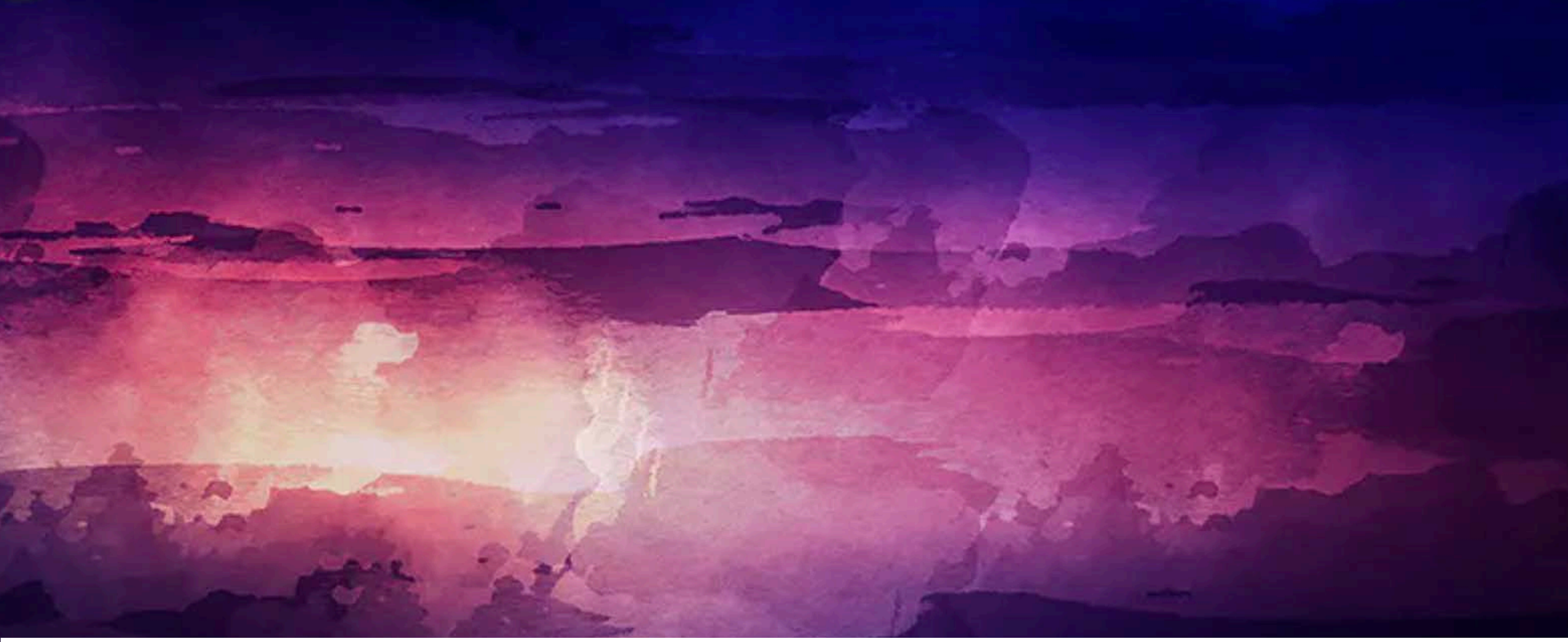


BRAIDS: CLASSICAL, VIRTUAL, AND WELDED, OH MY!

Nancy Scherich

Joint with Dror Bar-Natan

ICERM Feb 14, 2022

Classical

Braids

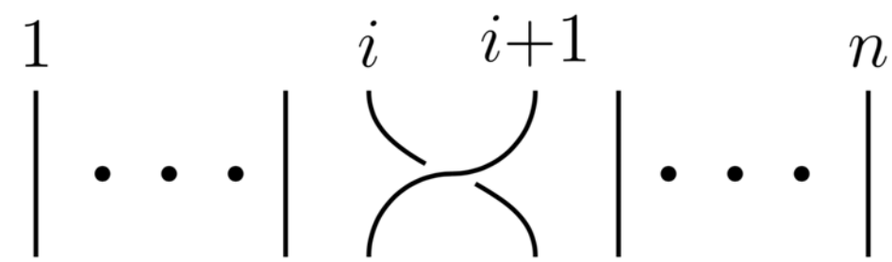
Virtual

Welded

THE classical BRAID GROUP B_n

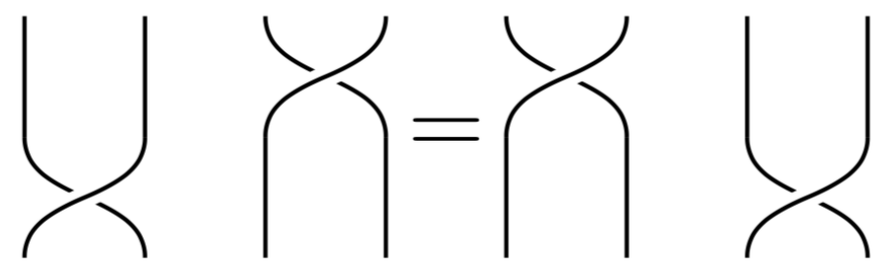
(Artin '25, '47)

Generators: $\sigma_1, \cdots, \sigma_{n-1}$

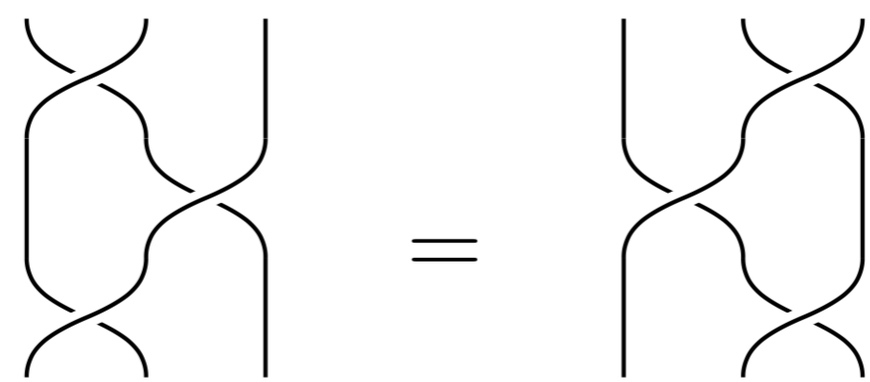


Relations:

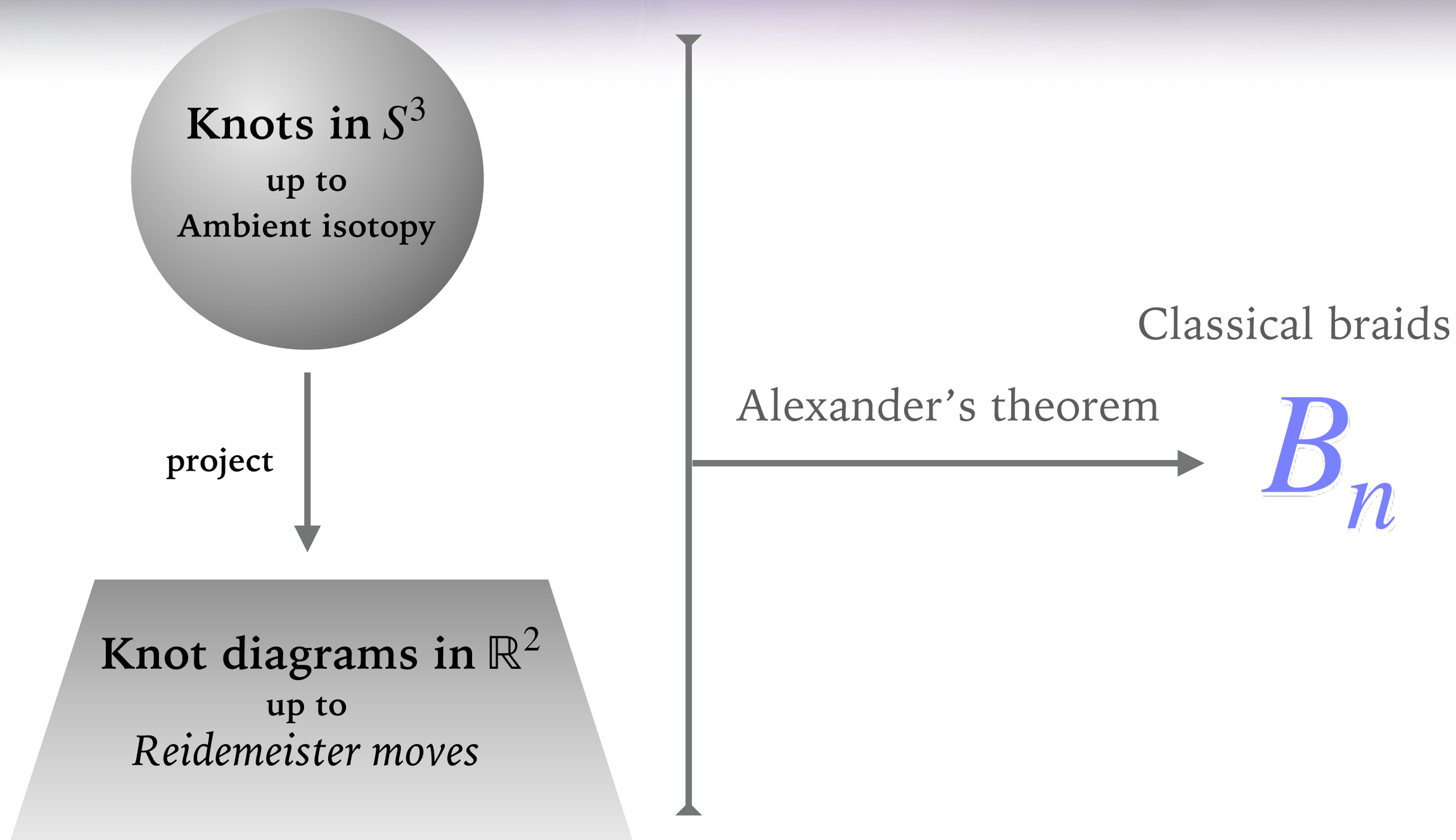
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad (j \neq i \pm 1)$$



