Log-Concavity of Littlewood–Richardson Coefficients

Avery St. Dizier

University of Illinois at Urbana–Champaign

Based on joint works with Alex Fink, June Huh, Jacob Matherne, and Karola Mészáros

ICERM Semester in Combinatorial Algebraic Geometry

Schubert Seminar Series

Outline

The Flag Manifold

A *complete flag* $F = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a sequence of vector subspaces

$$F_1 \subset F_2 \subset \cdots \subset F_n$$

where $\dim F_i = i$.

The *flag manifold* $\mathrm{Fl}(n)$ is the set of complete flags in $\mathbb{C}^n$.

Cohomology of the Flag Manifold

$\mathrm{Fl}(n)$ decomposes into a disjoint union of open cells X_w indexed by permutations w in S_n.

Their closures are the *Schubert varieties* $\overline{X}_w$.

$\{[\overline{X}_w]\}_{w \in S_n}$ is a linear basis of the cohomology ring $H^*(\mathrm{Fl}(n))$.

Borel's Isomorphism

Theorem (Borel 1953)

$$H^*(Fl(n)) \cong \mathbb{Z}[x_1, \ldots, x_n]/I_n,$$

where $I_n = \langle \textit{symmetric functions with zero constant term} \rangle$.

What is a good choice of polynomial representatives for the basis $\{[\overline{X}_w]\}$?

Schubert Polynomials

Schubert polynomials were defined by Lascoux and Schützenberger (1982) divided difference operators ∂_i:

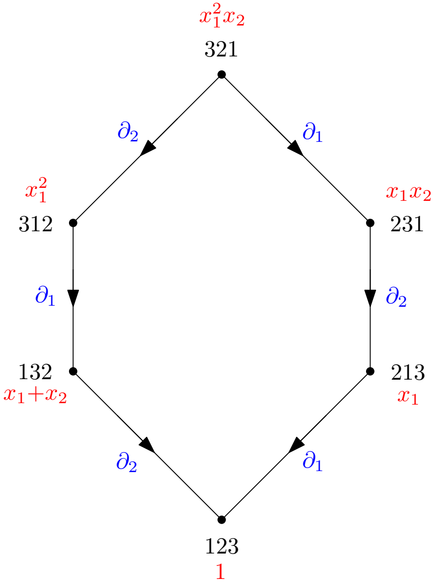
$$\partial_i f = \frac{f - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

Definition

The *Schubert polynomial* $\mathfrak{S}_w$ of $w \in S_n$ is defined by

$$\mathfrak{S}_w = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = n\,(n-1) \cdots 2\,1 \\ \partial_i \mathfrak{S}_{ws_i} & \text{if } w(i) < w(i+1) \end{cases}$$

Divided Difference Operators



$$\partial_1(x_1^2 x_2) = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2$$

$$\partial_2(x_1 x_2) = \frac{x_1 x_2 - x_1 x_3}{x_2 - x_3} = x_1$$

$$\partial_1(x_1) = \frac{x_1 - x_2}{x_1 - x_2} = 1$$

Nonnegativity of Schubert Polynomials

Theorem (Lascoux-Schützenberger 1982)

The coefficients of $\mathfrak{S}_w$ are nonnegative integers.

Proof.

The Transition Rule:

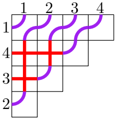
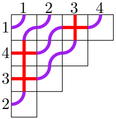
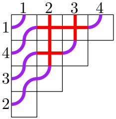
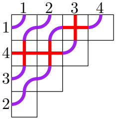
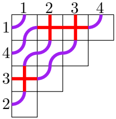
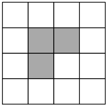
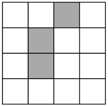
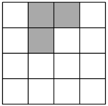
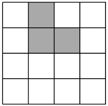
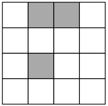
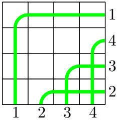
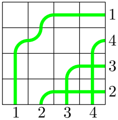
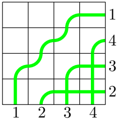
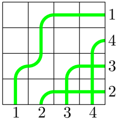
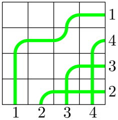
$$\mathfrak{S}_w = x_i \mathfrak{S}_v + \sum_{\substack{k<i \\ v\cdot(ki) \text{ covers } v}} \mathfrak{S}_{v(ki)}$$

□

Is there a combinatorial interpretation behind the nonnegativity?

