Log-Concavity of Littlewood–Richardson Coefficients

Avery St. Dizier

University of Illinois at Urbana–Champaign

Based on joint works with Alex Fink, June Huh, Jacob Matherne, and Karola Mészáros

ICERM Semester in Combinatorial Algebraic Geometry

Schubert Seminar Series
Outline

1. Schubert Polynomials
2. Schur Polynomials and Littlewood–Richardson Coefficients
3. Log-Concavity and Okounkov’s Conjecture
4. Newton Polytopes and Saturation
5. Lorentzian Polynomials
A complete flag $F = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a sequence of vector subspaces

$$F_1 \subset F_2 \subset \cdots \subset F_n$$

where $\text{dim } F_i = i$.

The flag manifold $\text{Fl}(n)$ is the set of complete flags in $\mathbb{C}^n$. 
\( \Fl(n) \) decomposes into a disjoint union of open cells \( X_w \) indexed by permutations \( w \) in \( S_n \).

Their closures are the \textit{Schubert varieties} \( \overline{X}_w \).

\[ \{ [\overline{X}_w] \}_{w \in S_n} \] is a linear basis of the cohomology ring \( H^*(\Fl(n)) \).
**Theorem (Borel 1953)**

\[ H^*(Fl(n)) \cong \mathbb{Z}[x_1, \ldots, x_n]/I_n, \]

where \( I_n = \langle \text{symmetric functions with zero constant term} \rangle \).

What is a good choice of polynomial representatives for the basis \( \{[X_w]\} \)?
Schubert polynomials were defined by Lascoux and Schützenberger (1982) divided difference operators $\partial_i$:

$$\partial_i f = \frac{f - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

**Definition**

The *Schubert polynomial* $\mathcal{S}_w$ of $w \in S_n$ is defined by

$$\mathcal{S}_w = \begin{cases} 
  x_1^{n-1}x_2^{n-2} \cdots x_{n-1} & \text{if } w = n \,(n-1) \cdots 2 \, 1 \\
  \partial_i \mathcal{S}_{ws_i} & \text{if } w(i) < w(i+1)
\end{cases}$$
Divided Difference Operators

\[ \partial_1(x_1^2 x_2) = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2 \]

\[ \partial_2(x_1 x_2) = \frac{x_1 x_2 - x_1 x_3}{x_2 - x_3} = x_1 \]

\[ \partial_1(x_1) = \frac{x_1 - x_2}{x_1 - x_2} = 1 \]
Theorem (Lascoux-Schützenberger 1982)

The coefficients of $\mathcal{S}_w$ are nonnegative integers.

Proof.

The Transition Rule:

$$\mathcal{S}_w = x_i \mathcal{S}_v + \sum_{k < i} \mathcal{S}_{v(ki)}$$

where $v(ki)$ covers $v$.

Is there a combinatorial interpretation behind the nonnegativity?

First answered by Billey–Jockusch–Stanley (1993), and re-answered in many interesting ways since.
A Sample of the Nonnegativity of $S_{1432}$

$$S_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3$$

<table>
<thead>
<tr>
<th>Compatible Sequences</th>
<th>Reduced Pipe Dreams</th>
<th>Kohnert Diagrams</th>
<th>Bumpless Pipe Dreams</th>
<th>Generalized Young Tableaux</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_3 s_2 s_3$ (2, 2, 3)</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
<td>Billey–Jockusch–Stanley</td>
</tr>
<tr>
<td>$s_3 s_2 s_3$ (1, 2, 3)</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td>Billey–Bergeron, Fomin–Kirilov</td>
</tr>
<tr>
<td>$s_3 s_2 s_3$ (1, 1, 2)</td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
<td>Kohnert, Winkel</td>
</tr>
<tr>
<td>$s_2 s_3 s_2$ (1, 2, 2)</td>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
<td><img src="image15" alt="Diagram" /></td>
<td><img src="image16" alt="Diagram" /></td>
<td>Lam–Lee–Shimozono</td>
</tr>
<tr>
<td>$s_3 s_2 s_3$ (1, 1, 3)</td>
<td><img src="image17" alt="Diagram" /></td>
<td><img src="image18" alt="Diagram" /></td>
<td><img src="image19" alt="Diagram" /></td>
<td><img src="image20" alt="Diagram" /></td>
<td>Magyar</td>
</tr>
</tbody>
</table>
Outline