$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$



THE classical BRAID GROUP B_n

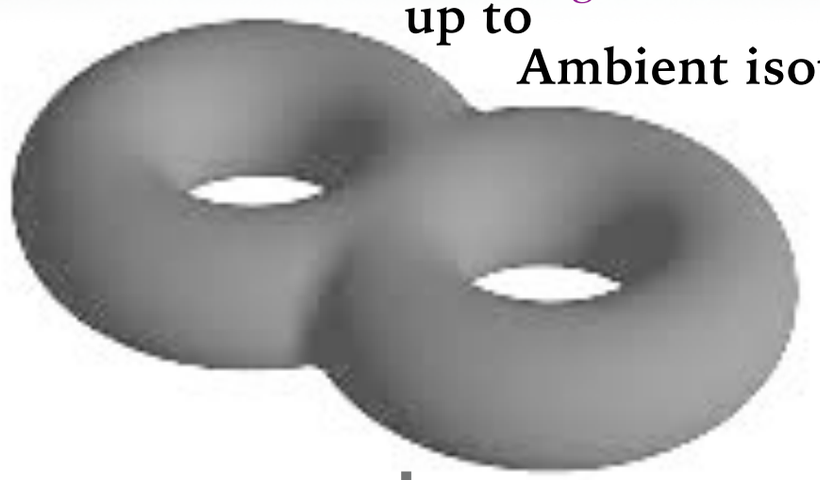


THE virtual BRAID GROUP vB_n

Knots in $S_g \times I$

up to

Ambient isotopy



project

Virtual

Knot diagrams in $\mathbb{R}^2$

up to

Virtual

Reidemeister moves

Virtual braids

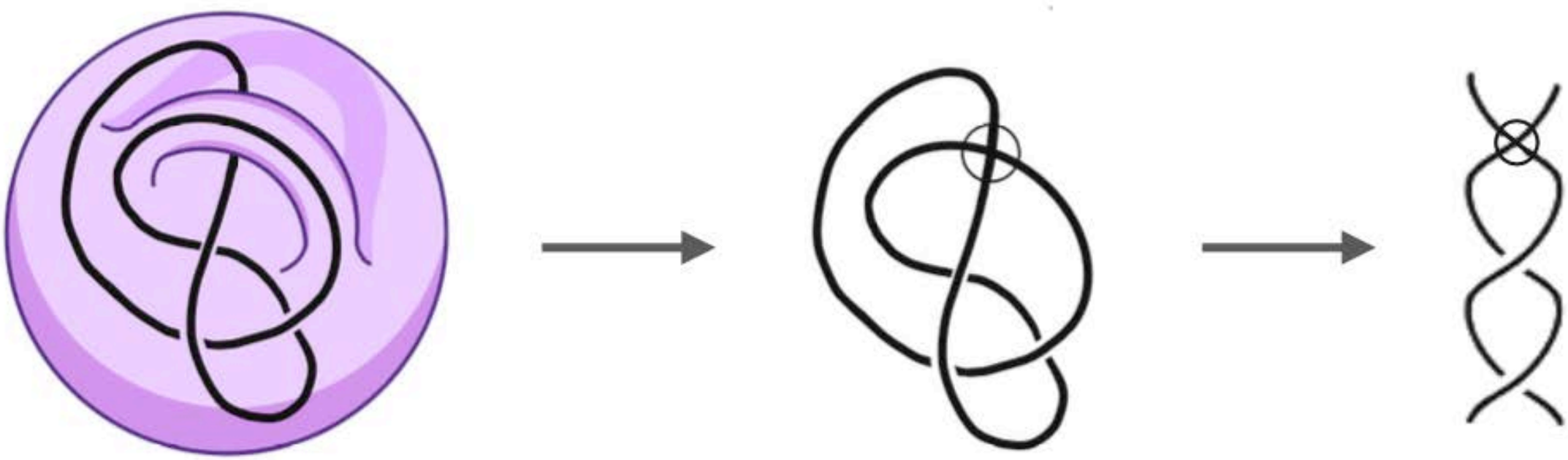
Virtual

Alexander's theorem

vB_n

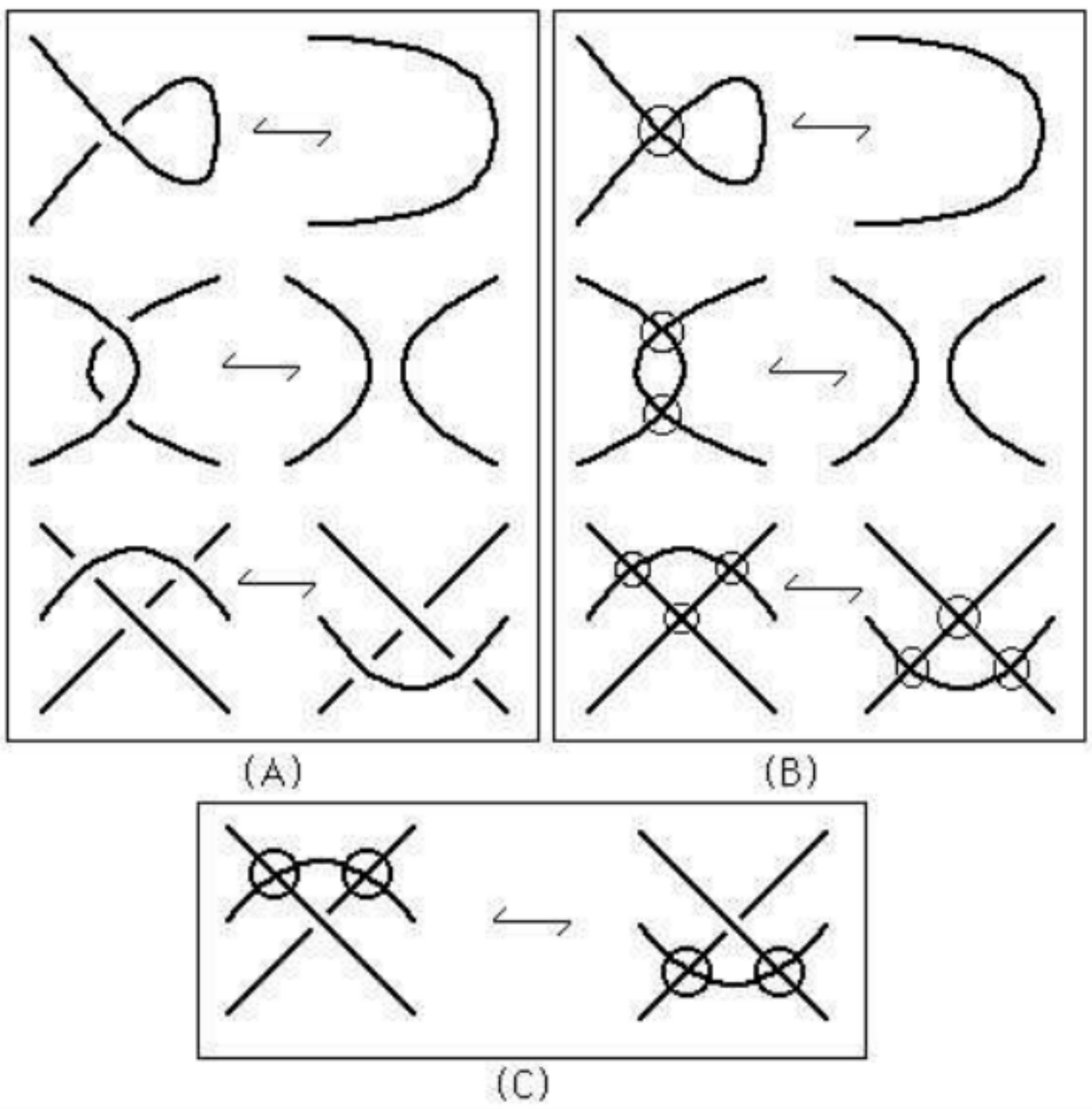
vB_n

THE virtual BRAID GROUP vB_n



THE virtual BRAID GROUP vB_n

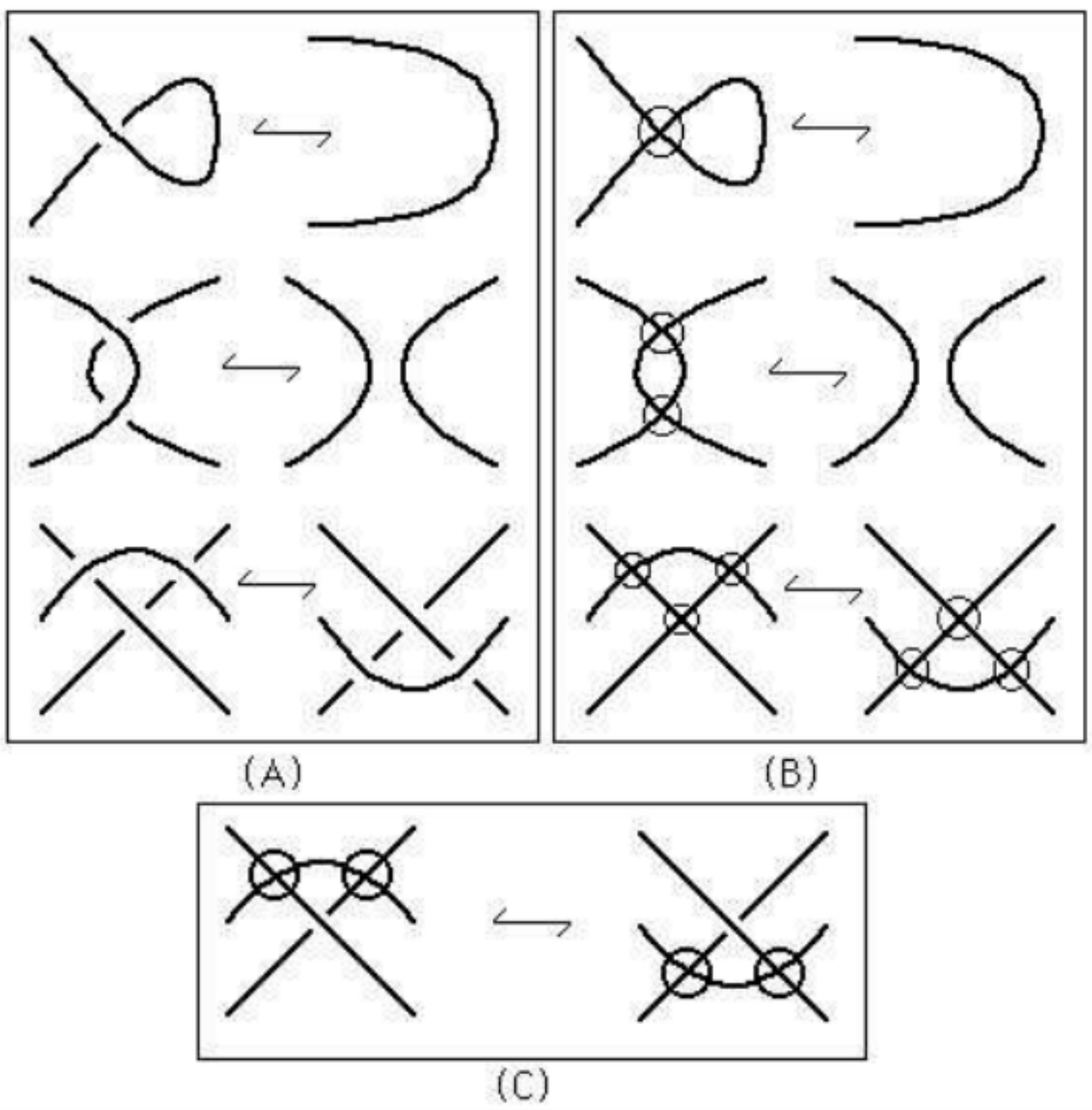
Virtual Reidemeister Moves

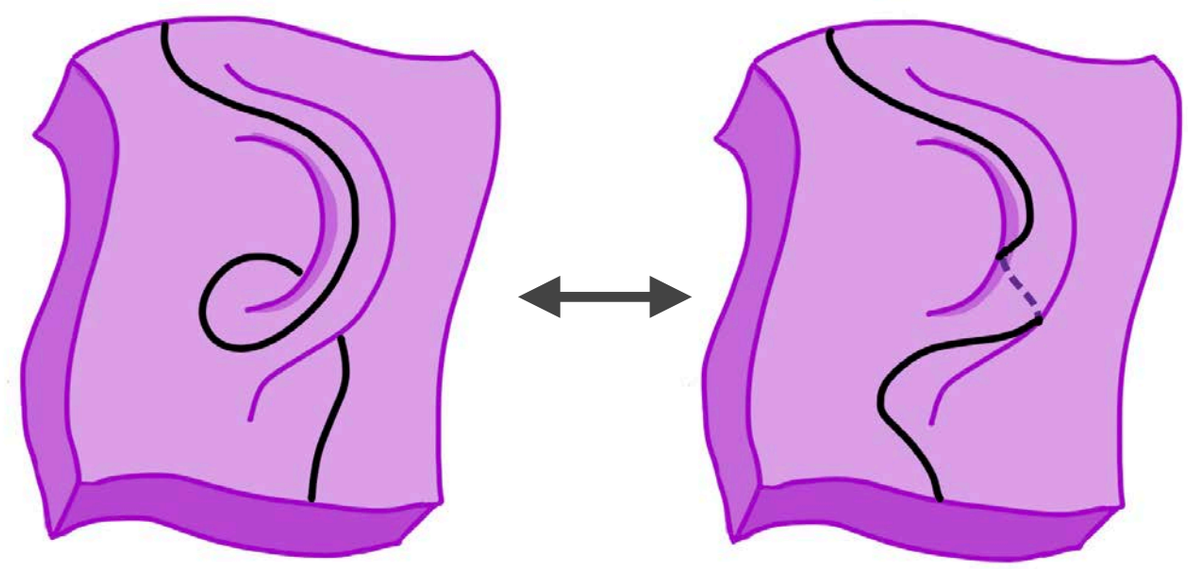


(Kauffman '99)

