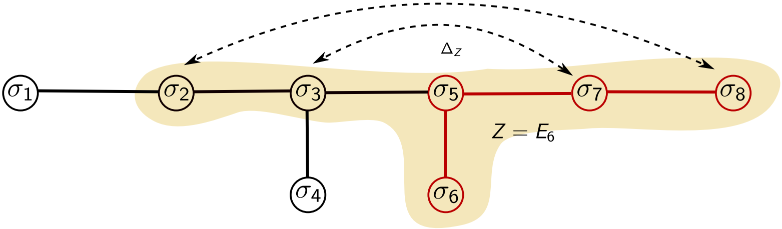

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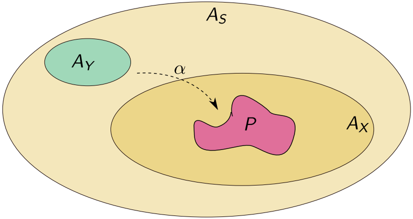
   That is, the conjugacy element depends on the "conjugacy paths" that we can combinatorially find in the Coxeter graph.

2. If $P \subset A_X$ are two parabolic subgroups of some Artin group A_S, P is also a parabolic subgroup of A_X.



These conjectures have been shown for spherical Artin groups [Paris '97], FC-type Artin groups [Godelle '03] and two-dimensional Artin groups [Godelle '07] and some Euclidean Artin groups [Haettel '21].

Parabolic closure

Conjecture 3

For every Artin group A_S and any element $g \in A_S$, there is a minimal (with respect to the inclusion) parabolic subgroup containing g. We denote this parabolic subgroup P_g and we call it **parabolic closure** of g.

We know this conjecture is true for spherical Artin groups [C., Gebhardt, González-Meneses, Wiest '19], some FC-type cases [Morris-Wright '20], large-type [C., Martin, Vaskou], some two-dimensional [Blufstein], some Euclidean [Haettel].

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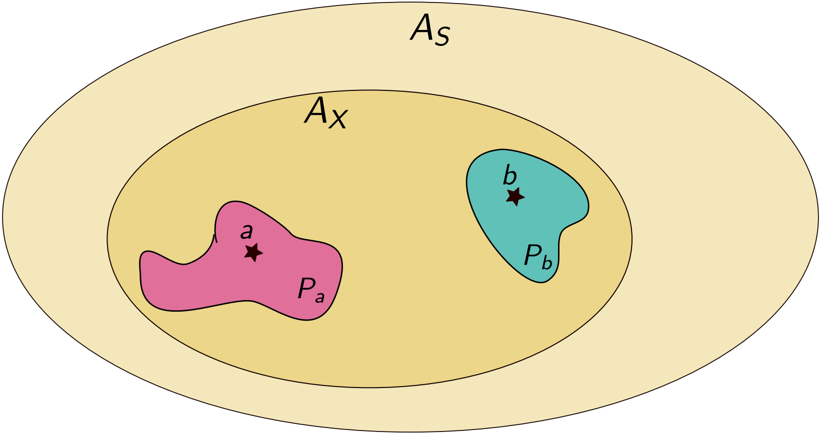
Lemma (C. 2021, preprint).

Let $a, b, c \in A_S$. If every element in A_S has a parabolic closure, then

$$c^{-1}ac = b \Rightarrow c^{-1}P_ac = P_b.$$

We (almost) have an algorithm!



