# Well-posedness and invariance of the Gibbs measures for periodic generalized Korteweg-de Vries equations

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# The generalized KdV equations

**Periodic generalized Korteweg-de Vries equations:**

$$\text{(gKdV)} \qquad \begin{cases} \partial_t u + \partial_x^3 u = \pm \partial_x(u^k), \\ u\big|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}$$

- Dispersive equations

- Hamiltonian equations: (gKdV) $\iff \partial_t u = \partial_x \frac{\delta H}{\delta u}$

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 \, dx \pm \frac{1}{k+1} \int_{\mathbb{T}} u^{k+1} \, dx$$

- conserved quantities: **mean** $\int_{\mathbb{T}} u \, dx$, **mass** $\int_{\mathbb{T}} u^2 \, dx$, **Hamiltonian** $H(u)$

**AIM:** Construct global-in-time dynamics in the support of the **Gibbs measure** $\mu$

$$d\mu \text{ “}=\text{” } Z^{-1} e^{-H(u)} du$$

and show invariance of $\mu$ under the flow

# Hamiltonian systems

Finite dimensional Hamiltonian system: $(p, q) \in \mathbb{R}^{2n}$ solve

$$\dot{p} = \frac{\partial H}{\partial q}, \quad \dot{q} = -\frac{\partial H}{\partial p}$$

with Hamiltonian $H = H(p, q)$.

▷ **Liouville's theorem + conservation of Hamiltonian** $\implies$ Gibbs measure $\mu$

$$d\mu = Z^{-1} \exp\big(-H(p,q)\big)\, dp\, dq$$

is **invariant** under the dynamics.

▷ Similarly for gKdV, the Gibbs measure formally defined as

$$d\mu \text{ "="} \; Z^{-1} e^{-H(u)} du$$

is expected to be **invariant:**

$$\mu\big(\Psi(-t)A\big) = \mu(A), \quad t \in \mathbb{R}, \; A \text{ measurable}$$

# Why do we care about Gibbs measures?

1. Invariance of Gibbs measure $\implies$ **typical** behaviour of solutions
   - Poincaré recurrence theorem
   - Furstenberg multiple recurrence theorem

2. **Local** well-posedness (LWP) + invariance $\implies$ **global** well-posedness
   - Bourgain's invariant measure argument ('94)
   - Main idea: use invariance of Gibbs measure as a *substitute* for a conservation law

**Problem 1:** How do we make sense of the measure $\mu$

$$d\mu \text{ “}=\text{” } Z^{-1} e^{-H(u)} du \quad ?$$

**Problem 2:** Need local well-posedness at low regularity (in the support of $\mu$)!

# Gaussian measure

Want to make sense of the Gibbs measure $\mu$ as a **weighted Gaussian measure**

$$d\mu = Z^{-1} e^{\mp \frac{1}{k+1} \int_{\mathbb{T}} u^{k+1}\, dx} e^{-\frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2\, dx} du$$

The **Gaussian measure** $\rho$ can be defined through the density

$$d\rho = Z_0^{-1}\, e^{-\frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2\, dx}\, du \quad (= \text{“} \lim_{N\to\infty} d\rho_N \text{”})$$

or equivalently, the induced probability measure under the map

$$\omega \in \Omega \mapsto u^\omega(x) = \sum_{n\neq 0} \frac{g_n(\omega)}{|n|} e^{inx}$$

where $\{g_n\}_{n\neq 0}$ are independent $\mathbb{C}$-valued standard Gaussian r.v.'s with $g_{-n} = \overline{g}_n$

$\triangleright$ $\rho$ is a **probability measure** on $H^s(\mathbb{T}) \iff s < \frac{1}{2}$

$$\mathbb{E}\big[\|u^\omega\|_{H^s}^2\big] = \mathbb{E}\Big[\sum_{n\neq 0} \frac{|n|^{2s}|g_n(\omega)|^2}{|n|^2}\Big] = \sum_{n\neq 0} |n|^{2(s-1)} \underbrace{\mathbb{E}\big[|g_n(\omega)|^2\big]}_{=1}$$

# Gibbs measure

• **defocusing case:** '+' in gKdV and $k$ odd

$$d\mu = Z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}\,dx} d\rho$$

▷ $u \in \operatorname{supp}\rho \implies u \in L^{k+1}(\mathbb{T})$ (Sobolev embedding)

• **non-defocusing case:** '−' in gKdV **or** $k$ even

Lebowitz-Rose-Speer '88 : $\quad d\mu = Z^{-1} \mathbb{1}_{\{\|u\|_{L^2}\le R\}} e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}\,dx} d\rho$

**Theorem** (Lebowitz-Rose-Speer '88, Bourgain '94, Oh-Sosoe-Tolomeo '21)

Let $k \geq 2$, $R > 0$ and $Z_{k,R} = \left\| \mathbb{1}_{\{\|u\|_{L^2}\le R\}} e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}\,dx} \right\|_{L^1(\rho)}$.