First answered by Billey–Jockusch–Stanley (1993), and re-answered in many interesting ways since.

A Sample of the Nonnegativity of $\mathfrak{S}_{1432}$

$\mathfrak{S}_{1432} =$	$x_2^2x_3$	$+\ x_1x_2x_3$	$+\ x_1^2x_2$	$+\ x_1x_2^2$	$+\ x_1^2x_3$	
Compatible Sequences	$s_3s_2s_3$ $(2,2,3)$	$s_3s_2s_3$ $(1,2,3)$	$s_3s_2s_3$ $(1,1,2)$	$s_2s_3s_2$ $(1,2,2)$	$s_3s_2s_3$ $(1,1,3)$	Billey–Jockusch–Stanley
Reduced Pipe Dreams						Billey–Bergeron, Fomin–Kirilov
Kohnert Diagrams						Kohnert, Winkel
Bumpless Pipe Dreams						Lam–Lee–Shimozono
Generalized Young Tableaux	2 2 3	1 2 3	1 1 2	1 2 2	1 1 3	Magyar

Outline

- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov's Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials

Semistandard Young Tableaux

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, a *semistandard Young tableaux* (SSYT) of shape λ is a filling of the Young diagram of λ with numbers from $[n]$ such that:

- The entries weakly increase left-to-right in each row,
- The entries strictly increase top-to-bottom in each column.

$\mathrm{SSYT}(2,1,0):$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur Polynomials

Definition

Given a partition $\lambda \in \mathbb{Z}^n_{\geq 0}$, the *Schur polynomial* s_λ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda)} x_1^{wt(T)_1} \cdots x_n^{wt(T)_n}$$

$\mathrm{SSYT}(2,1,0):$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$s_{210}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Kostka Numbers

The coefficient of x^α in s_λ is the *Kostka number*

$$K_{\lambda\alpha} = \# \text{ of SSYT with shape } \lambda \text{ and weight } \alpha$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

$$K_{(2,1,0),(1,1,1)} = 2$$

$$s_{210}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

VIPs (Very Important Polynomials)

Schur polynomials are symmetric polynomials, and form a basis for the space of symmetric polynomials.

Representation Theory:
The characters of the irreducible representations V_λ of GL_n are exactly Schur polynomials.

Geometry:
Schur polynomials form a linear basis for the cohomology ring of the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$.

Littlewood–Richardson Coefficients

The *Littlewood–Richardson coefficients* are $c^{\nu}_{\lambda\mu}$ defined by

$$s_\lambda s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu.$$

Representation Theory:

$c^{\nu}_{\lambda\mu}$ controls tensor products of GL_n irreducible representations via

$$V_\lambda \otimes V_\mu \cong \bigoplus_{\ell(\nu)\leq n} V_\nu^{c^{\nu}_{\lambda\mu}}$$

Geometry:

For σ_λ the cohomology class of the Schubert variety X_λ in the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$,

$$\sigma_\lambda \sigma_\mu = \sum_{\nu\subseteq k\times(n-k)} c^{\nu}_{\lambda\mu} \sigma_\nu$$

Outline

- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov's Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials

Log-Concavity

Continuous:

A function $f : \mathbb{R}^n \to \mathbb{R}_{>0}$ is *log-concave* if

$$\log f(\theta x + (1 - \theta y)) \geq \theta \log f(x) + (1 - \theta) \log f(y)$$
$$f(\theta x + (1 - \theta y)) \geq f(x)^{\theta} f(y)^{1-\theta}$$

Discrete:

A function $a : \mathbb{Z} \to \mathbb{R}_{>0}$ is *log-concave* if

$$a_i^2 \geq a_{i+1} a_{i-1}$$

Okounkov's Conjecture

Conjecture (Okounkov 2003)

The discrete function

$$(\lambda, \mu, \nu) \mapsto \log c_{\lambda\mu}^{\nu}$$

is a concave function of λ, μ, ν.