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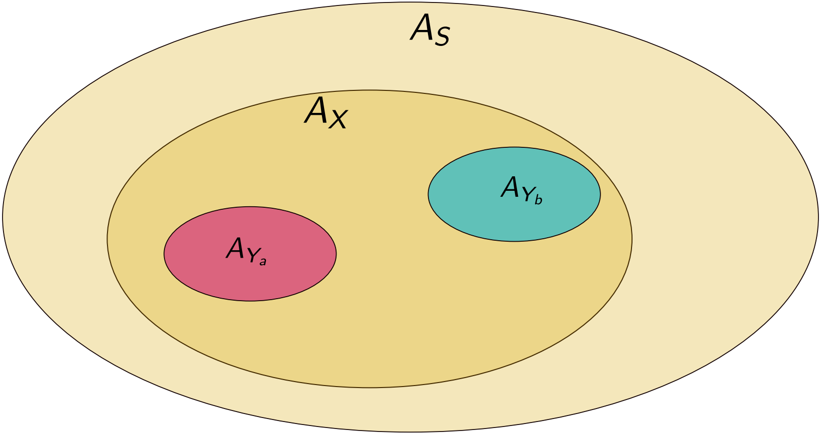

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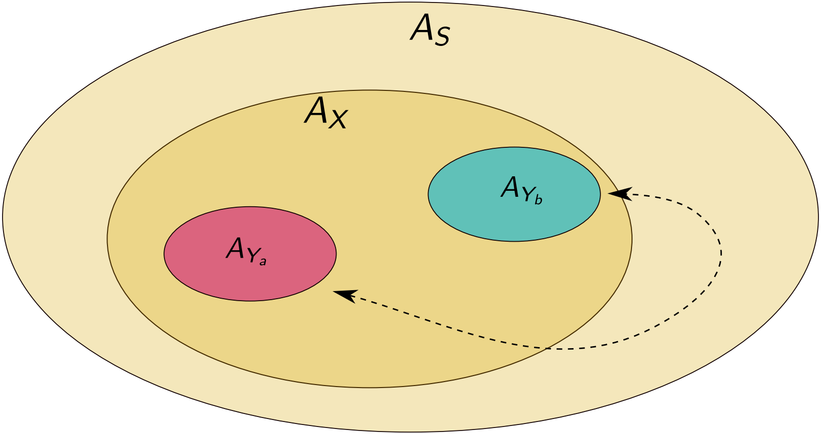

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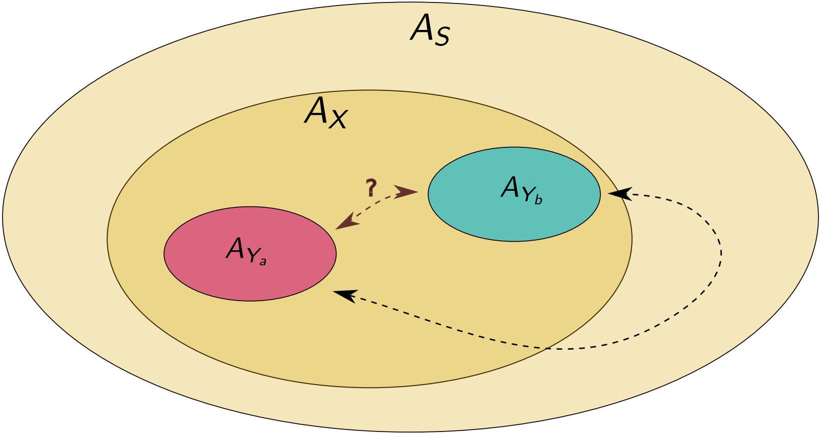

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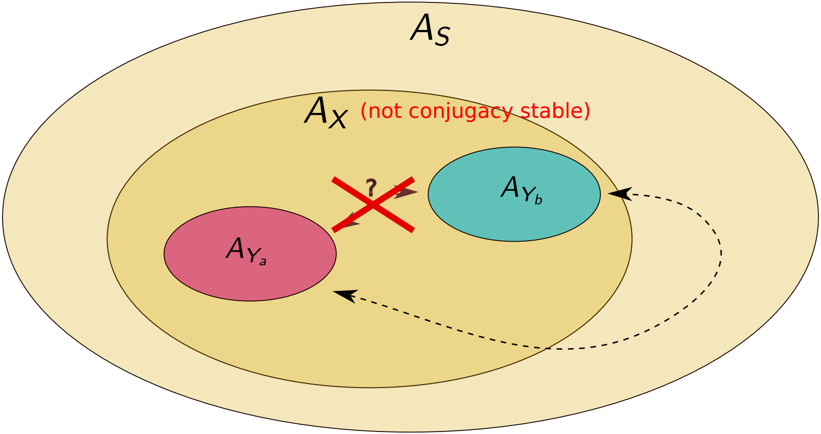