- Schubert Polynomials

- Schur Polynomials and Littlewood–Richardson Coefficients

- Log-Concavity and Okounkov’s Conjecture

- Newton Polytopes and Saturation

- Lorentzian Polynomials
Semistandard Young Tableaux

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, a semistandard Young tableaux (SSYT) of shape $\lambda$ is a filling of the Young diagram of $\lambda$ with numbers from $[n]$ such that:

- The entries weakly increase left-to-right in each row,
- The entries strictly increase top-to-bottom in each column.

SSYT(2, 1, 0):

$$
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 1 & 2 & 1 & 3 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
\end{array}
$$
Schur Polynomials

**Definition**

Given a partition \( \lambda \in \mathbb{Z}^n_{\geq 0} \), the *Schur polynomial* \( s_\lambda \) is defined by

\[
s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x_1^{\text{wt}(T)_1} \cdots x_n^{\text{wt}(T)_n}
\]

\[\text{SSYT}(2, 1, 0) : \]

\[
\begin{align*}
1 & \quad 1 \\
2 & \quad 2 \\
1 & \quad 1 \\
2 & \quad 2 \\
1 & \quad 3 \\
3 & \quad 3 \\
2 & \quad 2 \\
2 & \quad 3 \\
3 & \quad 3
\end{align*}
\]

\[
s_{210}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3 + x_2 x_3^2
\]
The coefficient of $x^\alpha$ in $s_\lambda$ is the \textit{Kostka number}

$$K_{\lambda\alpha} = \# \text{ of SSYT with shape } \lambda \text{ and weight } \alpha$$

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\
2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

$$K_{(2,1,0),(1,1,1)} = 2$$

$$s_{210}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$
Schur polynomials are symmetric polynomials, and form a basis for the space of symmetric polynomials.

**Representation Theory:**
The characters of the irreducible representations \( V_\lambda \) of \( \text{GL}_n \) are exactly Schur polynomials.

**Geometry:**
Schur polynomials form a linear basis for the cohomology ring of the Grassmannian \( \text{Gr}_{k,n}(\mathbb{C}) \).
The *Littlewood–Richardson coefficients* are $c_{\lambda \mu}^{\nu}$ defined by

$$s_{\lambda} s_{\mu} = \sum_{\nu} c_{\lambda \mu}^{\nu} s_{\nu}.$$ 

**Representation Theory:**

$c_{\lambda \mu}^{\nu}$ controls tensor products of $GL_n$ irreducible representations via

$$V_{\lambda} \otimes V_{\mu} \cong \bigoplus_{\ell(\nu) \leq n} V_{\nu}^{c_{\lambda \mu}^{\nu}}$$

**Geometry:**

For $\sigma_{\lambda}$ the cohomology class of the Schubert variety $X_{\lambda}$ in the Grassmannian $Gr_{k,n}(\mathbb{C})$,

$$\sigma_{\lambda} \sigma_{\mu} = \sum_{\nu \subseteq k \times (n-k)} c_{\lambda \mu}^{\nu} \sigma_{\nu}$$
Outline

- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov’s Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials
Continuous:
A function $f : \mathbb{R}^n \to \mathbb{R}_{>0}$ is log-concave if

$$\log f(\theta x + (1 - \theta y)) \geq \theta \log f(x) + (1 - \theta) \log f(y)$$

$$f(\theta x + (1 - \theta y)) \geq f(x)^\theta f(y)^{1-\theta}$$

Discrete:
A function $a : \mathbb{Z} \to \mathbb{R}_{>0}$ is log-concave if

$$a_i^2 \geq a_{i+1} a_{i-1}$$
Conjecture (Okounkov 2003)

The discrete function

\[(\lambda, \mu, \nu) \mapsto \log c_{\lambda\mu}^{\nu}\]

is a concave function of \(\lambda, \mu, \nu\).
Why would multiplicities be log-concave?