Why would multiplicities be log-concave ?

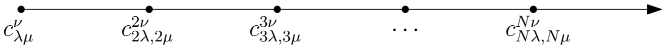
Andrei Okounkov

Abstract

It is a basic property of the entropy in statistical physics that is concave as a function of energy. The analog of this in representation theory would be the concavity of the logarithm of the multiplicity of an irreducible representation as a function of its highest weight. We discuss various situations where such concavity can be established or reasonably conjectured and consider some implications of this concavity. These are rather informal notes based on a number of talks I gave on the subject, in particular, at the 1997 International Press lectures at UC Irvine.

Okounkov's Conjecture

In the cone of partition tuples (λ, μ, ν), consider



Log-concavity would imply

$$(c^{N\nu}_{N\lambda,N\mu})^2 \geq c^{(N+1)\nu}_{(N+1)\lambda,(N+1)\mu} c^{(N-1)\nu}_{(N-1)\lambda,(N-1)\mu}.$$

Counterexample found by Chindris–Derksen–Weyman (2007):
$\lambda = \mu = (3^{21}, 2^{21}, 1^{21})$, $\nu = (4^{21}, 3^{42}, 2^{21})$

Okounkov's Conjecture

Okounkov's Conjecture is false.

Some consequences of it are true.

Okounkov proved log-concavity holds *asymptotically*.

Are there interesting special sets of tuples (λ, μ, ν) for which a version of Okounkov's conjecture holds?

A True Log-Concavity

Theorem (Huh–Matherne–Mészáros–S. 2019)

Choose any $i, j \geq 0$ and any partitions λ, μ, ν such that ν/μ is a skew shape with at most one box per column. Whenever all subscripts and superscripts are partitions,

$$\left(c^{\nu}_{\lambda,\mu}\right)^2 \geq c^{\nu+\varpi_i-\varpi_j}_{\lambda,\mu+\varpi_{i-1}-\varpi_{j-1}}\, c^{\nu-\varpi_i+\varpi_j}_{\lambda,\mu-\varpi_{i-1}+\varpi_{j-1}}$$

where ϖ_k is the kth fundamental weight $(1^k, 0^{n-k})$.

For example, if $n = 5$, $i = 2$, and $j = 4$, then

$$11^2 = \left(c^{87\color{red}{64}1}_{52100,7\color{blue}{64}10}\right)^2 \geq c^{87\color{red}{53}1}_{52100,7\color{blue}{53}10}\, c^{87\color{red}{75}1}_{52100,7\color{blue}{75}10} = 14 \times 4$$

Outline

- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov's Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials

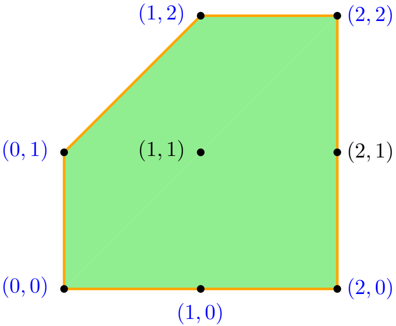
A Polytopal Perspective

Any polynomial $f = \sum_{z \in \mathbb{Z}^n} a_{\mathbf{z}} x^{\mathbf{z}} \in \mathbb{C}[x_1, \ldots, x_n]$ has an associated integer polytope called its *Newton polytope*:

$$\mathrm{Newton}(f) = \mathrm{Conv}(\mathbf{z} : a_{\mathbf{z}} \neq 0)$$

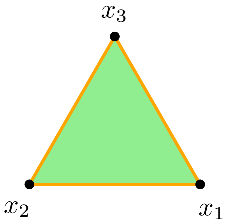
$\mathrm{Newton}(1 + x + y + x^2 + xy^2 + x^2y^2) =$

$\mathrm{Newton}(1 + y + xy + xy^2 + x^2 + x^2y^2) =$

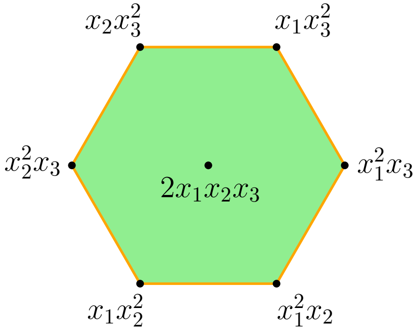


Schur Polynomial Newton Polytopes

What kind of polytopes are the Newton polytopes of Schur polynomials?