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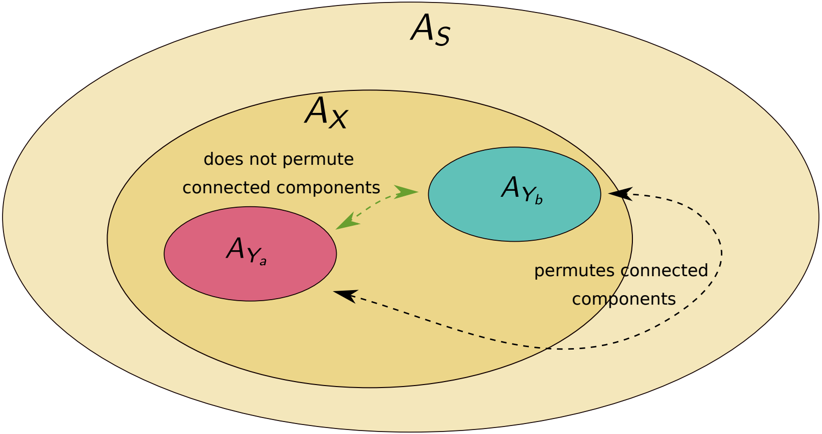

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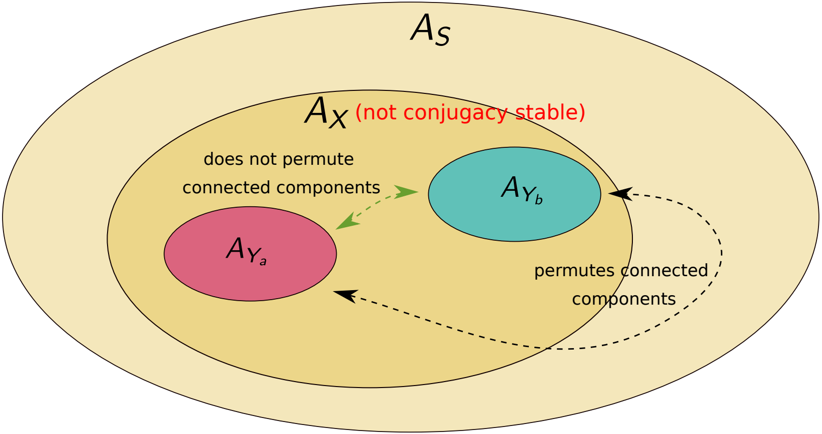

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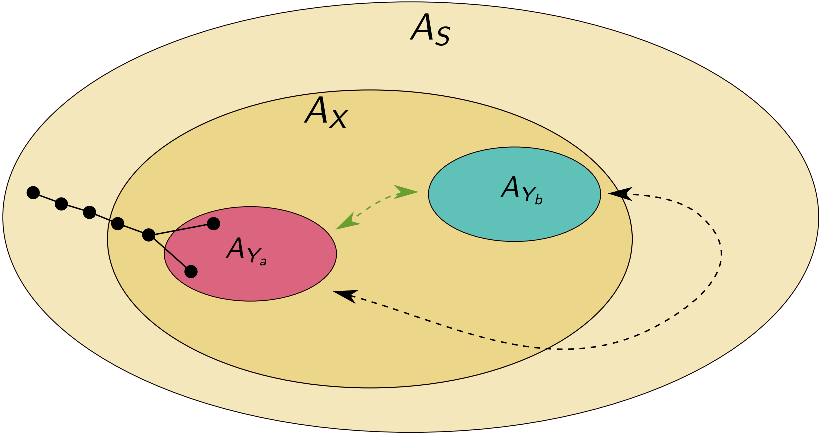