Andrei Okounkov

Abstract

It is a basic property of the entropy in statistical physics that is concave as a function of energy. The analog of this in representation theory would be the concavity of the logarithm of the multiplicity of an irreducible representation as a function of its highest weight. We discuss various situations where such concavity can be established or reasonably conjectured and consider some implications of this concavity. These are rather informal notes based on a number of talks I gave on the subject, in particular, at the 1997 International Press lectures at UC Irvine.
Okounkov’s Conjecture

In the cone of partition tuples \((\lambda, \mu, \nu)\), consider

\[c_{\lambda \mu}^{\nu} \quad c_{2\lambda,2\mu}^{2\nu} \quad c_{3\lambda,3\mu}^{3\nu} \quad \cdots \quad c_{N\lambda,N\mu}^{N\nu}\]

Log-concavity would imply

\[
\left( c_{N\lambda,N\mu}^{N\nu} \right)^2 \geq c_{(N+1)\lambda,(N+1)\mu}^{(N+1)\nu} c_{(N-1)\lambda,(N-1)\mu}^{(N-1)\nu}.
\]

Counterexample found by Chindris–Derksen–Weyman (2007):

\(\lambda = \mu = (3^{21}, 2^{21}, 1^{21}), \quad \nu = (4^{21}, 3^{42}, 2^{21})\)
Okounkov’s Conjecture is false.

Some consequences of it are true.

Okounkov proved log-concavity holds asymptotically.

Are there interesting special sets of tuples \((\lambda, \mu, \nu)\) for which a version of Okounkov’s conjecture holds?
Theorem (Huh–Matherne–Mészáros–S. 2019)

Choose any \(i, j \geq 0\) and any partitions \(\lambda, \mu, \nu\) such that \(\nu/\mu\) is a skew shape with at most one box per column. Whenever all subscripts and superscripts are partitions,

\[
\left( c_{\nu}^{\lambda, \mu} \right)^2 \geq c_{\nu+\varpi_i-\varpi_j}^{\lambda, \mu+\varpi_i-1-\varpi_j-1} c_{\nu-\varpi_i+\varpi_j}^{\lambda, \mu-\varpi_i-1+\varpi_j-1}
\]

where \(\varpi_k\) is the \(k\)th fundamental weight \((1^k, 0^{n-k})\).

For example, if \(n = 5\), \(i = 2\), and \(j = 4\), then

\[
11^2 = \left( c_{52100,76410}^{87641} \right)^2 \geq c_{52100,75310}^{87531} c_{52100,77510}^{87751} = 14 \times 4
\]
Outline

- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov’s Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials
Any polynomial \( f = \sum_{z \in \mathbb{Z}^n} a_z x^z \in \mathbb{C}[x_1, \ldots, x_n] \) has an associated integer polytope called its \textit{Newton polytope}: 

\[
\text{Newton}(f) = \text{Conv}(z : a_z \neq 0)
\]

\[
\text{Newton}(1 + x + y + x^2 + xy^2 + x^2y^2) = \{(0, 1), (1, 1), (2, 1), (2, 0), (1, 0)\}
\]

\[
\text{Newton}(1 + y + xy + xy^2 + x^2 + x^2y^2) = \{(0, 0), (1, 0), (2, 0), (1, 1), (2, 2)\}
\]
What kind of polytopes are the Newton polytopes of Schur polynomials?

\[ s_{(1,0,0)} = x_1 + x_2 + x_3 \]

\[ s_{(2,1,0)} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3 \]
The Newton polytope of $s_\lambda$ is the permutahedron $\mathcal{P}_\lambda$ (the convex hull of all permutations of $\lambda$).
How do the Kostka numbers within the Schur Newton polytope look?

