The Abelian/non-Abelian correspondence and mirror symmetry

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Motto

The Abelian/non-Abelian correspondence is a powerful tool both for:

- studying (type A) flag varieties;
- extending results for flag varieties to other more general varieties.

We'll see this through studying quantum cohomology and mirror symmetry constructions.

Outline

1. The Abelian/non-Abelian correspondence
2. Quiver flag varieties
3. Results

Main results

- A rim-hook removal rule for the (quantum) cohomology of type A flag varieties and quiver flag varieties.
- Plücker coordinate mirrors for type A flag varieties and beyond

GIT quotients: an example

Let $V = \mathbb{C}^n$. The torus $G = \mathbb{C}^*$ acts on V by scaling.

The naive quotient V/G is not very nice - it isn't even Hausdorff.

If instead of quotienting all of V, we delete the origin, we obtain

$$(\mathbb{C}^n - \{0\})/\mathbb{C}^* = \mathbb{P}^{n-1}.$$

Geometric invariant theory (GIT) generalizes this construction.

GIT quotients

Let V be a $\mathbb{C}$-vector space, and G a reductive linear algebraic group acting on V.

Given an extra piece of data,

$$\theta \in \chi(G) = \mathrm{Hom}(G, \mathbb{C}^*)$$

GIT tells you how to delete the bad locus in V. What remains is the semi-stable locus V^{ss}. This character is called the stability condition. The *GIT quotient* is

$$V//_\theta G = V^{ss}/G.$$

Assumptions

I'll make some assumptions on all GIT quotients here:

1. $V^{ss} = V^{s}$,
2. Stablizers of semi-stable points are trivial.
3. The unstable locus has codimension at least 2.

Abelian GIT quotients

If G is an Abelian group, i.e. G is a torus $T = (\mathbb{C}^*)^k$, then $V /\!/_\theta T$ is a toric variety. It behaves a lot like projective space:

- Homogeneous coordinates (Cox ring).
- Explicit description of cohomology and relations.
- Explicit description of Picard lattice, effective and ample cones.

When $V /\!/ T$ is Fano, we can say a lot about mirror symmetry (quantum cohomology, I-functions, J-functions, Laurent polynomial mirrors...).

Other GIT quotients

Type A flag varieties and Grassmannians are GIT quotients.

Example

Let $V = \mathrm{Mat}(r \times n)$, $G = \mathrm{GL}(r)$ acting by multiplication. Let

$$\theta = \det : G \to \mathbb{C}^*.$$

Then V^{ss} is the set of full rank matrices, so $V//_\theta G = Gr(n, r)$, the Grassmannian of r-dimensional quotients of $\mathbb{C}^n$.

Remark
Type A flag varieties can also be constructed as subvarieties of products of Grassmannians and quotients G/P.

Vector bundles on GIT quotients

Vector bundles on GIT quotients can be constructed from representations of G. Let E be a representation of G. Then consider

$$V^{ss} \times E \to V^{ss}$$

with G action. After quotienting, we get a representation theoretic vector bundle on $V/\!/G$:

$$(V^{ss} \times E)/G \to V/\!/G.$$

Given a character $\alpha \in \chi(G)$, this produces a line bundle

$$L_\chi \in \mathrm{Pic}(V/\!/G).$$

In fact,

$$\chi(G) \cong \mathrm{Pic}(V/\!/G).$$

The Abelian/non-Abelian correspondence

The Abelian/non-Abelian correspondence says that given a GIT quotient $V/\!/_\theta G$, one should instead consider the GIT quotient obtained by replacing G with a maximal torus: $V/\!/_\theta T$.
These two quotients are related by the following diagram:

$$\begin{tikzcd}
V^{ss}(G)/T \arrow[r, hook] \arrow[d, two heads] & V^{ss}(T)/T \\
V^{ss}(G)/G &
\end{tikzcd}$$

Example

If $V/\!/G$ is the Grassmannian $\mathrm{Gr}(n, r)$, then the Abelianization is $(\mathbb{P}^{n-1})^{\times r}$.