THE virtual BRAID GROUP vB_n

Virtual Reidemeister Moves

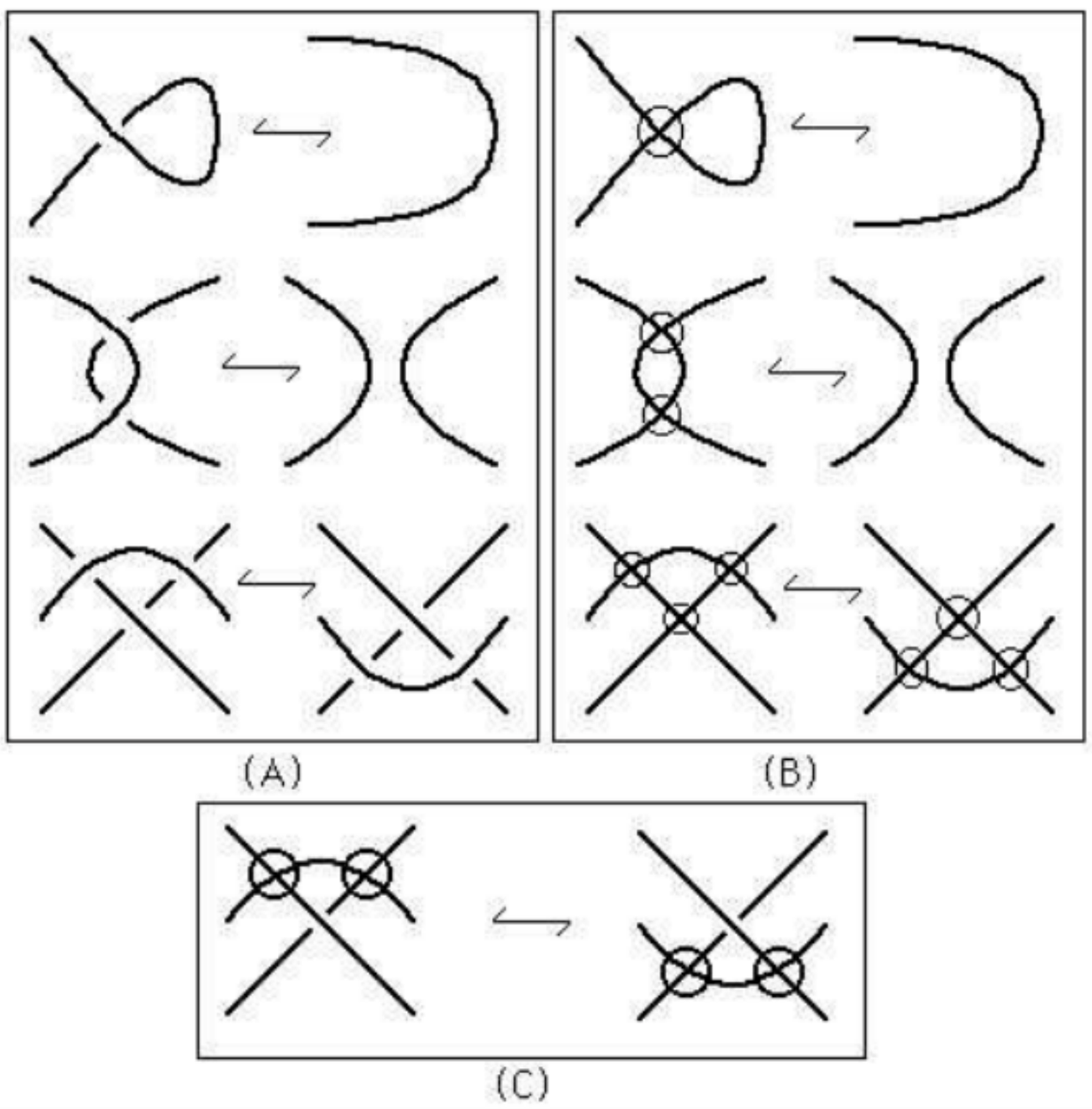


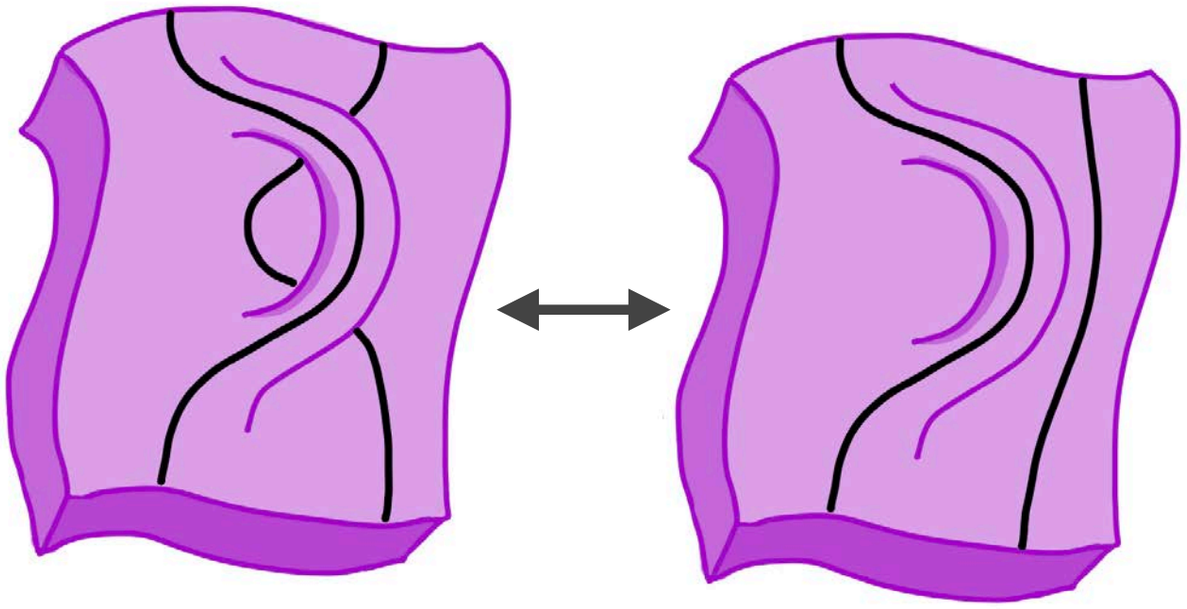


(Kauffman '99)