Question

Can there be zeros in the polytope?
Definition (Monical-Tokcan-Yong 2017)
A polynomial $f$ is said to have *saturated Newton polytope* (SNP) if every integer point in the Newton polytope corresponds to a monomial with non-zero coefficient in $f$.

Theorem (Monical-Tokcan-Yong 2017)
The following all have SNP:

- *Schur polynomials*
- *Skew-Schur polynomials*
- *Stanley symmetric functions*
- *(q, t) evaluations of symmetric Macdonald polynomials*
SNP says there are no zeros within the Newton polytope. How are the nonzero coefficients distributed?

\[ \text{Newton}(s_{(4,2,0)}) \]

\[
\begin{align*}
&\ e_1 - e_2 \\
&\ e_1 - e_3 \\
&\ 1 \\
&\ 1 \\
&\ 1 \\
&\ 1 \\
&\ 1 \\
&\ 2 \\
&\ 2 \\
&\ 2 \\
&\ 2 \\
&\ 1 \\
&\ 1 \\
\end{align*}
\]
SNP says there are no zeros within the Newton polytope. How are the nonzero coefficients distributed?

Idea: look along lines in root directions!
Unimodal: $a_0 \leq a_1 \leq \cdots \leq a_j$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some $j$

Log-concave: $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i$

Positive and log-concave implies unimodal

Question

Do the Schur coefficients form unimodal sequences along lines in root directions? Even better, are they log-concave?
The crystal structure on SSYT provides some insight into unimodality of Kostka numbers.

Define the $i$th crystal operator $f_i$ on $\text{SSYT}(\lambda)$ to be the function that changes an $i$ to an $i + 1$ in a tableau $T$ by the recipe:

- map $i \rightarrow \)$ and $i + 1 \rightarrow \)(
- read parentheses up columns
- iteratively remove matched pairs ()
- change the rightmost $)$ to a $\($

$$f_2 : \begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 4 \\
3 & 4 & 5 \\
4 &
\end{array} \rightarrow \begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 4 & 3 \\
3 & 4 & 5 & 3 \\
4 &
\end{array}$$
A Crystal Graph
Crystals: Some Unimodality

Newton\left(s_{(4,2,0)}\right)

\begin{align*}
&
\end{align*}
Crystals: Some Unimodality

\[ f_1 \]

\[
\begin{array}{cccc}
1 & 1 & 3 & 3 \\
2 & 2 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 2 & 3 \\
2 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
3 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
3 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
3 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
3 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
3 & 3 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
(0, 4, 2) & (1, 3, 2) & (2, 2, 2) & (3, 1, 2) & (4, 0, 2)
\end{array}
\]
Question

For any partition $\lambda$ and $\alpha \in \mathbb{Z}_{\geq 0}^n$, is

$$K_{\lambda \alpha}^2 \geq K_{\lambda, \alpha + e_i - e_j} K_{\lambda, \alpha - e_i + e_j}$$

for all $i, j \in [n]$?
But what about Littlewood–Richardson?

When the skew shape \( \nu/\mu \) has at most one box in each column,

\[
\begin{align*}
c^\nu_{\lambda\mu} &= K_{\lambda,(\nu-\mu)}. \\
\text{Conversely, for any partition } \lambda \text{ and any } \alpha,
K_{\lambda\alpha} &= c^\nu_{\lambda,\mu},
\end{align*}
\]

where \( \nu \) and \( \mu \) are the partitions given by

\[
\begin{align*}
\nu_i &= \sum_{j=i}^{n} \alpha_j \quad \text{and} \quad 
\mu_i &= \sum_{j=i+1}^{n} \alpha_j.
\end{align*}
\]
Discrete log-concavity of the Kostka numbers is a special case of Okounkov’s conjecture!
Outline