First application: cohomology of GIT quotients

Let A be the set of roots of G and A^+ be a choice of positive roots. Let W be the Weyl group and set

$$\omega := \prod_{\alpha \in A^+} c_1(L_\alpha) \in H^*(V/\!/T),$$

$$Ann(\omega) := (a \in H^*(V/\!/T) | a \cdot \omega = 0).$$

Theorem [Martin, Ellingsrud–Strømme]

There is an isomorphism

$$H^*(V/\!/T)^W / Ann(\omega) \xrightarrow{\sim} H^*(V/\!/G).$$

Cohomology of the Grassmannian

In this case, the Abelianization is $(\mathbb{P}^{n-1})^{\times r}$, so the cohomology is

$$H^*(V/\!/T) = \mathbb{C}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n).$$

The Weyl group Sym_r permutes the x_i and

$$\omega = \prod_{i<j}(x_i - x_j).$$

So the cohomology of the Grassmannian is

$$\mathbb{C}[x_1, \ldots, x_r]^W / Ann(\omega).$$

Schubert calculus from the A/NA correspondence

The annhilator of ω is

$$(f | f\omega \in (x_1^n, \ldots, x_r^n)).$$

Let $s_\lambda = s_\lambda(x_1, \ldots, x_r)$ denote the Schur polynomial associated to a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$.

$$s_\lambda \prod_{i<j}(x_i - x_j) = \det \begin{bmatrix} x_1^{r-1+\lambda_1} & \cdots & x_r^{r-1+\lambda_1} \\ \vdots & & \vdots \\ x_1^{1+\lambda_r} & \cdots & x_r^{1+\lambda_r} \\ x_1^{\lambda_r} & \cdots & x_r^{\lambda_r} \end{bmatrix}$$

Therefore,

$$s_\lambda \in Ann(\omega) \leftrightarrow \lambda_1 > n - r.$$

Schubert calculus from the A/NA correspondence

The Schur polynomial $s_\lambda(x_1, \ldots, x_r)$ represents the class of a Schubert variety. One multiplies two Schubert classes as Schur polynomials

$$[s_\lambda][s_\mu] = [s_\lambda s_\mu] = \sum_\kappa [c^\kappa_{\lambda\mu} s_\kappa]$$

then sets any partition which does not fit into an $r \times (n-r)$ box to zero.

Example

In $\mathrm{Gr}(4,2)$,

$$[s_{\ydiagram{1}}] \cdot [s_{\ydiagram{2,2}}] = [s_{\ydiagram{2,2,1}}] + [s_{\ydiagram{3,2}}] = 0.$$

Summary

The Abelian/non-Abelian correspondence relates non-Abelian GIT quotients (like Grassmannians) to Abelian GIT quotients

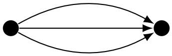
$$V /\!/ G \longrightarrow V /\!/ T.$$

By Martin's theorem, we can describe the cohomology of $V /\!/ G$. For example, we recover Schubert calculus for the Grassmannian. This isn't anything new, but what if we extend this

- to other GIT quotients?
- to quantum cohomology?

Quivers and quiver flag varieties

A quiver is a directed graph (made up of vertices and arrows).



If you assign a vector space to each vertex, you can study *representations of the quiver*. A representation of a quiver is a linear map for each arrow. For example, if we assign vector spaces $\mathbb{C}$ and $\mathbb{C}^2$ to the first and second verticies respectively, then we consider three linear maps:

$$\mathbb{C} \to \mathbb{C}^2, \mathbb{C} \to \mathbb{C}^2, \mathbb{C} \to \mathbb{C}^2 \text{ or equivalently, } \mathbb{C}^3 \to \mathbb{C}^2.$$

You might want to study all such maps, up to change of basis.