$s_{(1,0,0)} =$
$x_1 + x_2 + x_3$



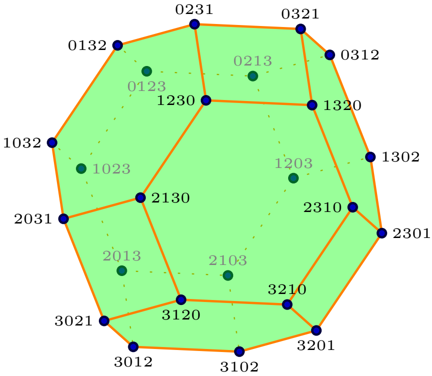
$s_{(2,1,0)} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3$

Schur Polynomial Newton Polytopes

Theorem (Rado 1952)

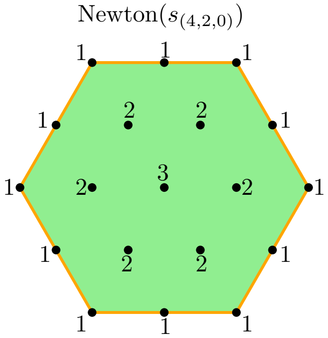
The Newton polytope of s_λ is the permutahedron $\mathcal{P}_\lambda$ (the convex hull of all permutations of λ).

$\mathrm{Newton}(s_{(3,2,1,0)})$



Coefficients and Newton Polytopes

How do the Kostka numbers within the Schur Newton polytope look?



Question

Can there be zeros in the polytope?

Saturated Newton Polytopes (SNP)

Definition (Monical-Tokcan-Yong 2017)

A polynomial f is said to have *saturated Newton polytope* (SNP) if every integer point in the Newton polytope corresponds to a monomial with nonzero coefficient in f.

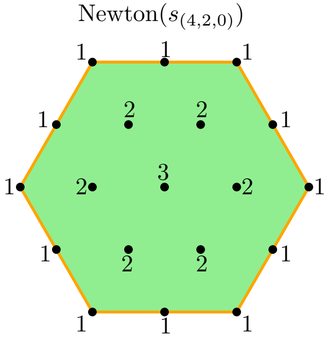
Theorem (Monical-Tokcan-Yong 2017)

The following all have SNP:

- *Schur polynomials*
- *Skew-Schur polynomials*
- *Stanley symmetric functions*
- (q, t) *evaluations of symmetric Macdonald polynomials*

Back to Kostka Numbers

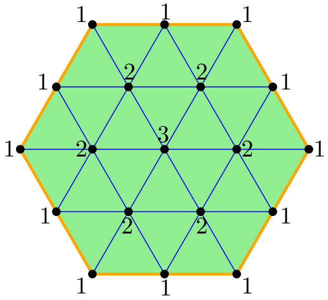
SNP says there are no zeros within the Newton polytope. How are the nonzero coefficients distributed?

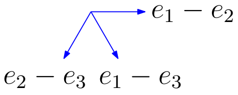


Back to Kostka Numbers

SNP says there are no zeros within the Newton polytope. How are the nonzero coefficients distributed?

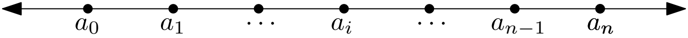
$\mathrm{Newton}(s_{(4,2,0)})$





Idea: look along lines in root directions!

Unimodal and Log-Concave Sequences



Unimodal: $a_0 \leq a_1 \leq \cdots \leq a_j$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some j

Log-concave: $a_i^2 \geq a_{i-1}a_{i+1}$ for all i

Positive and log-concave implies unimodal

Question

Do the Schur coefficients form unimodal sequences along lines in root directions? Even better, are they log-concave?