THE virtual BRAID GROUP vB_n

Virtual Reidemeister Moves

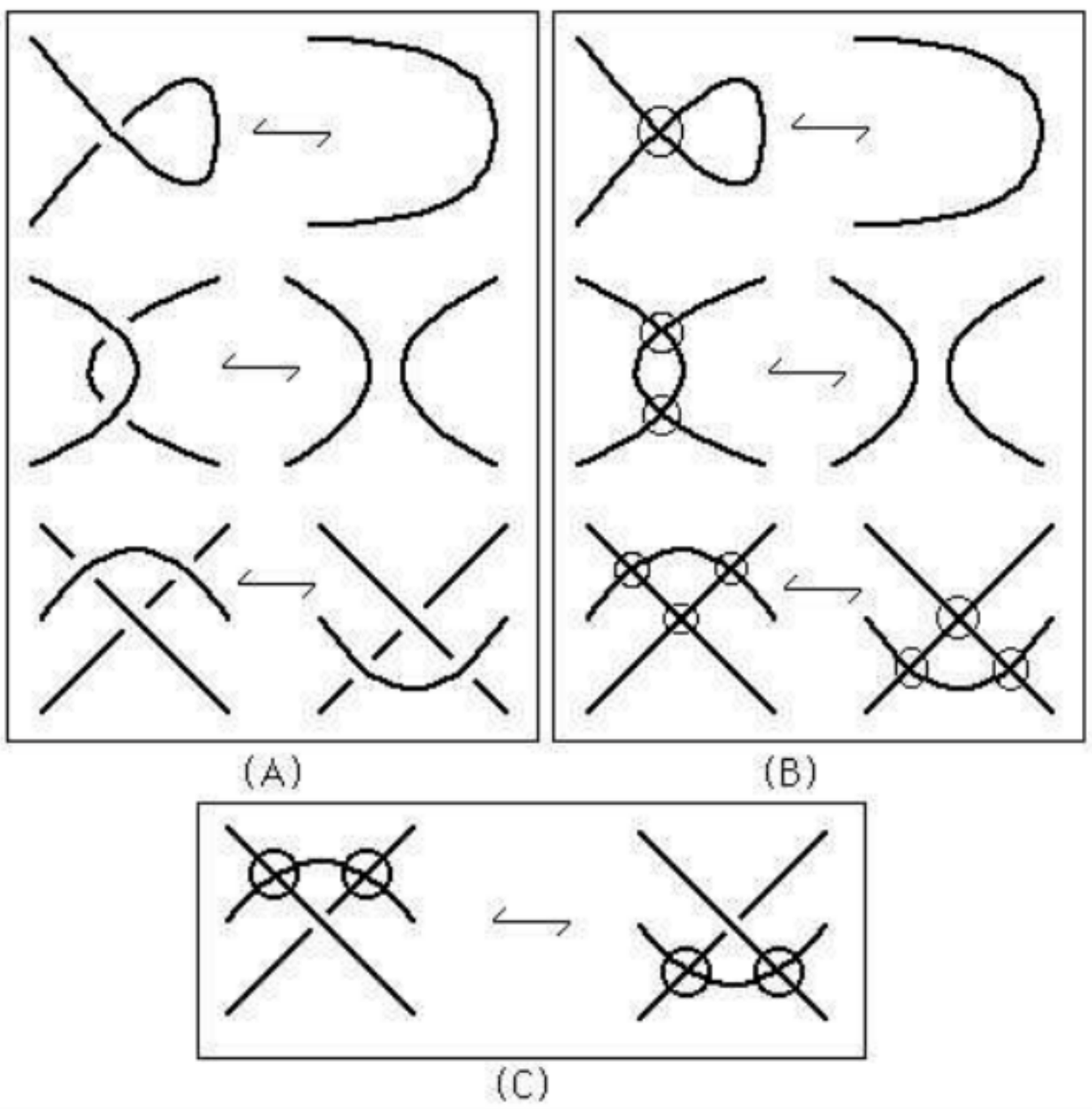


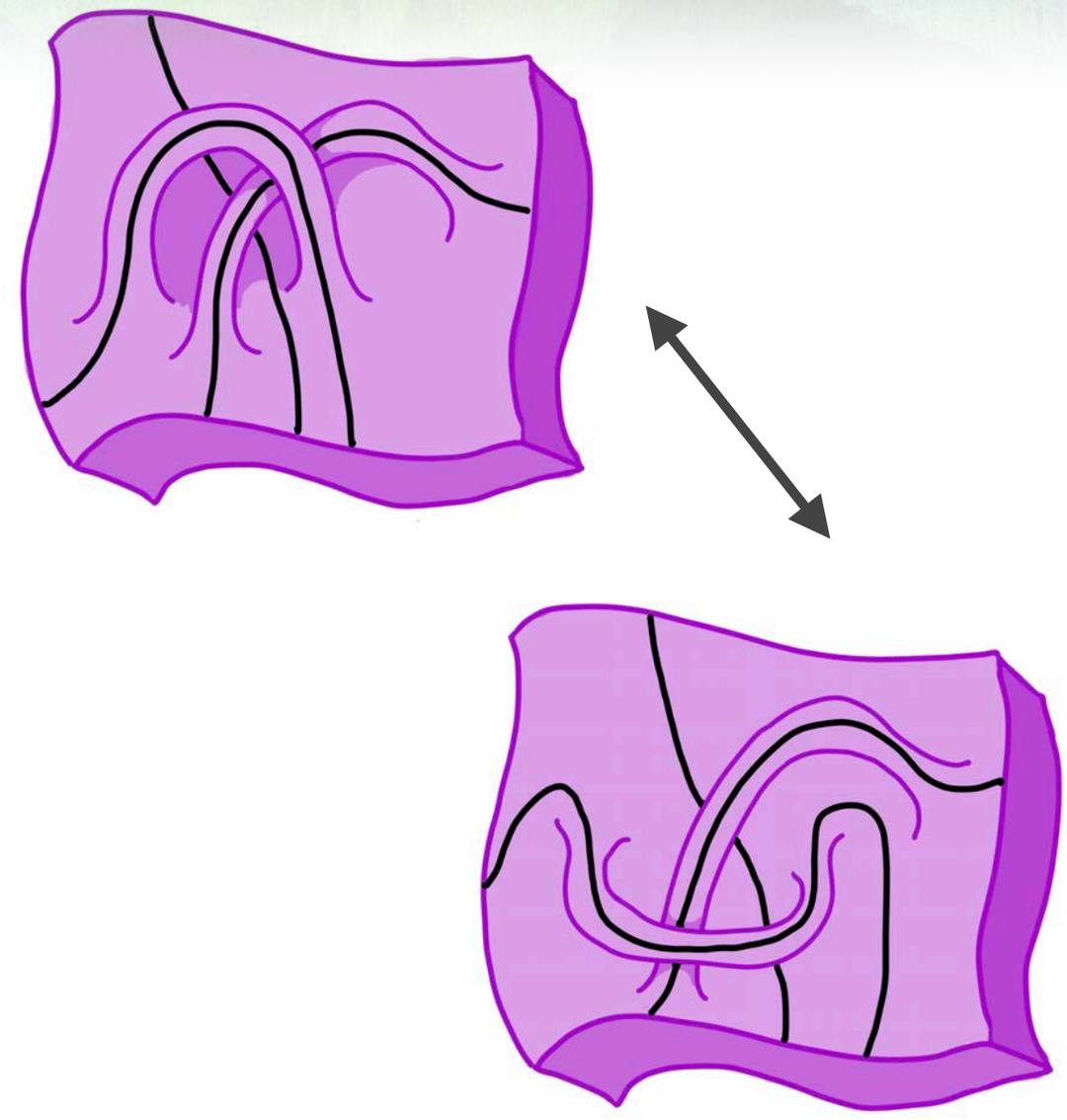


(Kauffman '99)

THE virtual BRAID GROUP vB_n

Virtual Reidemeister Moves

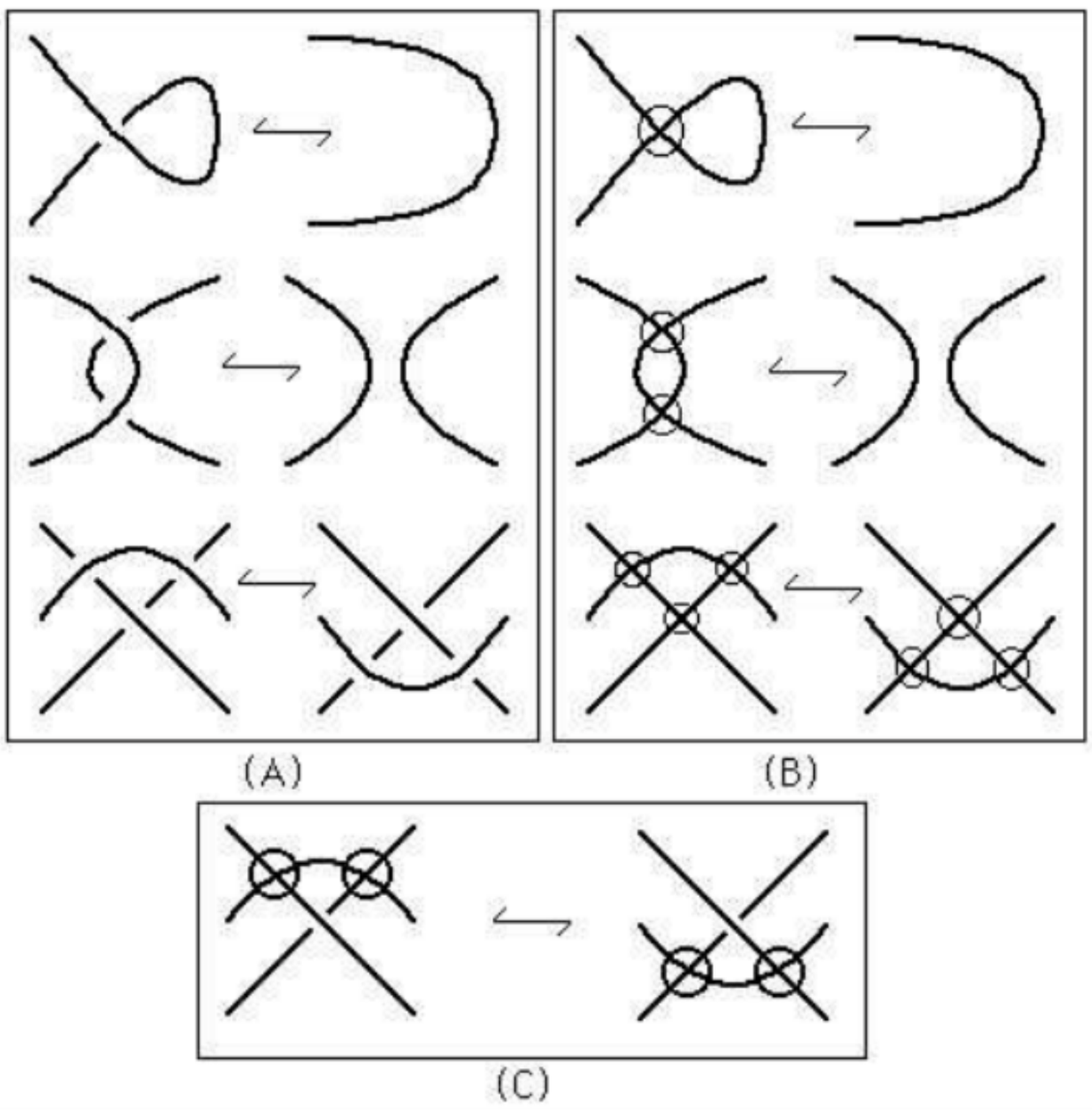


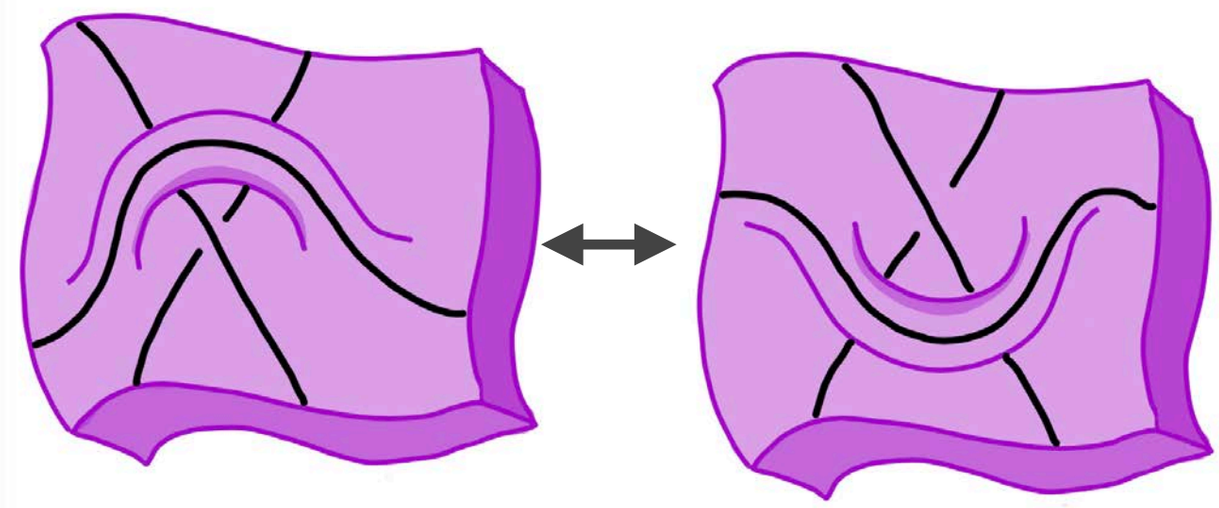


(Kauffman '99)

THE virtual BRAID GROUP vB_n

Virtual Reidemeister Moves





(Kauffman '99)

THE virtual BRAID GROUP vB_n

$vB_n = B_n * S_n /_{\sim}$ *Group presentation*

	B_n	S_n
Generators:	$\sigma_1, \cdots, \sigma_{n-1}$	$\tau_1, \cdots, \tau_{n-1}$
Relations:	$\sigma_i\sigma_j = \sigma_j\sigma_i \quad j \neq i \pm 1$	$\tau_i\tau_j = \tau_j\tau_i \quad j \neq i \pm 1$
	$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$	$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$
		$\tau_i^2 = e$
	$\sigma_i\tau_j = \tau_j\sigma_i \quad j \neq i \pm 1$	$\tau_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\tau_{i+1}$

THE welded BRAID GROUP

Let $F_n = \langle x_i \rangle$ be the free group of rank n

$$vB_n \to Aut(F_n)$$

$$B_n \subseteq Aut(F_n)$$

$$\sigma_i \mapsto \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j \quad \text{otherwise} \end{cases}$$

$$S_n \subseteq Aut(F_n)$$

$$\tau_i \mapsto \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad \text{otherwise} \end{cases}$$

THE welded BRAID GROUP

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(Fenn, Rimanyi, Rourke '93) *"Braid permutation group"* $\langle B_n, S_n \rangle \subseteq Aut(F_n)$