Quivers and quiver flag varieties

We want to study representations of the quiver

$$\begin{tikzcd} \underset{\mathbb{C}}{\bullet} \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \underset{\mathbb{C}^2}{\bullet} \end{tikzcd}$$

In other words, we want to study maps $\mathbb{C}^3 \to \mathbb{C}^2$ up to change of basis – this is a GIT quotient.

One natural choice of stability condition gives:

"The map $\mathbb{C}^3 \to \mathbb{C}^2$ must be surjective."

This GIT quotient is called the *quiver moduli space*.

$$\mathrm{Gr}(3,2) = \{\text{two dimensional quotients of } \mathbb{C}^3\}.$$

Quiver flag varieties

Quiver flag varieties are quiver moduli spaces for a specified stability condition, that arise from quivers which are acyclic with a unique source.

$$\mathbb{C} \xrightarrow{4} \mathbb{C}^2 \qquad \mathrm{Gr}(4,2)$$

$$\mathbb{C} \xrightarrow{4} \mathbb{C}^2 \longrightarrow \mathbb{C} \qquad \mathrm{Fl}(4,2,1) = \mathrm{Gr}(E,1)$$

$$\begin{tikzcd} & & \mathbb{C} \\ \mathbb{C} \arrow[r, "4"'] & \mathbb{C}^2 \arrow[ur] \arrow[dr] & \\ & & \mathbb{C} \end{tikzcd} \qquad \mathrm{Gr}(E,1) \times Gr(E,1)$$

Abelianization of quiver flag varieties

Properties of quiver flag varieties

Quiver flag varieties vastly generalize type A flag varieties, but share many of their important properties (Craw):

- They are smooth and projective.
- They are towers of Grassmannian bundles.
- They are fine moduli spaces and Mori dream spaces.

Another motivation for studying quiver flag varieties arises from the Fano classification program and mirror symmetry.

Fano varieties

Fano varieties are a basic building block in algebraic geometry in two senses:

- They are one of the 'atomic pieces' of algebraic varieties by the minimal model program.
- Many explicit constructions in algebraic geometry start with a Fano variety.

Example

Grassmannians and flag varieties are Fano.

Fano varieties are projective varieties whose anti-canonical line bundle is *ample*.

Classifying Fano varieties

Trying to classify Fano varieties makes sense because, up to deformation, there are only finitely many Fano varieties in each dimension (Kollár–Miyaoka–Mori).

Fano varieties are only classified up to dimension 3. In dimension 3, the classification is due to Iskovskih and Mori–Mukai.

Dimension	**Number of Fano varieties**
1	1
2	10
3	105
4	??

Low dimensional Fano varieties

As a consequence of the Iskovskih–Mori–Mukai classification, all Fano varieties of dimension three or less can be constructed as certain subvarieties* of either

- *toric varieties,*
- *quiver flag varieties.*

To understand the Fano fourfold landscape, studying subvarieties of toric varieties and quiver flag varieties is a good place to start.

Exhaustive computer searches by Coates–Kasprzyk–Prince (toric) and Coates–Kalashnikov–Kasprzyk (quiver flag varieties) give almost all known examples of smooth Fano fourfolds.

Classifying four dimensional Fano varieties

There is a program to use *mirror symmetry* to classify Fano varieties (Coates–Corti–Galkin–Golyshev–Kasprzyk–Tveiten and others).

Mirror symmetry for Fano varieties

Mirror symmetry for Fano varieties is a conjectural correspondence

$$\begin{array}{ccc} \{n\text{-dimensional Fano varieties}\} & & \mathbb{P}^2 \\ \updownarrow & \textit{up to deformation} & \updownarrow \\ \{\text{special Laurent polynomials in } n \text{ variables}\} & & x + y + \frac{1}{xy} \\ & \textit{up to mutation} & \end{array}$$

Mirror symmetry and classification

The idea is to classify Fano varieties by classifying their mirror partners instead. This approach should work in any dimension.