- Schubert Polynomials
- Schur Polynomials and Littlewood–Richardson Coefficients
- Log-Concavity and Okounkov’s Conjecture
- Newton Polytopes and Saturation
- Lorentzian Polynomials
A subset $J \subseteq \mathbb{Z}^n$ is **M-convex** if for any index $i$ and any $\alpha, \beta \in J$ with $\alpha_i > \beta_i$, there is an index $j$ satisfying

$$\alpha_j < \beta_j, \quad \alpha - e_i + e_j \in J, \quad \text{and} \quad \beta + e_i - e_j \in J.$$
The support of a polynomial $f$ is M-convex if and only if $f$ has SNP and $\text{Newton}(f)$ is a generalized permutahedron.
Lorentzian Polynomials

Definition (Brändén–Huh 2019)

A homogeneous polynomial $f$ of degree $d$ with nonnegative coefficients is **Lorentzian** if

- $f$ has M-convex support
- $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$ has at most one positive eigenvalue.

Lorentzian polynomials generalize volume polynomials of irreducible projective varieties from algebraic geometry and stable polynomials from optimization.
Example and Nonexample of Lorentzian-ness

Nonexample:
\[ f = x_1^2 + x_1 x_2 + x_2^2 \]
Matrix: \[
\begin{pmatrix}
1 & 1/2 \\
1/2 & 1
\end{pmatrix}
\]
Eigenvalues: 1/2 and 3/2

Example:
\[ g = \frac{x_1^2}{2} + x_1 x_2 + \frac{x_2^2}{2} \]
Matrix: \[
\begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2
\end{pmatrix}
\]
Eigenvalues: 0 and 1
Definition
Let $N$ be the \textit{normalization operator} defined by $N(x^\alpha) = \frac{x^\alpha}{\alpha!}$.

Theorem (Brändén–Huh 2019)
If $f = \sum_\alpha c_\alpha x^\alpha$ is nonzero and $N(f)$ is Lorentzian, then

$$c_\alpha^2 \geq c_\alpha + e_i - e_j c_\alpha - e_i + e_j$$

for every $\alpha$ and $1 \leq i, j \leq n$. 
Theorem (Brändén–Huh 2019)

If $f = \sum \alpha c_{\alpha} x^\alpha$ is nonzero and $N(f)$ is Lorentzian, then

$$c_{\alpha}^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$$

for every $\alpha$ and $1 \leq i, j \leq n$.

Proof idea:

$$\frac{\partial^{\alpha-e_1-e_2}}{\partial x^{\alpha-e_1-e_2}} N(f) \bigg|_{x_3=\cdots=x_n=0} = \frac{1}{2} c_{\alpha+e_1-e_2} x_1^2 + c_{\alpha} x_1 x_2 + \frac{1}{2} c_{\alpha-e_1+e_2} x_2^2$$

$$\det \begin{pmatrix} c_{\alpha+e_1-e_2} & c_{\alpha} \\ c_{\alpha} & c_{\alpha-e_1+e_2} \end{pmatrix} \leq 0$$
Theorem (Brändén–Huh 2019)

If \( f = \sum_{\alpha} c_{\alpha} x^{\alpha} \) is nonzero and \( N(f) \) is Lorentzian, then the function \( \log(N(f)) \) is concave on \( \mathbb{R}^n_{>0} \).

The Lorentzian property is actually equivalent to being strongly log-concave or completely log-concave.
Theorem (Huh–Matherne–Mészáros–S. 2019)

For any permutation \( w \in S_n, \)

\[
N((x_1 \cdots x_n)^{n-1} S_w(x_1^{-1}, \ldots, x_n^{-1}))
\]

is Lorentzian.

Proof Idea:

Knutson–Miller (2001): Schubert polynomials are multidegrees of matrix Schubert varieties
By a symmetry argument:

$$(x_1 \cdots x_n)^{n-1} s_\lambda(x_1^{-1}, \ldots, x_n^{-1}) \approx s_\mu$$
Schur Polynomial Conclusions

**Theorem (Huh–Matherne–Mészáros–S. 2019)**

For any partition $\lambda$, $N(s_\lambda)$ is Lorentzian.