THE welded BRAID GROUP

Let $F_n = \langle x_i \rangle$ be the free group of rank n

$$vB_n \to Aut(F_n)$$

Not injective

$$B_n \subseteq Aut(F_n)$$

$$\sigma_i \mapsto \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j \quad \text{otherwise} \end{cases}$$

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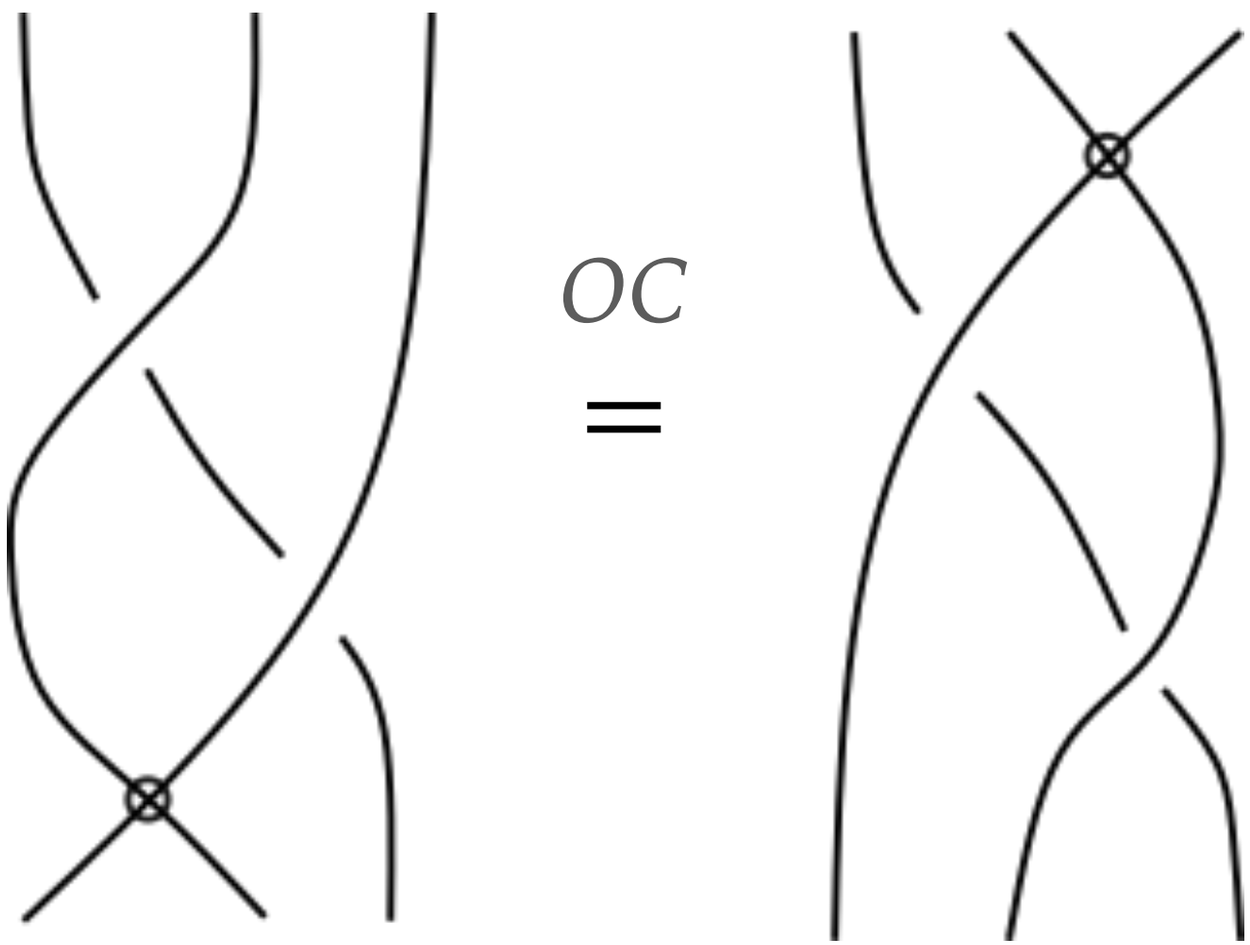
(Fenn, Rimanyi, Rourke '93) *"Braid permutation group"* $\langle B_n, S_n \rangle \subseteq Aut(F_n)$

THE welded BRAID GROUP

$$wB_n = vB_n/_{OC}$$



"Over crossings Commute"

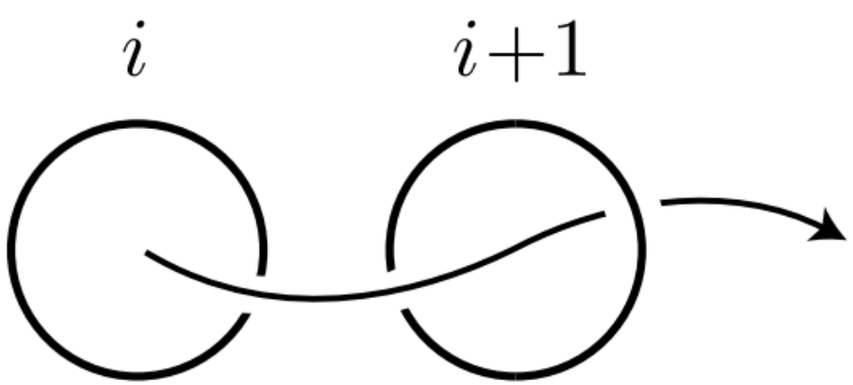
$$\tau_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \tau_{i+1}$$

THE welded BRAID GROUP wB_n

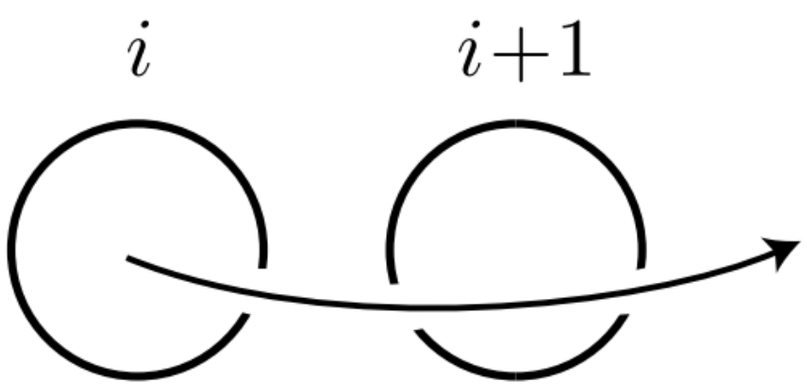
(Brendle-Hatcher 2010)

"Knotted tubes in $\mathbb{R}^4$"

wB_n is the fundamental group of configurations of circles in $\mathbb{R}^3$



σ_i



τ_i

THE welded BRAID GROUP wB_n

(Brendle-Hatcher 2010)

"Knotted tubes in $\mathbb{R}^4$"

$$\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$$

THE welded BRAID GROUP wB_n

	B_n	S_n
Generators:	$\sigma_1, \cdots, \sigma_{n-1}$	$\tau_1, \cdots, \tau_{n-1}$
Relations:	$\sigma_i\sigma_j = \sigma_j\sigma_i \quad j \neq i \pm 1$	$\tau_i\tau_j = \tau_j\tau_i \quad j \neq i \pm 1$
	$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$	$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$
		$\tau_i^2 = e$
	$\sigma_i\tau_j = \tau_j\sigma_i \quad j \neq i \pm 1$	$\tau_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\tau_{i+1}$

$$\tau_i\sigma_{i+1}\sigma_i \overset{OC}{=} \sigma_{i+1}\sigma_i\tau_{i+1}$$

LOOP THE welded BRAID GROUP wB_n BRAID-PERMUTAION

	B_n	S_n
Generators:	$\sigma_1, \cdots, \sigma_{n-1}$	$\tau_1, \cdots, \tau_{n-1}$
Relations:	$\sigma_i\sigma_j = \sigma_j\sigma_i \quad j \neq i \pm 1$	$\tau_i\tau_j = \tau_j\tau_i \quad j \neq i \pm 1$
	$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$	$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$
		$\tau_i^2 = e$
	$\sigma_i\tau_j = \tau_j\sigma_i \quad j \neq i \pm 1$	$\tau_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\tau_{i+1}$

$$\tau_i\sigma_{i+1}\sigma_i \overset{OC}{=} \sigma_{i+1}\sigma_i\tau_{i+1}$$

B_n

Classical

Braids

Virtual

$vB_n = B_n * S_n \big/ \text{Mixed R-moves}$

Welded

$wB_n = vB_n \big/ \text{OC}$

THE virtual BRAID GROUP

B_n	vB_n

THE virtual BRAID GROUP

B_n	vB_n
infinite	infinite
torsion-free	has torsion

THE virtual BRAID GROUP

B_n	vB_n
infinite torsion-free	infinite has torsion
$B_n \xrightarrow{\pi} S_n$	$vB_n \xrightarrow{\pi} S_n$
$\sigma_i \mapsto \tau_i = (i, i+1)$	$\sigma_i \mapsto \tau_i$
	$\tau_i \mapsto \tau_i$

THE virtual BRAID GROUP

B_n

infinite torsion-free

$$B_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_{i=(i,i+1)}$$

pure braid group

$$P_n = \ker(\pi)$$

vB_n

infinite has torsion

$$vB_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_i$$

$$\tau_i \mapsto \tau_i$$

pure virtual braid group

$$vP_n = \ker(\pi)$$

THE virtual BRAID GROUP

B_n

infinite torsion-free

$$B_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_i = (i, i+1)$$

pure braid group

$$P_n = \ker(\pi)$$

$$1 \to P_n \to B_n \to S_n \to 1$$

short exact sequence

*Does **NOT** split*

vB_n

infinite has torsion

$$vB_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_i$$

$$\tau_i \mapsto \tau_i$$

pure virtual braid group

$$vP_n = \ker(\pi)$$

THE virtual BRAID GROUP

B_n

infinite torsion-free

$$B_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_{i=(i,i+1)}$$

pure braid group

$$P_n = \ker(\pi)$$

$$1 \to P_n \to B_n \to S_n \to 1$$

short exact sequence

*Does **NOT** split*

vB_n

infinite has torsion

$$vB_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_i$$

$$\tau_i \mapsto \tau_i$$

pure virtual braid group

$$vP_n = \ker(\pi)$$

$$1 \to vP_n \to vB_n \to S_n \to 1$$

Splits!

$$vB_n = vP_n \rtimes S_n$$

THE virtual BRAID GROUP

$$vB_n = vP_n \rtimes S_n$$

The bulk of vB_n

finite group

$$vB_n = B_n * S_n /_{\sim}$$

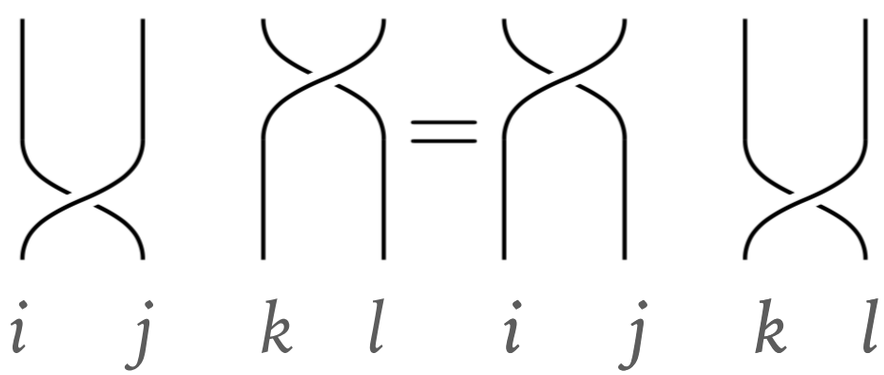
THE PURE virtual BRAID GROUP

(McCool '86, Bardakov 2004)