This conjecture is modeled on the theorems that are known for Fano toric varieties and Fano toric completely intersections.

The mirror of $\mathbb{P}^2$

Let $X = \mathbb{P}^2$. Let $f := x_1 + x_2 + \frac{q}{x_1 x_2}$.

- The Jacobi ring of f
$$\mathbb{C}[x]/(x^3 - q)$$
is isomorphic to the quantum cohomology ring of $\mathbb{P}^2$.
- On the critical locus, $f \sim 3x = -K_X$.
- The spanning fan of the Newton polytope of f is the fan of $\mathbb{P}^2$.
- The classical period of f (setting $q = 1$) is
$$\sum_{k=0}^{\infty} \frac{(3k)!}{(k!)^3} t^{3k}.$$
This is the regularized quantum period of $\mathbb{P}^2$.

Mirror symmetry for Fano varieties

The quantum period is a power series built out of genus 0 Gromov–Witten invariants; in general it is very difficult to compute.

Let X be a smooth n-dimensional Fano variety. A Laurent polynomial f is mirror to X if the classical period of f equals the regularized quantum period of X.

The mirror theorem for Fano toric varieties and complete intersections is due to work of Hori–Vafa, Batyrev, Givental and Lian–Liu–Yau.

Summary

Quiver flag varieties are a family of spaces that generalize type A flag varieties. They appear in the Fano classification program as potential ambient spaces of Fano varieties.

Mirror symmetry provides a way of attacking the Fano classification problem, and is well-understood for toric varieties.

What about mirrors for other Fano varieties? The natural family to consider next is Fano subvarieties of Grassmannians or more generally quiver flag varieties.

One approach is via the Abelian/non-Abelian correspondence.

Mirror symmetry and toric degenerations

If a Fano variety X is mirror to f, then it is expected that

$$X \xrightarrow{\text{toric degeneration}} Y_P \quad .$$

Batyrev–Ciocan-Fontanine–Kim–van Straten write down conjectural mirrors to complete intersections in type A flag varieties using a toric degeneration of the ambient space.

I generalized this construction to zero loci in Y-shaped quiver flag varieties.

Verifying conjectural mirrors

Ciocan-Fontanine–Kim–Sabbah conjectured an Abelian/non-Abelian correspondence for Gromov–Witten invariants. In particular, they conjectured that the quantum period of a GIT quotient $V /\!/ G$ can be obtained from its Abelianization.

C-FKS proved their conjecture in the case of flag varieties.

Theorem

[Kal19] This conjecture holds for quiver flag varieties and their subvarieties.

This result was proved by Webb for all GIT quotients $V /\!/ G$ using different methods.

This allows us to verify (up to say 20 terms) candidate Laurent polynomial mirrors of subvarieties of quiver flag varieties and varieties.

The Grassmannian and the Plücker coordinate mirror

Marsh–Rietsch use the quantum cohomology ring of the Grassmannian to build the *Plücker coordinate mirror*. This is a rational function on the Grassmannian. They show it is a partial compactification of the mirror obtained via toric degeneration.

Theorem

[Marsh–Rietsch] The mirror of the Grassmannian obtained via toric degenerations has the correct period sequence.

Towards mirror theorems

The goal for the rest of the talk is to explain how to

- Use the Abelian/non-Abelian correspondence to compute in the quantum cohomology ring of flag varieties and quiver flag varieties.
- Use these results to produce candidate Plücker coordinate mirrors for flag varieties and in examples quiver flag varieties.

Quantum version of Martin's theorem

Theorem [Gu–Kalashnikov]

Let X_G be a Fano quiver flag variety and X_T the Abelianization. Then

$$QH(X_G) \cong \overline{QH}(X_T)^W / I_q$$

where I_q is the ideal generated by f such that $\omega * f \in I_q^{ab}$, and I_q^{ab} is the ideal of quantum relations of the Abelianization, with quantum parameters specialized.