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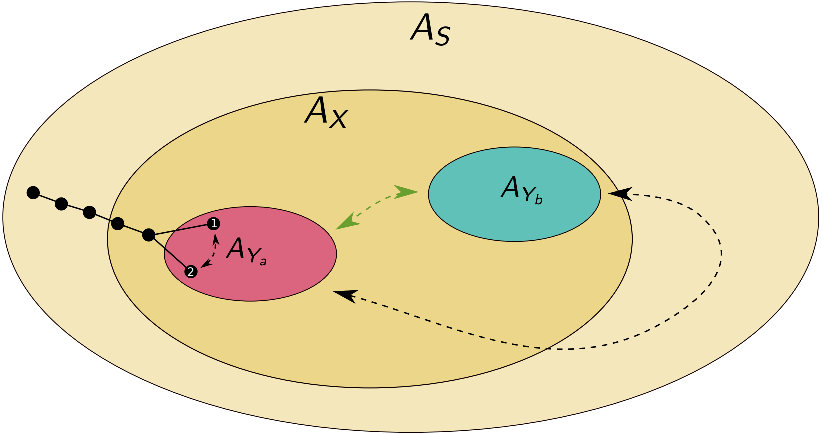

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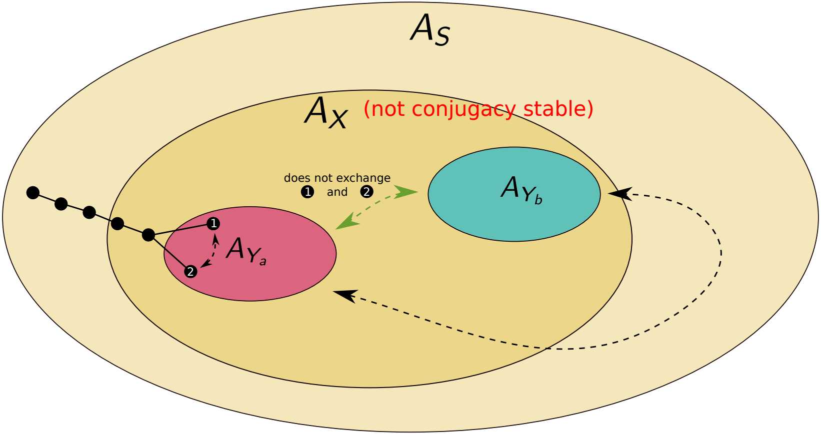

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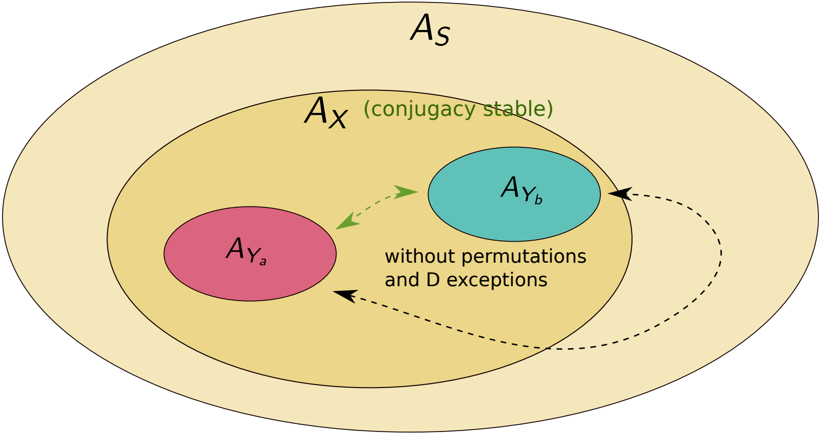

We write this algorithm!

<u>INPUT</u>: An Artin group A_S satisfying our 3 conjectures and a irreducible parabolic subgroup A_X.