The SSYT Crystal

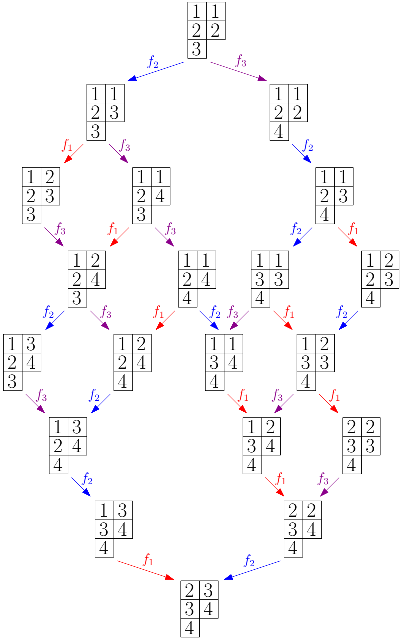
The crystal structure on SSYT provides some insight into unimodality of Kostka numbers.

Define the ith *crystal operator* f_i on $\mathrm{SSYT}(\lambda)$ to be the function that changes an i to an $i+1$ in a tableau T by the recipe:

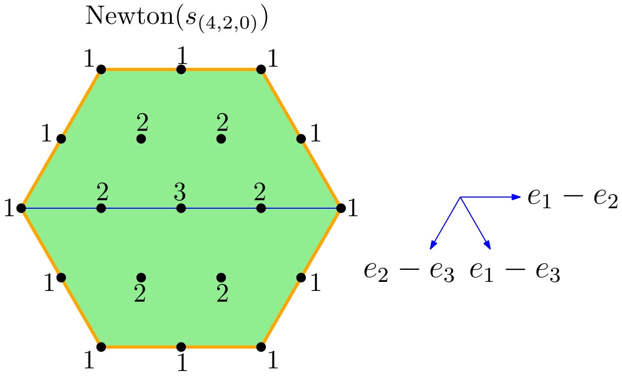
- map $i \mapsto)$ and $i+1 \mapsto ($
- read parentheses up columns
- iteratively remove matched pairs ()
- change the rightmost) to a (

$$f_2 : \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \longrightarrow ()()))(\to)((\longrightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

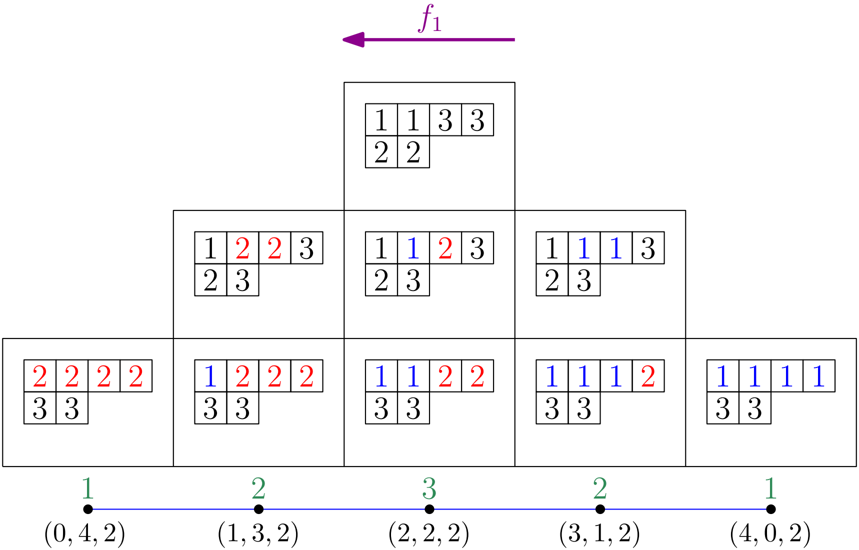
A Crystal Graph



Crystals: Some Unimodality



Crystals: Some Unimodality



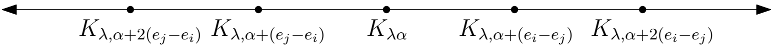
Log-Concavity of Kostka Numbers

Question

For any partition λ and $\alpha \in \mathbb{Z}^n_{\geq 0}$, is

$$K^2_{\lambda\alpha} \geq K_{\lambda,\alpha+e_i-e_j} K_{\lambda,\alpha-e_i+e_j}$$

for all $i, j \in [n]$?



But what about Littlewood–Richardson?