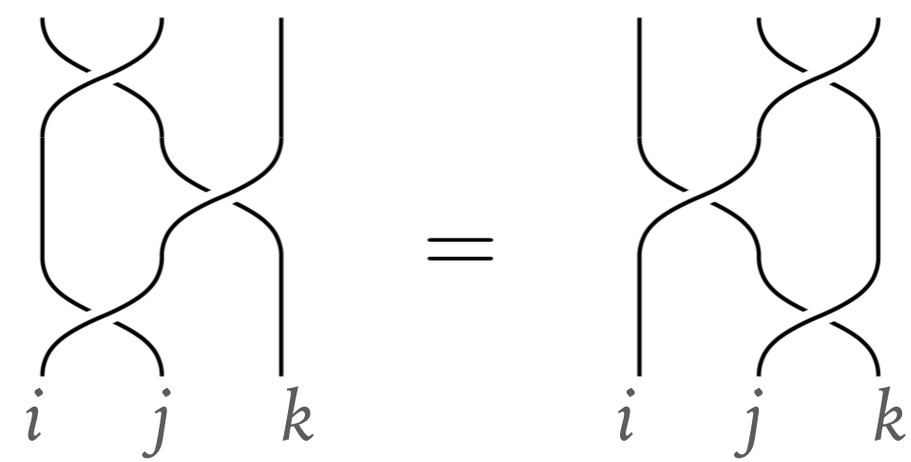
$$PvB_n = vPB_n = vP_n$$

Generators: $\sigma_{i,j}$ $\quad j \neq i,\ 1 \leq i, j \leq n$ "Strand i crosses over strand j"

Relations: $\sigma_{i,j}\sigma_{k,l} = \sigma_{k,l}\sigma_{i,j}$ $\quad i \neq j \neq k \neq l$



$\sigma_{i,j}\sigma_{i,k}\sigma_{j,k} = \sigma_{j,k}\sigma_{i,k}\sigma_{i,j}$



THE PURE virtual BRAID GROUP

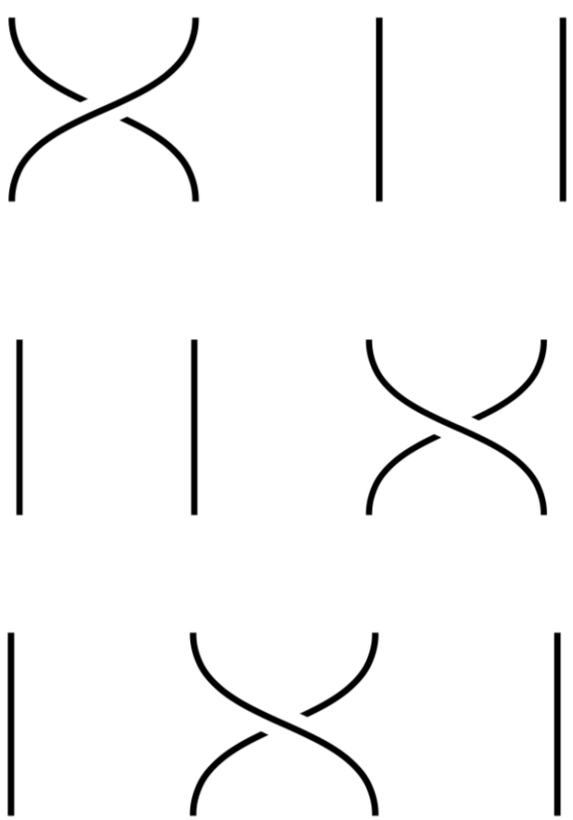
Geography

Strands get re-named after they change position

σ_1

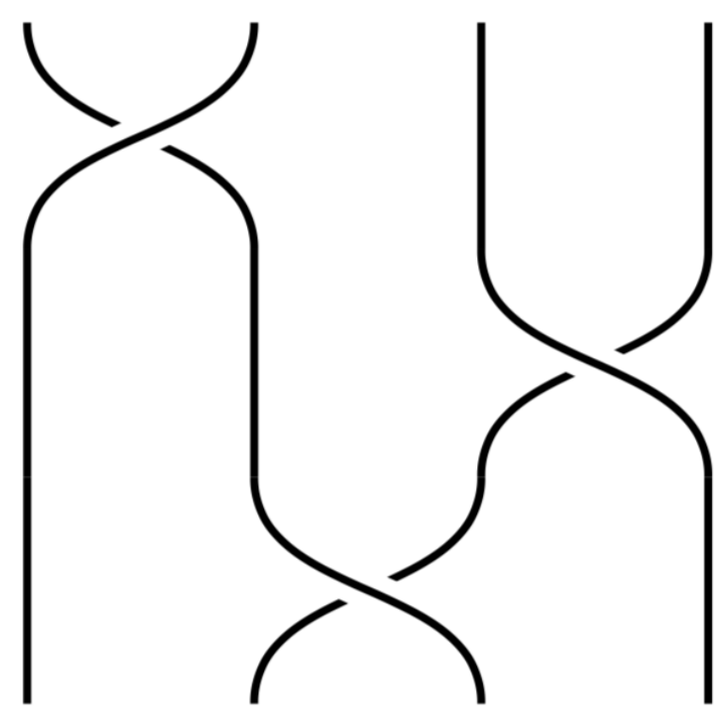
σ_3^{-1}

σ_2^{-1}



Identity

Strands keep their names after they change position



$\sigma_{1,3}$

$\sigma_{2,4}^{-1}$

$\sigma_{2,3}^{-1}$

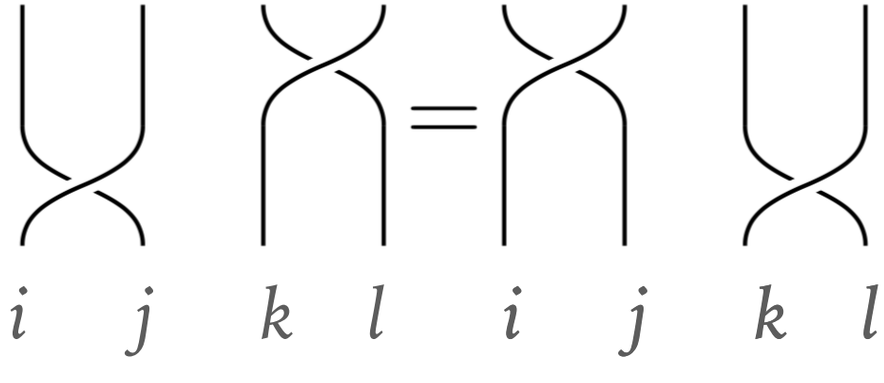
THE PURE virtual BRAID GROUP

$$PvB_n = vPB_n = vP_n$$

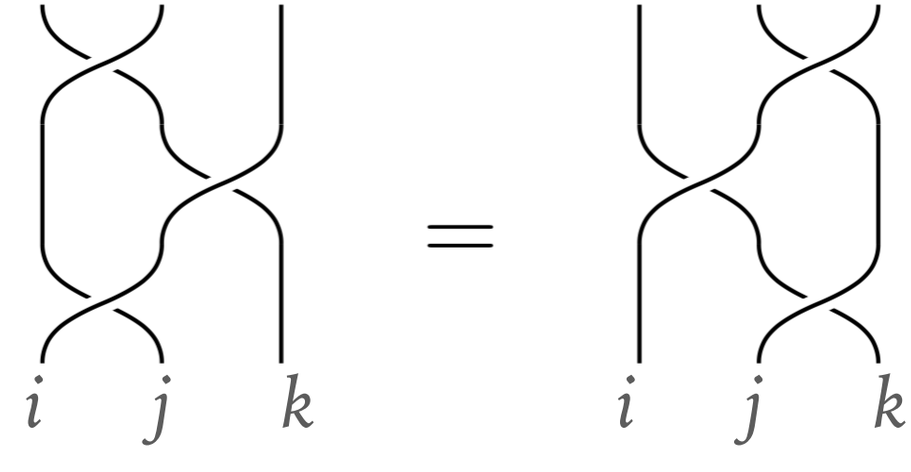
Generators: $\sigma_{i,j}$ $\quad j \neq i, \quad 1 \leq i, j \leq n$