The proof relies on work on the Abelian/non-Abelian correspondence for Gromov–Witten invariants cited above.

Gu–Sharpe mirror

Gu–Sharpe propose a mirror for Fano GIT quotients $V /\!/ G$ obtained by specializing the Hori–Vafa mirror of the Abelianization $V /\!/ T$.

Corollary

The Gu–Sharpe mirror computes the quantum cohomology of a quiver flag variety.

This is not a Laurent polynomial mirror in the sense of the mirror symmetry conjectures above, and it remains a mystery what relation this proposal has to the others.

Example: Quantum cohomology of the Grassmannian

The quantum relations of the Abelianization of the Grassmannian, $(\mathbb{P}^{n-1})^{\times r}$, are

$$(x_1^n - q_1, \dots, x_r^n - q_r).$$

To obtain I_q^{ab} we set $q_i := (-1)^{r-1} q$.

$$QH^*(\mathrm{Gr}(n,r)) = \mathbb{C}[x_1, \dots, x_r, q]^{\mathrm{Sym}_r} / (f | \omega * f \in I_q^{ab}).$$

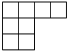
Recall that $\omega = \prod_{i<j}(x_i - x_j)$. As before, one multiplies Schubert classes as

$$[s_\lambda][s_\mu] = \sum_\kappa [c_{\lambda\mu}^\kappa s_\kappa].$$

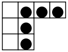
We can use I_q^{ab} to reduce too-wide partitions.

Rim-hook removals

A rim-hook of a partition λ of length n is a connected path of n boxes in λ starting from the top right box, and staying within the rim or boundary of λ. For example, the partition



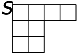
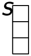
has a rim-hook of length 5 given by the marked boxes:

 .

The height of a rim-hook is the number of rows in which it appears: in the example above, it is of height 3. If μ is the partition without the rim-hook, then if μ is a partition,

$$RH_n(s_\lambda) := (-1)^{h+1} s_\mu,$$

otherwise we set $RH_n(s_\lambda) = 0$.

$RH_5(s$  $) = s$  .

Rim-hook removals for the Grassmannian

The main lemma in Bertram–Ciocan-Fontanine–Fulton can be stated as

Theorem

Let λ be a partition that is too wide to fit into an $r \times (n-r)$ box. Then

$$[s_\lambda] = q[RH_n(s_\lambda)]$$

This also follows immediately from the Abelian/non-Abelian correspondence.

Essentially, the rim-hook removal rule reduces quantum calculations to classical calculations.

Cohomology of quiver flag varieties

Let X_G be a quiver flag variety, with vertices $Q_0 := \{0, 1, \ldots, \rho\}$ and dimension vector $(r_0, \ldots, r_\rho)$.

The quiver flag variety is a tower of Grassmannian bundles. For each vertex $i > 0$, there is a step in the tower $\mathrm{Gr}(E_i, r_i)$. Let $s_i = \mathrm{rank}(E_i)$ and let $x_{i1}, \ldots, x_{ir_i}$ be the Chern roots of the tautological quotient bundle. Let s^i_λ denote

$$s_\lambda(x_{i1}, \ldots, x_{ir_i}).$$

A rim–hook removal rule for quiver flag varieties

Theorem

[Gu-Kal20] There is a rim-hook removal rule for the (quantum) cohomology of any quiver flag variety.

We obtain this result via the Abelian/non-Abelian correspondence. Like the original rim-hook removal rule, reduces computations to Littlewood–Richardson coefficients. A basis for the cohomology is given by

$$s^1_{\lambda^1} \cdots s^\rho_{\lambda^\rho}$$

where λ^i fits into an $r_i \times (s_i - r_i)$ box.