Immediately implies discrete and continuous log-concavity:

**Corollary (Huh–Matherne–Mészáros–S. 2019)**

- $K_{\lambda\alpha}^2 \geq K_{\lambda,\alpha+e_i-e_j} K_{\lambda,\alpha-e_i+e_j}$
- $\log(N(s_\lambda))$ is a concave function on $\mathbb{R}_0^n$

What about Schubert polynomials?
What about Schubert Polynomials?

Theorem (Fink-Mészáros-S. 2017)

$\mathcal{G}_w$ has SNP and $\text{Newton}(\mathcal{G}_w)$ is a generalized permutahedron.

$$\text{Newton}(\mathcal{G}_{21543}) = P(U_{1,1}) + P(U_{2,4}) + P(U_{1,3})$$
Let $\mathcal{G}_w = \sum_{\alpha} C_{w,\alpha} x^{\alpha}$. Consider

$$N((x_1 \cdots x_n)^{n-1} \mathcal{G}_w(x_1^{-1}, \ldots, x_n^{-1})).$$

**Theorem (Huh–Matherne–Mészáros–S. 2019)**

*For any* $w \in S_n$ *and* $i, j \in [n],$

$$C_{w,\alpha}^2 \geq C_{w,\alpha+e_i-e_j} C_{w,\alpha-e_i+e_j}.$$

Discrete log-concavity holds for Schubert polynomials.

(Combinatorial proof?)
What about Schubert Polynomials?

Even if $N(f)$ is Lorentzian,

$$N(x^\alpha f(x_1^{-1}, \ldots, x_n^{-1}))$$

may not be.

**Conjecture (Huh–Matherne–Mészáros–S. 2019)**

*For any permutation $w$, $N(\mathcal{G}_w)$ is Lorentzian.*
Theorem (Huh–Matherne–Mészáros–S. 2019)

If \( w \) avoids 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642, then \( N(\mathcal{G}_w) \) is Lorentzian.

Theorem (Huh–Matherne–Mészáros–S. 2019)

If \( w \) avoids 1432 and 1423, then \( \mathcal{G}_w \) and \( N(\mathcal{G}_w) \) are Lorentzian.

Question

What other interesting families of polynomials have Lorentzian normalizations?
Conjecturally SNP

Conjecture (Monical-Tokcan-Yong 2017)

The following all have the saturated Newton polytope property:

- Schubert polynomials
- Double Schubert polynomials
- Grothendieck polynomials
- Key polynomials
Conjecturally Lorentzian

**Conjecture (Huh–Matherne–Mészáros–S. 2019)**

*With appropriate modifications, the following all have Lorentzian normalizations:*

- Double Schubert polynomials
- Grothendieck polynomials
- Key polynomials
- Skew-Schur polynomials
- Characters of (infinite dimensional) irreducible $GL_n$ representations
Thanks for Listening!

Newton($s_{(4,2,0)}$)
Another perspective is the log-concavity properties of the function $\lambda \mapsto s_\lambda$.

For symmetric functions $f$ and $g$, say $f \geq_s g$ if $f - g$ expands positively in the Schur basis.

**Theorem (Lam–Postnikov–Pylyavskyy 2005)**

- $(s_{\frac{\lambda+\nu}{2}})^2 \succeq_s s_{\lambda/\mu}s_{\nu/\rho}$ (Okounkov)
- $s_{\text{sort}_1(\lambda,\mu)}s_{\text{sort}_2(\lambda,\mu)} \succeq_s s_\lambda s_\mu$ (Fomin–Fulton–Li–Poon)
- $\prod_{i=1}^n s_{\lambda[i,n]} \succeq_s \prod_{i=1}^m s_{\lambda[i,m]}$ (Lascoux–Leclerc–Thibon)
- $s_{\lambda/\mu} \vee (\nu/\rho)s_{\lambda/\mu} \wedge (\nu/\rho) \succeq_s s_{\lambda/\mu}s_{\nu/\rho}$ (Lam–Pylyavskyy)