When the skew shape ν/μ has at most one box in each column,

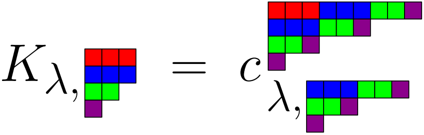
$$c^{\nu}_{\lambda\mu} = K_{\lambda,(\nu-\mu)}.$$

Conversely, for any partition λ and any α,

$$K_{\lambda\alpha} = c^{\nu}_{\lambda,\mu},$$

where ν and μ are the partitions given by

$$\nu_i = \sum_{j=i}^{n} \alpha_j \quad \text{and} \quad \mu_i = \sum_{j=i+1}^{n} \alpha_j.$$



Kostka and Littlewood–Richardson

$$\left(c^{\nu}_{\lambda,\mu}\right)^2 \geq c^{\nu+\varpi_i-\varpi_j}_{\lambda,\mu+\varpi_{i-1}-\varpi_{j-1}} c^{\nu-\varpi_i+\varpi_j}_{\lambda,\mu-\varpi_{i-1}+\varpi_{j-1}}$$

$$\Big\Updownarrow$$

$$K^2_{\lambda\alpha} \geq K_{\lambda,\alpha+e_i-e_j} K_{\lambda,\alpha-e_i+e_j}$$

Discrete log-concavity of the Kostka numbers is a special case of Okounkov's conjecture!

Outline

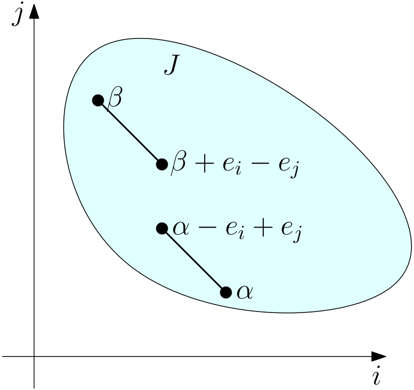
- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov's Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials

M-Convexity

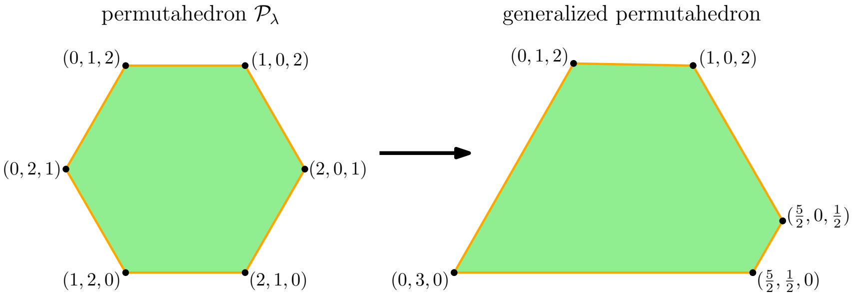
Definition

A subset $J \subseteq \mathbb{Z}^n$ is *M-convex* if for any index i and any $\alpha, \beta \in J$ with $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j, \quad \alpha - e_i + e_j \in J, \quad \text{and} \quad \beta + e_i - e_j \in J.$$



Generalized Permutahedra



Fact

The support of a polynomial f is M-convex if and only if f has SNP and $\mathrm{Newton}(f)$ is a *generalized permutahedon*.

Lorentzian Polynomials

Definition (Brändén–Huh 2019)

A homogeneous polynomial f of degree d with nonnegative coefficients is *Lorentzian* if

- f has M-convex support
- $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$ has at most one positive eigenvalue.

Lorentzian polynomials generalize volume polynomials of irreducible projective varieties from algebraic geometry and stable polynomials from optimization.

Example and Nonexample of Lorentzian-ness

Nonexample:

$f = x_1^2 + x_1x_2 + x_2^2$

Matrix: $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

Eigenvalues: $1/2$ and $3/2$

Example:

$g = \frac{x_1^2}{2} + x_1x_2 + \frac{x_2^2}{2}$

Matrix: $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

Eigenvalues: 0 and 1

Lorentzian Polynomials and Discrete Log-Concavity

Definition

Let N be the *normalization operator* defined by $N(x^\alpha) = \frac{x^\alpha}{\alpha!}$.