Type A flag varieties

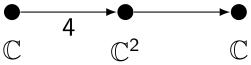
Suppose $M_\theta(Q, \mathrm{r})$ is the flag variety of quotients $\mathbb{C}^n$, $\mathrm{Fl}(n; r_1, \ldots, r_\rho)$. Then s^i_λ is the Schur polynomial in the Chern roots of the i^{th} tautological quotient bundle on the flag variety (the one of rank r_i). The quantum rim-hook removal rule in this case states that if $\lambda_1 \geq r_{i-1} - r_i + 1$, then

$$s^i_\lambda = \sum_{k=1}^{r_{i-1}} (-1)^{k+1} RH^{r_{i-1}}(s_{\lambda+\{r_{i-1}-k\}}) s^{i-1}_{(k)^t} \\ + (-1)^{r_i-1} q_i \sum_{k=0}^{r_{i+1}} (-1)^{r_{i+1}-k} RH^{r_{i-1}}(s_{\lambda+\{r_{i+1}-k\}}) s^{i+1}_{(k)^t}.$$

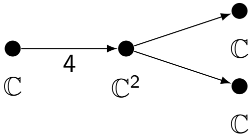
The basis is indexed by tuples of partitions $(\lambda^1, \ldots, \lambda^\rho)$, where λ^i fits into an $r_i \times (r_{i-1} - r_i)$ box. These tuples also index the Schubert basis.

Rim-hook removal rules

$$[s^1_{\ydiagram{3,2}}] = q_1[s^1_{\ydiagram{1}}]$$



$$[s^1_{\ydiagram{3,2}}] = q_1[s^1_{\ydiagram{1}} s^2_{\ydiagram{1}} - s^1_{\ydiagram{1,1}}].$$



$$[s^1_{\ydiagram{3,2}}] = q_1[s^1_{\ydiagram{1}} s^2_{\ydiagram{1}} s^3_{\ydiagram{1}} - s^1_{\ydiagram{1,1}}(s^2_{\ydiagram{1}} + s^3_{\ydiagram{1}})].$$

The Marsh–Rietsch mirror

There are two key ingredients in the construction of the Plücker coordinate mirror:

First, a representation of the anti-canonical class as a special sum of ratios of Schubert classes using quantum cohomology. For $\mathrm{Gr}(4,2)$, note that

$$s_{\ydiagram{1}} s_{\varnothing} = s_{\ydiagram{1}}, \quad s_{\ydiagram{1}} s_{\ydiagram{1,1}} = s_{\ydiagram{2,1}}, \quad s_{\ydiagram{1}} s_{\ydiagram{2}} = s_{\ydiagram{2,1}}, \quad s_{\ydiagram{1}} s_{\ydiagram{2,2}} = q s_{\ydiagram{1}}.$$

So

$$-K_{\mathrm{Gr}(4,2)} = 4 s_{\ydiagram{1}} = \frac{s_{\ydiagram{1}}}{s_{\varnothing}} + \frac{s_{\ydiagram{2,1}}}{s_{\ydiagram{1,1}}} + \frac{s_{\ydiagram{2,1}}}{s_{\ydiagram{2}}} + \frac{q s_{\ydiagram{1}}}{s_{\ydiagram{2,2}}}$$

The Marsh–Rietsch mirror

Secondly, to re-interpret this as a rational function on the Grassmannian. Plücker coordinate are also indexed by partitions fitting into an $r \times (n-r)$ box.

$$\frac{p_{\ydiagram{1}}}{p_{\varnothing}} + \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}} + \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{2}}} + \frac{q p_{\ydiagram{1}}}{p_{\ydiagram{2,2}}}.$$

Using the cluster structure of the coordinate ring, this gives many Laurent polynomial mirrors of the Grassmannian. The denominators are *frozen variables*.

Plücker coordinate mirrors for type A flag varieties and beyond

From this perspective on the quantum cohomology of quiver flag varieties and type A flag varieties, there's a natural way to generalize the Marsh–Rietsch mirror.