- $2 \leq k \leq 4$: $Z_{k,R} < \infty$ for any $R > 0$
- $k = 5$: $Z_{5,R} < \infty$ if $R \leq \|Q\|_{L^2(\mathbb{R})}$; $\quad Z_{5,R} = \infty$ if $R > \|Q\|_{L^2(\mathbb{R})}$

▷ **Also:** $\mu$ is well approximated by $\mu_N$ defined by

$$d\mu_N = Z^{-1} \mathbb{1}_{\{\|\mathbf{P}_{\le N} u\|_{L^2}\le R\}} e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} (\mathbf{P}_{\le N} u)^{k+1}\,dx} d\rho$$

**Bourgain '94:** Construct **local-in-time** solutions and exploit the (formal) invariance of the Gibbs measure $\mu$ as a substitute for a conservation law

▷ **General strategy:**

show **local well-posedness** in the support of $\mu$

$+$

**Bourgain's invariant measure argument**

▷ **Main difficulty:** The support of the Gibbs measure $\mu$ is rough

$$\operatorname{supp} \mu \subset H^{\frac{1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}),\ \varepsilon > 0$$

# Known results

| | | |
|---|---|---|
| KdV ($k=2$) | Bourgain '93 | GWP in $L^2(\mathbb{T})$ |
| mKdV ($k=3$) | Bourgain '94 | a.s. GWP in $H^{\frac{1}{2}-}(\mathbb{T}) \cap \mathcal{F}L^{1-,\infty}(\mathbb{T})$ |
| gKdV ($k \geq 4$) | Bourgain '93, Staffilani '97 | |
| | Colliander-Keel-Staffilani -Takaoka-Tao '04 | LWP in $H^{\frac{1}{2}}(\mathbb{T})$ |
| | | GWP in $H^{\frac{13}{14}-\frac{2}{7(k-1)}}(\mathbb{T})$ |
| | | $C^k$ failure in $H^{\frac{1}{2}-}(\mathbb{T})$ |
| | Richards '16 ($k=4$) | a.s. LWP in $H^{\frac{1}{2}-}(\mathbb{T})$, a.s. GWP |
| | | invariance under $\mathcal{G}$-gKdV |
| | Oh-Richards-Thomann '16 | a.s. GWP in $H^{\frac{1}{2}-}(\mathbb{T})$ (no uniqueness) |
| | | "mild" invariance† under gKdV |

† following a compactness argument due to Burq-Thomann-Tzvetkov '18, they show that $\mathcal{L}\big(u(t)\big) = \mu$ for all $t \in \mathbb{R}$

**Problem:** $\operatorname{supp}\mu \subset H^{\frac{1}{2}-}(\mathbb{T})$ where gKdV is mildly ill-posed

**Alternative 1:** Show *probabilistic* well-posedness in $H^{\frac{1}{2}-}(\mathbb{T})$

- Consider random initial data $u^\omega(x) = \sum_{n\neq 0} \frac{g_n(\omega)}{|n|} e^{inx}$

**Alternative 2:** Show *deterministic* local well-posedness in different spaces

- **Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$:**

$$\|f\|_{\mathcal{F}L^{s,p}} = \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^p_n}, \qquad \langle n\rangle = (1+n^2)^{\frac{1}{2}}$$

- $\mathcal{F}L^{s,2}(\mathbb{T}) = H^s(\mathbb{T})$
- $\operatorname{supp}\mu \subset \mathcal{F}L^{s,p}(\mathbb{T}) \iff s < 1 - \frac{1}{p}$

$$\mathbb{E}\big[\|u^\omega\|^p_{\mathcal{F}L^{s,p}}\big] \sim \sum_{n\neq 0} |n|^{(s-1)p} < \infty \iff (s-1)p < -1$$

- $\mathcal{F}L^{s,p}(\mathbb{T}) \hookrightarrow L^{k+1}(\mathbb{T}),\ s > 1 - \frac{1}{p} - \frac{1}{k+1}$

# Main results

▷ **Gauged gKdV equation:**

($\mathcal{G}$-gKdV)
$$\begin{cases} \partial_t u + \partial_x^3 u = \pm \partial_x \big( u^k - k\mathbf{P}_0(u^{k-1}) u \big) \\ u|_{t=0} = u_0 \end{cases}$$