Theorem (Brändén–Huh 2019)

If $f = \sum_\alpha c_\alpha x^\alpha$ is nonzero and $N(f)$ is Lorentzian, then

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$$

for every α and $1 \leq i, j \leq n$.

Lorentzian Polynomials and Discrete Log-Concavity

Theorem (Brändén–Huh 2019)

If $f = \sum_\alpha c_\alpha x^\alpha$ is nonzero and $N(f)$ is Lorentzian, then

$$c_\alpha{}^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$$

for every α and $1 \leq i, j \leq n$.

Proof idea:

$$\frac{\partial^{\alpha-e_1-e_2}}{\partial x^{\alpha-e_1-e_2}} N(f)\Big|_{x_3=\cdots=x_n=0} = \frac{1}{2} c_{\alpha+e_1-e_2} x_1^2 + c_\alpha x_1 x_2 + \frac{1}{2} c_{\alpha-e_1+e_2} x_2^2$$

$$\det \begin{pmatrix} c_{\alpha+e_1-e_2} & c_\alpha \\ c_\alpha & c_{\alpha-e_1+e_2} \end{pmatrix} \leq 0$$

Lorentzian Polynomials and Continuous Log-Concavity

Theorem (Brändén–Huh 2019)

If $f = \sum_\alpha c_\alpha x^\alpha$ is nonzero and $N(f)$ is Lorentzian, then the function $\log(N(f))$ is concave on $\mathbb{R}^n_{>0}$.

The Lorentzian property is actually equivalent to being *strongly log-concave* or *completely log-concave*.

A Lorentzian Polynomial

Theorem (Huh–Matherne–Mészáros–S. 2019)

For any permutation $w \in S_n$,

$$N((x_1 \cdots x_n)^{n-1}\mathfrak{S}_w(x_1^{-1}, \ldots, x_n^{-1}))$$

is Lorentzian.

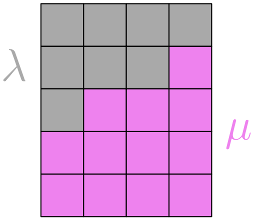
Proof Idea:

Knutson–Miller (2001): Schubert polynomials are multidegrees of matrix Schubert varieties

A Very Convenient Trick

By a symmetry argument:

$$(x_1 \cdots x_n)^{n-1} s_\lambda(x_1^{-1}, \dots, x_n^{-1}) \approx s_\mu$$



Schur Polynomial Conclusions

Theorem (Huh–Matherne–Mészáros–S. 2019)

For any partition λ, $N(s_\lambda)$ is Lorentzian.

Immediately implies discrete and continuous log-concavity:

Corollary (Huh–Matherne–Mészáros–S. 2019)

- $K_{\lambda\alpha}^2 \geq K_{\lambda,\alpha+e_i-e_j} K_{\lambda,\alpha-e_i+e_j}$
- $\log(N(s_\lambda))$ *is a concave function on* $\mathbb{R}^n_{>0}$

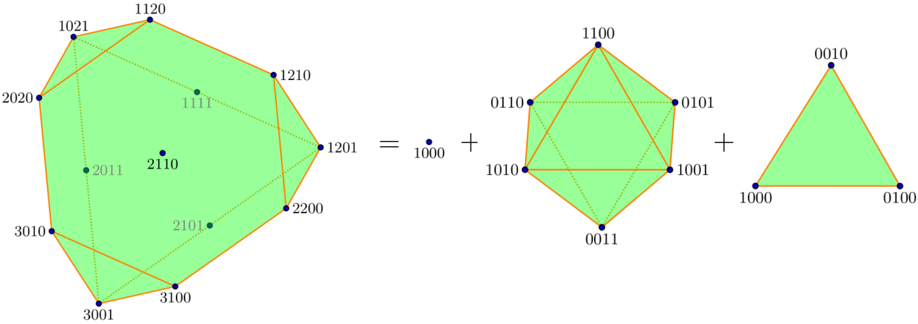
What about Schubert polynomials?

What about Schubert Polynomials?