<u>OUTPUT</u>: A_X *is conj. stable in* A_S or A_X *is not conj. stable in* A_S.

For every pair (A_Y, A_Z) of standard parabolic subgroups in A_X:

- **If** there are D exceptions, **return** A_X *is not conj. stable in* A_S;
- **If** they are conjugate in A_S:
  - **If** they are not conjugate in A_X, **return** A_X *is not conj. stable in* A_S;
  - **If** they conjugate in A_X but we cannot do the same permutation of components as in A_S, **return** A_X *is not conj. stable in* A_S;

return A_X *is conj. stable in* A_S;

We can now solve the conjugacy stability problem for new families of Artin groups...

FC-type

- Conjectures 1 and 2 [Godelle '03] ✓
- Partial results for Conjecture 3 [CGGW '19, Morris-Wright '20, C. '21].
- We can use the algorithm to know whether a spherical parabolic subgroup is conjugacy stable or not.

2-dimensional

- Conjectures 1 and 2 [Godelle '07] ✓
- Conjecture 3 for **large** [C., Martin, Vaskou, '20] and **(2,2)-free** [Blufstein, '21].

Euclidean

- All conjectures for types $\tilde{A}$ and $\tilde{C}$ [Haettel '21].

Algorithm 2: Algorithm to check the D_{2k}, $k>2$, exceptions described in the proof of Theorem 14

Input : The Coxeter graph Γ_S of an Artin group A_S and three subgraphs $\Gamma_X \subset \Gamma_S$, $\Gamma_{Y'} \subset \Gamma_Y \subset \Gamma_X$ such that A_X and A_S satisfies the hypotheses of Theorem 14 and Γ'_Y is a connected component of Γ_Y of type D_{2k}.
Output: 1 (if we know that A_X is not conjugacy stable) or 0.

Label the elements $s_1, s_2, \dots, s_{2k}$ of Y as in Figure 1.
for $t \in \mathrm{Adj}(\{s_{2k}\}) \cap (S \setminus X)$ **do**
  if *the connected component of* $\Gamma_{Y\cup\{t\}}$ *containing* Y' *(and* t*) is of type* D_{2m+1}*, for some* m **then**
    for $t' \in \mathrm{Adj}(\{s_{2k}\}) \cap X$ **do**
      if *the connected component of* $\Gamma_{Y\cup\{t'\}}$ *containing* Y' *(and* t'*) is of type* $D_{2m'+1}$*, for some* m' **then**
        return 0;
    return 1;
return 0

Algorithm 3: Algorithm to check the D_4 exceptions described in the proof of Theorem 14

Input : The Coxeter graph Γ_S of an Artin group A_S and two subgraphs $\Gamma_X \subset \Gamma_S$, $\Gamma_{Y'} \subset \Gamma_Y \subset \Gamma_X$ such that A_X and A_S satisfy the hypotheses of Theorem 14 and $\Gamma_{Y'}$ is a connected component of Γ_Y of type D_4.
Output: 1 (if we know that A_X is not conjugacy stable) or 0.

Label the elements s_1, s_2, s_3, s_4 of Y as in Figure 1.
$Z = \{s_1, s_2, s_3\}$;
for $s \in Z$ **do**
  for $t \in \mathrm{Adj}(\{s\}) \cap (S \setminus X)$ **do**
    $p = 0$; $q = 0$;
    if *the connected component of* $\Gamma_{Y\cup\{t\}}$ *containing* Y' *(and* t*) is of type* D_{2m}*, for some* m **then**
      $p = 1$; $q = 1$;
      for $t' \in \mathrm{Adj}(\{s\}) \cap X$ **do**
        if *the connected component of* $\Gamma_{Y\cup\{t'\}}$ *containing* Y' *(and* t'*) is of type* $D_{2m'+1}$*, for some* m' **then**
          $p = 0$; break loop;
      if $p = 1$ **then**
        for $t_1 \in \mathrm{Adj}(Z \setminus \{s\}) \cap X$ **do**
          if *the connected component of* $\Gamma_{Y\cup\{t_1\}}$ *containing* Y' *(and* t_1*) is of type* D_{2m_1+1}*, for some* m_1 **then**
            for $t_2 \in \mathrm{Adj}(Z \setminus \{s, t_1\}) \cap X$ **do**
              if *the connected component of* $\Gamma_{Y\cup\{t_2\}}$ *containing* Y' *(and* t_2*) is of type* D_{2m_2+1}*, for some* m_1 **then**
                $p = 0$; break loop;
          if $p = 0$ **then**
            break loop;
    if $p = 1$ **then**
      return 1;
    if $q = 1$ **then**
      break loop;
return 0