▷ **Gauge transform:** $\mathcal{G}[u](t,x) = u\Big(t, x \mp k \int_0^t \mathbf{P}_0(u^{k-1}(t'))\, dt'\Big)$

**Theorem (Local well-posedness of $\mathcal{G}$-gKdV) (C.-Kishimoto '21)**

Let $k \geq 3$ and $2 < p < \infty$. Then, there exists $\varepsilon_0 = \varepsilon_0(k,p) \in (0, \frac{1}{k+1})$ such that $\mathcal{G}$-gKdV is locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{T})$ for any $s > \max(\frac{1}{2}, 1 - \frac{1}{p} - \varepsilon_0)$.

- Fully deterministic result
- **Key ingredients:** multilinear Strichartz estimates adapted to the Fourier-Lebesgue setting

# Main results

**Theorem (a.s. GWP of $\mathcal{G}$-gKdV and invariance of $\mu$) (C.-Kishimoto '21)**

Let $k \geq 3$ and assume that the measure $\mu$ is well-defined. Then, there exists $\Sigma \subset \bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T})$ of full $\mu$-measure s.t.

1. For every $u_0 \in \Sigma$, there exists a **unique global-in-time** solution $u \in \bigcap_{s<1-\frac{1}{p}} C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ of $\mathcal{G}$-gKdV
2. The solution map $\Phi(t)$ for $\mathcal{G}$-gKdV is $\mu$-measurable, $\Phi(t)\Sigma = \Sigma$, and $\Phi(t+s) = \Phi(t)\Phi(s)$
3. The Gibbs measure $\mu$ is invariant under the flow of $\mathcal{G}$-gKdV

By inverting the gauge transform and using the invariance of the Gibbs measure under translations, we obtain the following:

**Theorem (C.-Kishimoto '21)**

Under the assumptions of the previous theorem, the (ungauged) gKdV equation is a.s. globally well-posed and the Gibbs measure is invariant under its flow.

**Linear propagator:** $\mathcal{F}_x\big(S(t)u_0\big)(n) = e^{itn^3}\widehat{u}_0(n)$

**Duhamel formulation:**

$$u(t) = S(t)u_0 \pm \int_0^t S(t-t')\underbrace{\partial_x\big(u^k - k\mathbf{P}_0(u^{k-1})u\big)(t')}_{\mathcal{N}[u](t')}\,dt'$$

**Interaction representation:** $v(t) = S(-t)u(t)$ solves

$$\widehat{v}(t,n) = \widehat{u}_0(n) \pm \int_0^t \sum_{n=n_1+\ldots+n_k} e^{-it'\phi_k(\bar{n})}\,in\prod_{j=1}^k \widehat{v}(t',n_j)\,dt'$$

**Phase function:** $\phi_k(\bar{n}) = n^3 - n_1^3 - \ldots - n_k^3$

$\triangleright$ **Lack of smoothing** effect in the periodic setting forces us to use the phase function to control the derivative $n$

**GOOD:** $\leftarrow$ non-resonant

**BAD:** $\leftarrow$ resonant

# Fourier restriction norm method

**Duhamel formulation:** $$u(t) = \underbrace{S(t)u_0}_{\text{linear solution}} \pm \int_0^t S(t-t')\mathcal{N}[u](t')\,dt'$$

**Fourier restriction spaces $X^{s,b}_{p,q}$:** (Bourgain '93, Grünrock '04, G.-Herr '08)

$$\|u\|_{X^{s,b}_{p,q}} := \Big\| \langle n\rangle^s \big\| \langle \tau - n^3\rangle^b \widehat{u}(\tau,n)\big\|_{L^q_\tau} \Big\|_{\ell^p_n}$$

▷ $X^{s,\frac{1}{2}}_{p,2} \cap X^{s,0}_{p,1}$ $\big(\hookrightarrow C\big(\mathbb{R};\mathcal{F}L^{s,p}(\mathbb{T})\big)\big)$

**Main nonlinear estimate:** $$\big\|\mathcal{N}[u_1,\dots,u_k]\big\|_{X^{s,-\frac{1}{2}}_{p,2}\cap X^{s,-1}_{p,1}} \lesssim \prod_{j=1}^k \|u_j\|_{X^{s,\frac{1}{2}}_{p,2}\cap X^{s,0}_{p,1}}$$