Theorem (Fink-Mészáros-S. 2017)

$\mathfrak{S}_w$ has SNP and $\mathrm{Newton}(\mathfrak{S}_w)$ is a generalized permutahedron.

$$\mathrm{Newton}(\mathfrak{S}_{21543}) = P(U_{1,1}) + P(U_{2,4}) + P(U_{1,3})$$



Log-Concavity of Schubert Coefficients

Let $\mathfrak{S}_w = \sum_{\alpha} C_{w\alpha} x^{\alpha}$. Consider

$$N((x_1 \cdots x_n)^{n-1} \mathfrak{S}_w(x_1^{-1}, \ldots, x_n^{-1})).$$

Theorem (Huh–Matherne–Mészáros–S. 2019)

For any $w \in S_n$ and $i, j \in [n]$,

$$C_{w\alpha}^2 \geq C_{w,\alpha+e_i-e_j} C_{w,\alpha-e_i+e_j}.$$

Discrete log-concavity holds for Schubert polynomials.

(Combinatorial proof?)

What about Schubert Polynomials?

Even if $N(f)$ is Lorentzian,

$$N(x^\alpha f(x_1^{-1}, \ldots, x_n^{-1}))$$

may not be.

Conjecture (Huh–Matherne–Mészáros–S. 2019)

For any permutation w, $N(\mathfrak{S}_w)$ is Lorentzian.

Known Lorentzian Schubert Polynomials

Theorem (Huh–Matherne–Mészáros–S. 2019)

If w avoids 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, *and* 315642, *then $N(\mathfrak{S}_w)$ is Lorentzian.*

Theorem (Huh–Matherne–Mészáros–S. 2019)

If w avoids 1432 *and* 1423, *then $\mathfrak{S}_w$ and $N(\mathfrak{S}_w)$ are Lorentzian.*

Question

What other interesting families of polynomials have Lorentzian normalizations?

Conjecturally SNP

Conjecture (Monical-Tokcan-Yong 2017)

The following all have the saturated Newton polytope property:

- *Schubert polynomials*
- *Double Schubert polynomials*
- *Grothendieck polynomials*
- *Key polynomials*

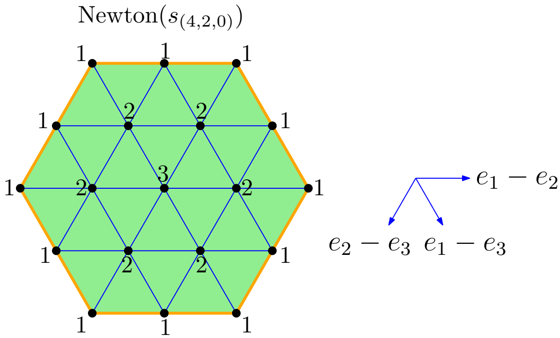
Conjecturally Lorentzian

Conjecture (Huh–Matherne–Mészáros–S. 2019)

With appropriate modifications, the following all have Lorentzian normalizations:

- *Double Schubert polynomials*
- *Grothendieck polynomials*
- *Key polynomials*
- *Skew-Schur polynomials*
- *Characters of (infinite dimensional) irreducible* GL_n *representations*

Thanks for Listening!



Other Log-Concavity

Another perspective is the log-concavity properties of the function $\lambda \mapsto s_\lambda$.

For symmetric functions f and g, say $f \geq_s g$ if $f - g$ expands positively in the Schur basis.

Theorem (Lam–Postnikov–Pylyavskyy 2005)

- $\left(s_{\frac{\lambda+\nu}{2}/\frac{\mu+\rho}{2}}\right)^2 \geq_s s_{\lambda/\mu} s_{\nu/\rho}$ *(Okounkov)*
- $s_{\mathrm{sort}_1(\lambda,\mu)} s_{\mathrm{sort}_2(\lambda,\mu)} \geq_s s_\lambda s_\mu$ *(Fomin–Fulton–Li–Poon)*
- $\prod_{i=1}^n s_{\lambda^{[i,n]}} \geq_s \prod_{i=1}^m s_{\lambda^{[i,m]}}$ *(Lascoux–Leclerc–Thibon)*
- $s_{(\lambda/\mu)\vee(\nu/\rho)} s_{(\lambda/\mu)\wedge(\nu/\rho)} \geq_s s_{\lambda/\mu} s_{\nu/\rho}$ *(Lam–Pylyavskyy)*