Relations: $\sigma_{i,j}\sigma_{k,l} = \sigma_{k,l}\sigma_{i,j}$

$i \neq j \neq k \neq l$

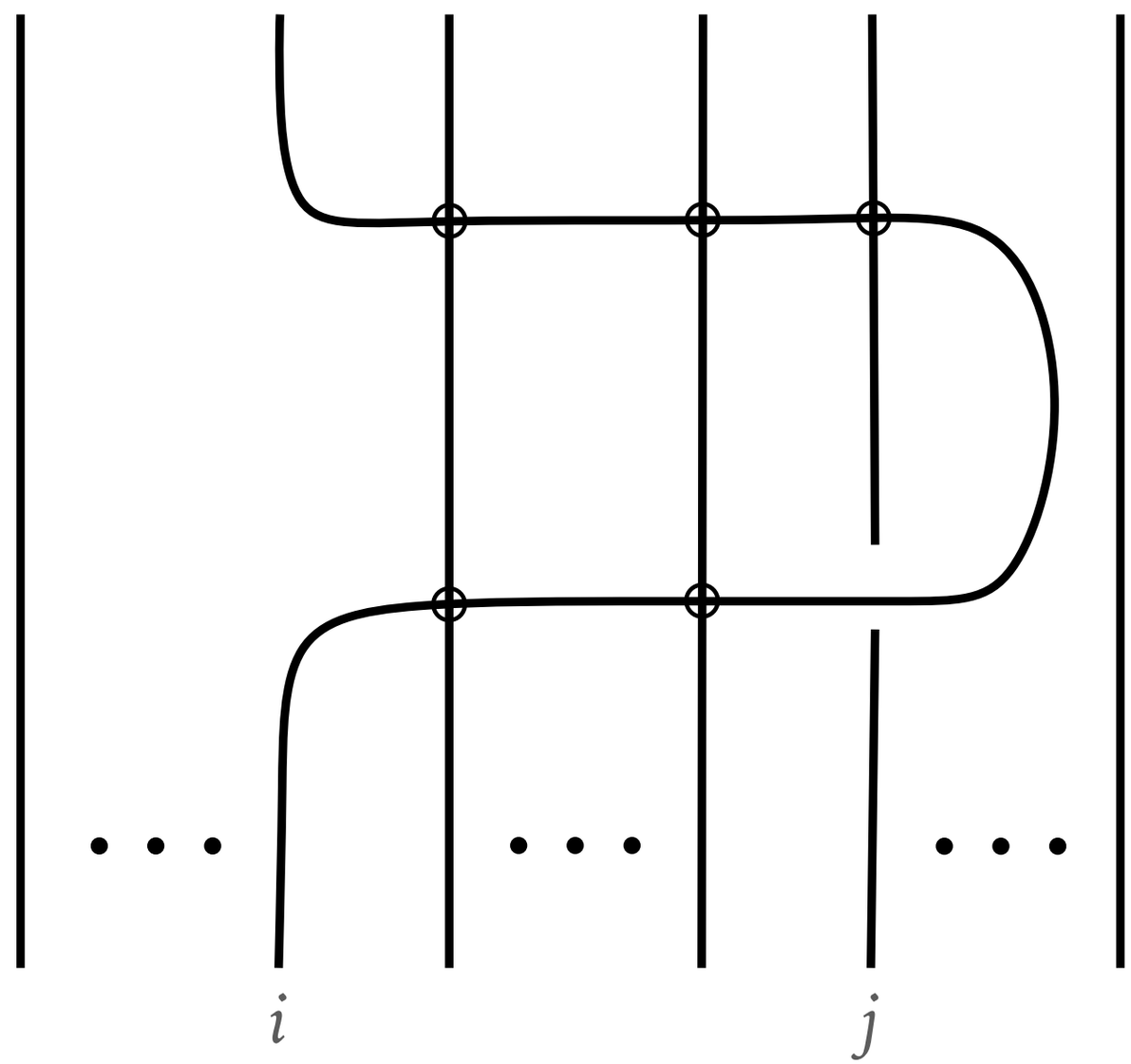


$\sigma_{i,j}\sigma_{i,k}\sigma_{j,k} = \sigma_{j,k}\sigma_{i,k}\sigma_{i,j}$



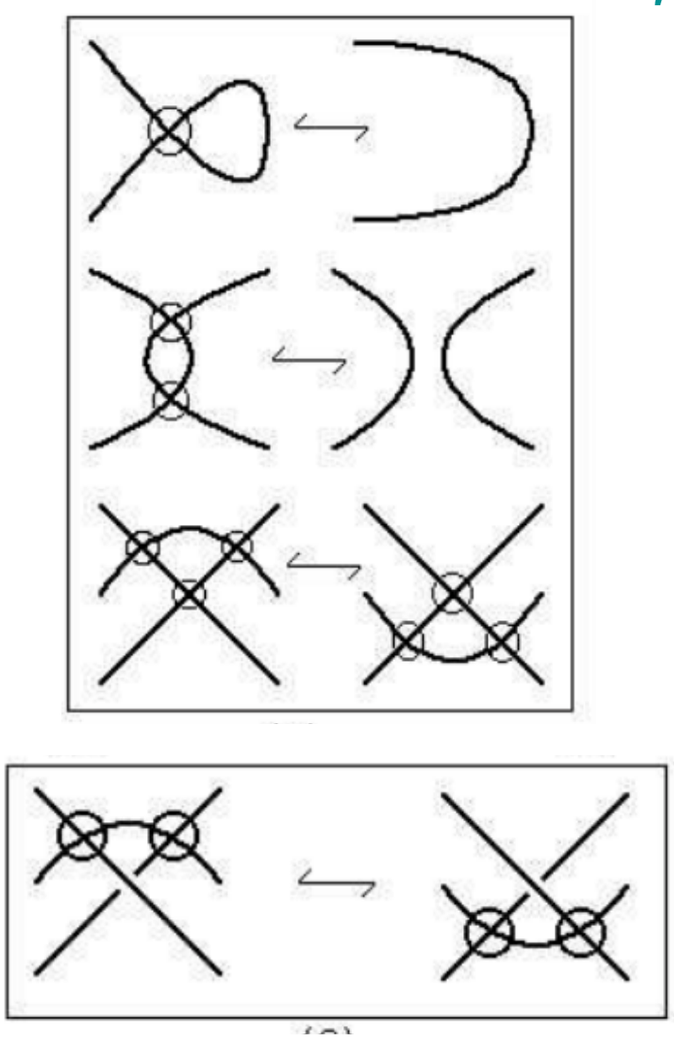
THE PURE virtual BRAID GROUP

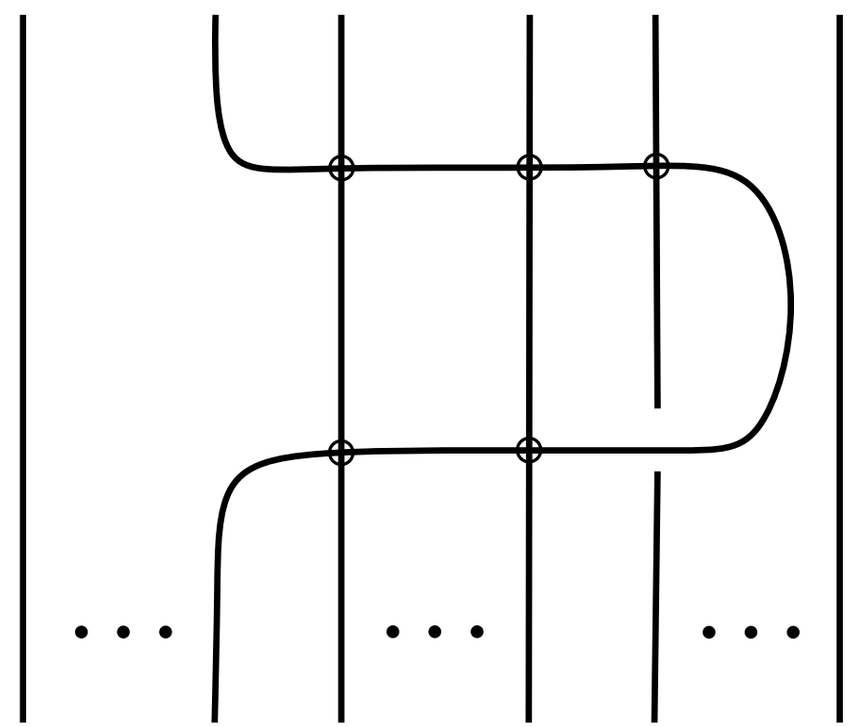
$$\sigma_{i,j} \leftrightarrow \tau_i\tau_{i+1}\cdots\tau_{j-2}\tau_{j-1}\sigma_{j-1}\tau_{j-2}\cdots\tau_{i+1}\tau_i$$



THE virtual BRAID GROUP

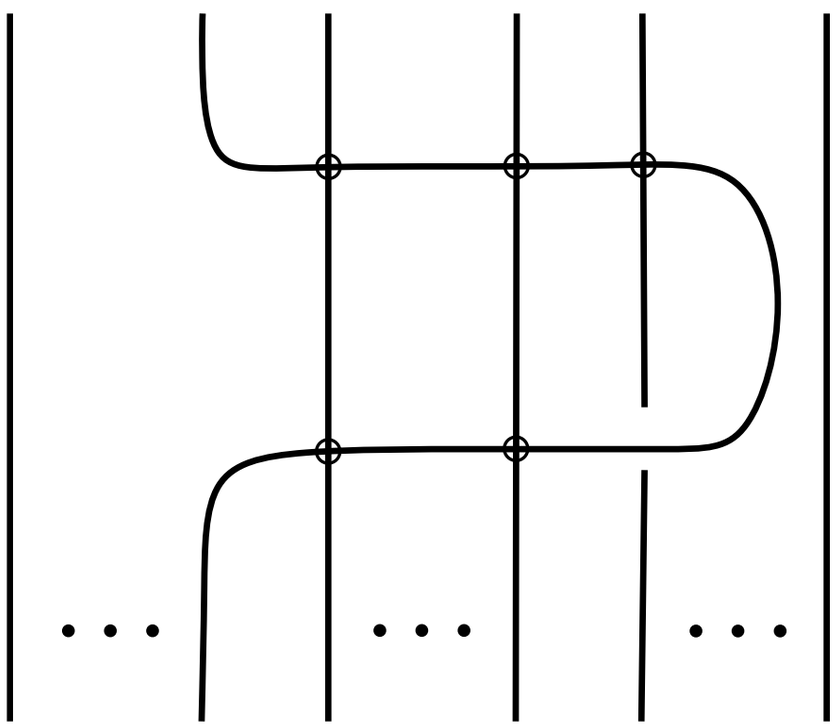
$$B_n * S_n / \sim = vB_n = vP_n \rtimes S_n$$