**Main ideas:** Use smoothing in time to help control the spatial derivative

$$|\underbrace{n^3 - n_1^3 - \dots - n_k^3}_{\phi_k}| = |(\tau - n^3) - (\tau_1 - n_1^3) - \dots - (\tau_k - n_k^3)| \lesssim \max(|\tau - n^3|, |\tau_j - n_j^3|)$$

and exploit *dispersion* through Strichartz estimates

**Phase function $\phi_k$:**

$$|\phi_k| \lesssim \max(|\tau - n^3|, |\tau_j - n_j^3|)$$

**KdV** $\quad \phi_2 = n^3 - n_1^3 - n_2^3 \overset{n=n_1+n_2}{=} 3nn_1n_2$

**mKdV** $\quad \phi_3 = n^3 - n_1^3 - n_2^3 - n_3^3 \overset{n=n_1+n_2+n_3}{=} 3(n-n_1)(n-n_2)(n-n_3)$

**gKdV** $(k \geq 4)$ $\quad \phi_k = n^3 - n_1^3 - \ldots - n_k^3 \overset{n=n_1+\ldots+n_k}{=}$ ??

**Problem:** no known factorization for $\phi_k$ when $k \geq 4$! (assume $|n_1| \geq \ldots \geq |n_k|$)

**Idea:** compare $\phi_k$ with $\phi_2$ and $\phi_3$

$$\begin{aligned}\phi_k &= n^3 - n_1^3 - (n-n_1)^3 + (n-n_1)^3 - n_2^3 - \ldots - n_k^3 \\ &= 3nn_1(n-n_1) + \mathcal{O}(|n_2n_3| \max(|n_4|, |n-n_1|))\end{aligned}$$

$$\begin{aligned}\phi_k &= n^3 - n_1^3 - n_2^3 - (n-n_1-n_2)^3 + (n_3+\ldots+n_k)^3 - n_3^3 - \ldots - n_k^3 \\ &= 3(n_1+n_2)(n-n_1)(n-n_2) + \mathcal{O}(|n_3n_4| \max(|n_5|, |n-n_1-n_2|))\end{aligned}$$

**But first need to consider resonances:** $\phi_2 = nn_1(n-n_1) = 0$
$\phi_3 = (n_1+n_2)(n-n_1)(n-n_2) = 0$

**Gauged nonlinearity:** $\mathcal{N}[u_1, \ldots, u_k] = \partial_x \Big( u_1 \cdots u_k - \sum_{j=1}^{k} \mathbf{P}_0 \big( \prod_{i \neq j} u_i \big) u_j \Big)$

$$\implies \quad \mathcal{F}_x \mathcal{N}[u_1, \ldots, u_k](n) = \sum_{\substack{n = n_1 + \ldots + n_k \\ n \cdot n_1 \cdots n_k \neq 0}} \Big( 1 - \sum_{j=1}^{k} \mathbb{1}_{\{n = n_j\}} \Big) in \widehat{u}_1(n_1) \cdots \widehat{u}_k(n_k)$$

$$\mathcal{F}_x \mathcal{N}_0[u_1, \ldots, u_k](n) = \sum_{\substack{n = n_1 + \ldots + n_k \\ n \cdot n_1 \cdots n_k \neq 0}} \mathbb{1}_{\bigcap_{j=1}^{k} \{n \neq n_j\}} in \widehat{u}_1(n_1) \cdots \widehat{u}_k(n_k)$$

$$\mathcal{F}_x \mathcal{R}[u_1, \ldots, u_k](n) = \sum_{\substack{n = n_1 + \ldots + n_k \\ n \cdot n_1 \cdots n_k \neq 0}} \sum_{\substack{J \subset \{1, \ldots, k\} \\ |J| \geq 2}} (-1)^{|J|} \, \mathbb{1}_{\bigcap_{j \in J} \{n = n_j\}} in \widehat{u}_1(n_1) \cdots \widehat{u}_k(n_k)$$

Then, $\boxed{\mathcal{N} = \mathcal{N}_0 + \mathcal{R}}$

**Estimate for resonant part $\mathcal{R}$:** Consider, e.g., the case $n = n_1 = n_2$

$$\langle n \rangle^s \, n \, \widehat{u}_1(n) \widehat{u}_2(n) \sum_{-n = n_3 + \ldots + n_k} \widehat{u}_3(n_3) \cdots \widehat{u}_k(n_k)$$

▷ Assuming $|n_3| \geq \ldots \geq |n_k|$, we can use $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$ to control the derivative

**Estimate for the non-resonant part $\mathcal{N}_0$:**

$$\begin{aligned}\phi_k &= 3nn_1(n-n_1) + \mathcal{O}(|n_2 n_3| \max(|n_4|, |n-n_1|)) \\ &= 3(n_1+n_2)(n-n_1)(n-n_2) + \mathcal{O}(|n_3 n_4| \max(|n_5|, |n-n_1-n_2|))\end{aligned}$$