Algorithm 4: Algorithm that tell us if a parabolic subgroup is conjugacy stable or not.

Input : The Coxeter graph Γ_S of an Artin group A_S and a $\Gamma_X \subset A_S$ such that A_X and A_S satisfy the hypotheses of Theorem 14.
Output: "A_X is conjugacy stable" or "A_X is not conjugacy stable".

for $(X_1, X_2) \subset (X, X)$ *such that* $|X_1| = |X_2|$ **do**
  if Γ_{X_1} *is of type* D_{2k} **then**
    if $k > 2$ **then**
      run algorithm 2;
      if *algorithm 2 returns* 1 **then**
        return "A_X is not conjugacy stable";
    if $k = 2$ **then**
      run algorithm 3;
      if *algorithm 3 returns* 1 **then**
        return "A_X is not conjugacy stable";
  $\Gamma_{X'_1}, \Gamma_{X'_2}, \dots, \Gamma_{X'_m} :=$ components of Γ_{X_1};
  $C := \{(X'_1, X'_2, \dots, X'_m)\}$;
  if $X_1 = X_2$ **then**
    $D := \{(X'_1, X'_2, \dots, X'_m)\}$;
  else
    $D := \{\emptyset\}$;
  for $(Y_1, Y_2, \dots, Y_m) \in C$ **do**
    $Y := Y_1 \cup Y_2 \cup \dots \cup Y_m$;
    for $t \in X \cap \mathrm{Adj}(Y)$ **do**
      if *the connected component* $\Gamma_{Y'}$ *of* $\Gamma_{Y\cup\{t\}}$ *containing* t *is twistable* **then**
        $Z = \Delta_{Y'}^{-1} Y \Delta_{Y'}$;
        $T = (\Delta_{Y'}^{-1} Y_1 \Delta_{Y'}, \Delta_{Y'}^{-1} Y_2 \Delta_{Y'}, \dots, \Delta_{Y'}^{-1} Y_m \Delta_{Y'})$;
        if $T \notin C$ **then**
          $C = C \cup \{T\}$;
          if $Z = X_2$ *and* $T \notin D$ **then**
            $D = D \cup \{T\}$;
  for $(Y_1, Y_2, \dots, Y_m) \in C$ **do**
    $Y := Y_1 \cup Y_2 \cup \dots \cup Y_m$;
    for $t \in \mathrm{Adj}(Y)$ **do**
      if *the connected component* $\Gamma_{Y'}$ *of* $\Gamma_{Y\cup\{t\}}$ *containing* t *is twistable* **then**
        $Z = \Delta_{Y'}^{-1} Y \Delta_{Y'}$
        $T = (\Delta_{Y'}^{-1} Y_1 \Delta_{Y'}, \Delta_{Y'}^{-1} Y_2 \Delta_{Y'}, \dots, \Delta_{Y'}^{-1} Y_m \Delta_{Y'})$
        if $T \notin C$ **then**
          $C = C \cup \{T\}$;
          if $Z = X_2$ *and* $T \notin D$ **then**
            return *"A_X is not conjugacy stable"*;
return *"A_X is conjugacy stable"*;

Thank you!