THE welded BRAID GROUP wB_n

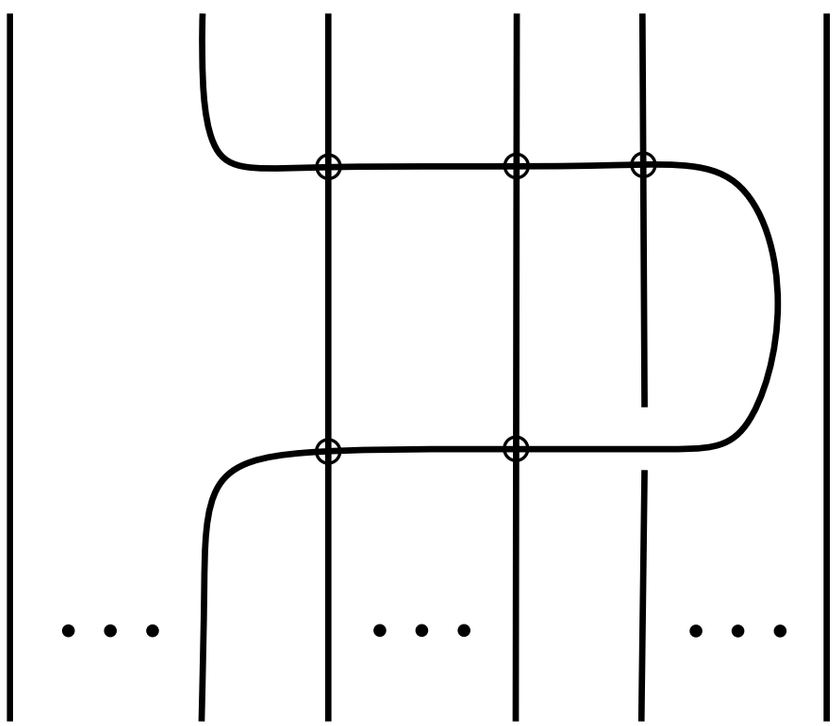
$$wP_n = vP_{n/OC}$$



$$\sigma_{i,j}\sigma_{i,k} = \sigma_{i,k}\sigma_{i,j}$$

THE welded BRAID GROUP wB_n

$$wP_n = vP_{n/OC}$$



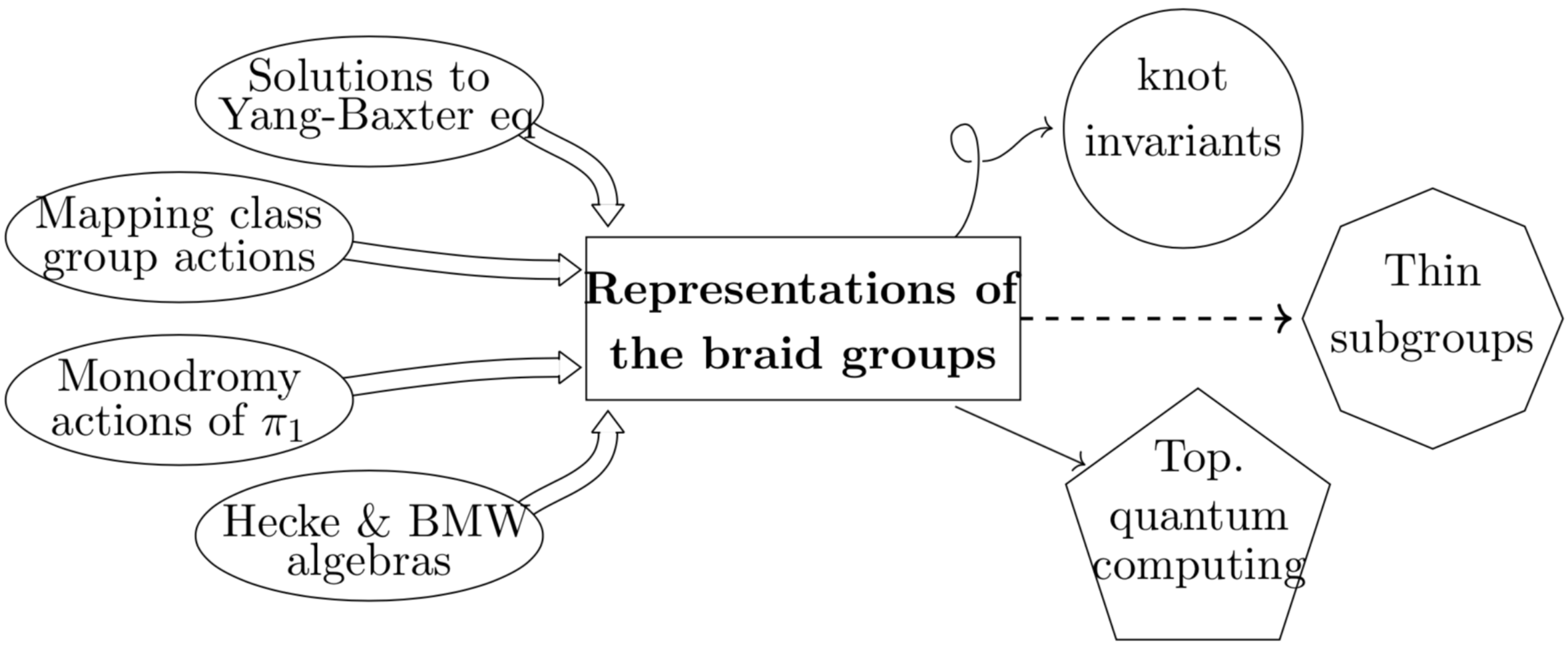
$$\sigma_{i,j}\sigma_{i,k} = \sigma_{i,k}\sigma_{i,j}$$

i

THE welded BRAID GROUP

$$\begin{tikzcd}
 & v\mathcal{B}_n & \\
\mathcal{B}_n \arrow[ur, "\text{subgroup}"] & v\mathcal{P}_n \arrow[u, "\text{subgroup}"] \arrow[dr, "\text{quotient}\ \text{OC relation}"'] & \\
 & & w\mathcal{B}_n \\
 & & w\mathcal{P}_n \arrow[u, "\text{subgroup}"'] \arrow[uu, bend right]
\end{tikzcd}$$

REPRESENTATIONS OF B_n



WHEN DO REPRESENTATIONS "EXTEND"?

Given: $B_n \xrightarrow{\phi} GL_m(V)$

Does there exist?: $vB_n \xrightarrow{\hat{\phi}} GL_m(V)$

So that $\hat{\phi}|_{B_n} = \phi$

But really we seek only this:

$$vP_n \xrightarrow{\hat{\phi}} GL_m(V)$$

THE BURAU REPRESENTATION

The (*reduced*) Burau representation $\rho_n : B_n \longrightarrow GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & Id_{n-3} \end{array}\right), \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & t & -t \end{array}\right)$$

$$\sigma_i \mapsto \left(\begin{array}{c|ccc|c} Id_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & Id_{n-i-2} \end{array}\right) \quad for\ 2 \leq i \leq n-2$$

THE BURAU REPRESENTATION

What do we need to extend Burau?

$$\rho_n : \quad B_n \longrightarrow GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$$

$$S_n \nearrow$$

"Play nice together"

Satisfies the mixed relations

$$\tau_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \tau_{i+1}$$

$$\sigma_i \tau_j = \tau_j \sigma_i$$

THE BURAU REPRESENTATION

Burau extends!

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & Id_{n-3} \end{array}\right), \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & t & -t \end{array}\right)$$

$$\sigma_i \mapsto \left(\begin{array}{c|ccc|c} Id_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & Id_{n-i-2} \end{array}\right) \; for\ 2 \leq \boldsymbol{i} \leq n-2$$

$$\tau_1 \mapsto \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & Id_{n-3} \end{array}\right), \tau_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 1 & -1 \end{array}\right)$$

$$\tau_i \mapsto \left(\begin{array}{c|ccc|c} Id_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & Id_{n-i-2} \end{array}\right) \; for\ 2 \leq \boldsymbol{i} \leq n-2$$

THE BURAU REPRESENTATION

Burau extends!

(unreduced)

Let V be a $\mathbb{Z}[t^{\pm 1}]$-module with basis $v_1, \cdots, v_n$

vP_n acts on V by

$$\sigma_{i,j}(v_k) = \begin{cases} v_k & \text{if } k \neq j \\ tv_j - (t-1)v_i & \text{if } k = j \end{cases}$$

THE GASSNER REPRESENTATION

Gassner extends!

Let V be a $\mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$-module with basis $v_1, \cdots, v_n$

vP_n acts on V by

$$\sigma_{i,j}(v_k) = \begin{cases} v_k & \text{if } k \neq j \\ t_i v_j - (t_i - 1)v_i & \text{if } k = j \end{cases}$$

THE LKB REPRESENTATION

What about Lawrence-Krammer-Bigelow?

Let V be a $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$-module with basis $v_{i,j}$, where $1 \leq j < k \leq n$

B_n acts on V by

$$\sigma_i \cdot v_{j,k} = \begin{cases} v_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qv_{i,k} + (q^2 - q)v_{i,j} + (1-q)v_{j,k} & i = j-1 \\ v_{j+1,k} & i = j \neq k-1, \\ qv_{j,i} + (1-q)v_{j,k} - (q^2-q)tv_{i,k} & i = k-1 \neq j, \\ v_{j,k+1} & i = k, \\ -tq^2 v_{j,k} & i = j = k-1. \end{cases}$$

Long-Moody Construction

Start with a representation V of B_{n+1} with dimension m

Create a new representation of B_n with dimension $n \cdot m$

$$F_n = \langle g_i \rangle_{i=1}^n \text{ free group of rank } n$$

$$B_n \subseteq Aut(F_n) \qquad \beta(f) := f^\beta$$

$$I_n = \langle g_i - 1 \rangle_{i=1}^n \text{ augmentation ideal in } \mathbb{Q}[F_n]$$

B_n acts on $I_n \otimes_{\mathbb{Q}[F_n]} V$

$$\beta(i \otimes v) = i^\beta \otimes \beta \,.\, v$$

$$\uparrow$$

well defined because $F_n \rtimes B_n \subseteq B_{n+1}$

Long-Moody Construction

Start with the 1-dim representation of B_{n+1} that sends $\sigma_i \mapsto [t]$

Creates Burau rep on B_n

Long-Moody Construction

Start with the 1-dim representation of B_{n+1} that sends $\sigma_i \mapsto [t]$

Creates Burau rep on B_n

LM construction works for welded braids (Bellingeri-Soulie 2020)

(Need $F_n \rtimes wB_n \subseteq wB_{n+1}$ and this needs OC relation)

Start with the 1-dim representation of wB_{n+1} that sends $\sigma_i \mapsto [t]$

Creates Burau rep on wB_n

Long-Moody Construction with I_n^k

Start with a representation V of B_{n+1} (or wB_{n+1}) with dimension m

Create a new representation of B_n (or wB_n) with dimension $k \cdot n \cdot m$

$$I_n^k = \langle \prod_{j=1}^{k} (g_{i_j} - 1) \rangle \; k\text{'th power of the augmentation ideal in } F_n$$

B_n acts on $I_n^k \otimes_{\mathbb{Q}[F_n]} V$

$$\beta(i \otimes v) = i^\beta \otimes \beta \,.\, v$$

Start with the 1-dim representation of B_{n+1} that sends $\sigma_i \mapsto [t]$

Creates variant Burau rep on B_n

Long-Moody Construction with I_n^2

Start with the 1-dim representation of B_{n+1} that sends $\sigma_i \mapsto [t]$

$$\sigma_i \mapsto \begin{bmatrix} B \otimes Id_{ni-n} & & & \\ & 0 & A & \\ & B & C & \\ & & & B \otimes Id_{n^2-ni-n} \end{bmatrix}$$

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} & i \quad i{+}1 & \end{array} \\ \left[\begin{array}{c|cc|c} Id_{i-1} & & & \\ \hline & 0 & t & \\ \hline t\text{-}1 \cdots t\text{-}1 & t & 0 & t\text{-}1 \cdots t\text{-}1 \\ \hline & & & Id_{n-i-1} \end{array}\right] \end{array} \begin{array}{c} \\ i \\ i{+}1 \\ \\ \end{array}$$

Equivalent to Unreduced Burau

$$B = \left[\begin{array}{c|cc|c} Id_{i-1} & & & \\ \hline & 0 & t & \\ & 1 & 1\text{-}t & \\ \hline & & & Id_{n-i-1} \end{array}\right]$$

$$C = \begin{bmatrix} & 0 & \\ t(1\text{-}t) & \cdots\cdots & t(1\text{-}t) \\ (t-1)^2 & \cdots\cdots & (t-1)^2 \\ & 0 & \end{bmatrix} \begin{array}{c} \\ i \\ i{+}1 \\ \\ \end{array}$$

Long-Moody Construction with I_n^2

Start with the 1-dim representation of B_{n+1} that sends $\sigma_i \mapsto [t]$

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Equivalent to Unreduced Burau

$$\rho \qquad B = \begin{bmatrix} Id_{i-1} & & \\ & \begin{matrix} 0 & t \\ 1 & 1\text{-}t \end{matrix} & \\ & & Id_{n-i-1} \end{bmatrix}$$

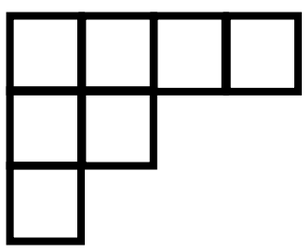
$$C = \begin{bmatrix} & 0 & \\ t(1\text{-}t) & \cdots\cdots & t(1\text{-}t) \\ (t-1)^2 & \cdots\cdots & (t-1)^2 \\ & 0 & \end{bmatrix} \begin{array}{c} \\ i \\ i{+}1 \\ \\ \end{array}$$

$n = 5$

ϕ is stronger than Burau: $k \in ker(\rho)$ *but* $\phi(k) \neq Id$

QUESTIONS I HAVE FOR YOU!

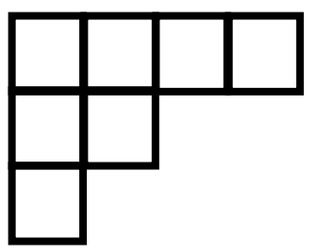
- If I gave you a Young diagram, can you compute the corresponding Jones representation of the classical braid group?



$$\sigma_i \longrightarrow \overset{\textit{Irreducible rep of } B_7}{\begin{bmatrix} & ? & \end{bmatrix}}$$

QUESTIONS I HAVE FOR YOU!

- If I gave you a Young diagram, can you compute the corresponding Jones representation of the classical braid group?



Irreducible rep of B_7

$$\sigma_i \longrightarrow \begin{bmatrix} & ? & \end{bmatrix}$$

- Does the Burau representation show up in your research?

  (I'm writing a survey paper about Burau!)

THANK YOU!