1. Give a representation of the anti-canonical class as a special sum of ratios of Schur polynomials classes using quantum cohomology

$$-K_{\mathrm{Fl}(4,2,1)} = \frac{s^1_{\square}}{s^1_{\varnothing}} + \frac{s^1_{(2,1)} + q_1}{s^1_{(2)}} + \frac{s^1_{(2,1)}}{s^1_{(1,1)}} + \frac{q_1(s^1_{\square}s^2_{\square} - s^1_{(1,1)})}{s^1_{(2,2)}} + \frac{s^2_{\square}}{s^2_{\varnothing}} + \frac{q_2 + s^1_{\square}s^2_{\square} - s^1_{(1,1)}}{s^2_{\square}} - s^1_{\square}.$$

2. Re-interpret this as a rational function on a *product of Grassmannians*, where the denominators are frozen variables.

This doesn't work. But (bizarrely) it almost works! The problem is the negative structure constants.

Plücker coordinate mirrors for type A flag varieties

For a type A flag variety, there's a natural fix: re-write this in the Schubert basis.

In [Kal20], I propose a Plücker coordinate mirror for type A flag varieties. For example, the mirror of $\mathrm{Fl}(4,2,1)$ is the rational function

$$\frac{p^1_{\ydiagram{1}}}{p^1_{\varnothing}} + \frac{p^1_{\ydiagram{2,1}} + q_1}{p^1_{\ydiagram{2}}} + \frac{p^1_{\ydiagram{2,1}}}{p^1_{\ydiagram{1,1}}} + \frac{q_1 p^1_{\ydiagram{1}} p^2_{\ydiagram{1}}}{p^1_{\ydiagram{2,2}}} + \frac{p^2_{\ydiagram{1}}}{p^2_{\varnothing}} + \frac{q_2}{p^2_{\ydiagram{1}}}$$

on $\mathrm{Gr}(4,2) \times \mathrm{Gr}(2,1)$.

A first check

Theorem

[Kal20] The Plücker coordinate mirror is a partial compactification of the Laurent polynomial mirror obtained via the toric degeneration approach.

Using the cluster structure of the Grassmannian, this produces many different candidate Laurent polynomial mirrors for flag varieties.

In examples of other quiver flag varieties, strategically 'shifting' the basis of the cohomology results in Plücker coordinate mirrors with the correct period sequences (up to 15 terms) .

Summary

The Plücker coordinate mirror for $\mathrm{Gr}(n, r)$ is the key to verifying the toric degeneration mirror (Marsh–Rietsch).

The Abelian/non-Abelian correspondence generalizes the rim-hook removal rule from Grassmannians to quiver flag varieties.

The Marsh–Rietsch construction can be generalized to type A flag varieties, and it agrees with the toric degeneration approach [Kal20].

- The fibration structure of the flag variety is trivialized under mirror symmetry; everything is encoded in the mirror.
- Intriguing new connection between mirror symmetry and positivity.

Thank you!

A rim–hook removal rule for quiver flag varieties

There is a rim-hook removal rule for the (quantum) cohomology of any quiver flag variety.

Theorem [Gu-Kalashnikov]

Let λ a partition such that $\lambda_1 \geq s_i - r_i + 1$. Then

$$s^i_\lambda = \sum_{k=1}^{s_i} \sum_{\sum k_a = k} (-1)^{k+1} RH^{s_i}(s_{\lambda + \{s_i - k\}}) \prod_{\substack{a \in Q_1 \\ t(a) = i}} s^{s(a)}_{(k_a)^t}$$

$$+ (-1)^{r_i - 1} q_i \sum_{k=0}^{s'_i} \sum_{\sum k_a = k} (-1)^{s'_i - k} RH^{s_i}(s_{\lambda + \{s'_i - k\}}) \prod_{\substack{a \in Q_1 \\ s(a) = i}} s^{t(a)}_{(k_a)^t}.$$