Further splitting:

$$\begin{array}{lll} \text{if} & n_1+n_2 \neq 0 & \to \quad \mathcal{N}_1 \\ \text{if} & n_1+n_2 = 0,\ n_3+n_4 \neq 0 & \to \quad \mathcal{N}_3 \\ \text{if} & n_1+n_2 = n_3+n_4 = 0,\ n_5+n_6 \neq 0 & \to \quad \mathcal{N}_5 \\ \vdots & & \qquad \vdots \end{array}$$

Then, $\mathcal{N}_0 = \mathcal{N}_1 + \mathcal{N}_3 + \ldots$

For $1 \leq \alpha \leq k$ odd, then $n = n_\alpha + \ldots + n_k$ with $n_\alpha + n_{\alpha+1} \neq 0$, so we consider

$$\begin{aligned}\phi_k &= 3nn_\alpha(n-n_\alpha) + \ldots \\ &= 3(n_\alpha + n_{\alpha+1})(n-n_\alpha)(n-n_{\alpha+1}) + \ldots\end{aligned}$$

**Example:** $|n| \sim |n_1| \gg |n_2| \geq \ldots \geq |n_k|$

$$\phi_k = 3nn_1(n-n_1) + \mathcal{O}(|n_2|, |n_3|, \max(|n_4|, |n-n_1|))$$

- **Case 1:** $|\phi_k| \gtrsim |nn_1(n-n_1)| \quad (\lesssim \max\{\langle \tau - n^3\rangle, \langle \tau_j - n_j^3\rangle\} =: \sigma_{\max})$

▷ Use 'time derivatives' to control spatial derivatives!

- **Case 2:** $|\phi_k| \ll |nn_1(n-n_1)|$

$$\implies |n_1|^2|n-n_1| \lesssim |n_2 n_3| \max(|n_4|, |n-n_1|) \lesssim |n_2 n_3 n_4|$$

$$\implies \boxed{\langle n\rangle^s |n| \lesssim \frac{\langle n_1\rangle^s |n_2 n_3 n_4|^{\frac{1}{2}}}{|n-n_1|^{\frac{1}{2}}}}$$

▷ **Key ingredients:** bilinear ($\sim L^4$) and trilinear ($\sim L^6$) Strichartz estimates adapted to the Fourier-Lebesgue setting!

**(Linear) Strichartz estimates** (Bourgain '93):

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{3}}_{2,2}}, \qquad \|u\|_{L^6_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}}_{2,2}}$$

Multilinear Strichartz estimates (C.-Kishimoto '21)

For any $2 \leq p < \infty$, $b > \max(\frac{1}{3}, \frac{3p-2}{8p})$, we have

$$\left\|\mathbf{P}_{\neq 0}\left(\mathbf{P}_{\neq 0} u_1 \mathbf{P}_{\neq 0} u_2\right)\right\|_{X^{0,0}_{p,2}} \lesssim \|u_1\|_{X^{0,b}_{p,2}} \|u_2\|_{X^{0,b}_{2,2}} \qquad (p = 2: \sim L^4)$$

$$\left\|\mathbf{P}_{\phi_3 \neq 0}(u_1 u_2 u_3)\right\|_{X^{0,0}_{p,2}} \lesssim \|u_1\|_{X^{0+,\frac{1}{2}}_{p,2}} \|u_2\|_{X^{0+,\frac{1}{2}}_{2,2}} \|u_3\|_{X^{0+,\frac{1}{2}}_{2,2}} \qquad (p = 2: \sim L^6)$$

**Bilinear estimate for $p \geq 2$:**

Bilinear estimate (C.-Kishimoto '21)

For $2 \leq p < \infty$ and $b > \max(\frac{1}{3}, \frac{3p-2}{8p})$, we have

$$\left\|\mathbf{P}_{\neq 0}\left(\mathbf{P}_{\neq 0} u_1 \mathbf{P}_{\neq 0} u_2\right)\right\|_{X^{0,-\frac{1}{2}+}_{p,2}} \lesssim \|u_1\|_{X^{0,b}_{p,2}} \|u_2\|_{X^{0,0}_{p',2}}$$

$$|n| \sim |n_1| \gg |n_2| \geq \cdots |n_k| \text{ and } |\phi_k| \ll |nn_1(n-n_1)|$$

$$\left\| \mathbf{P}_{\neq 0}\left(\mathbf{P}_{\neq 0} u_1 \, \mathbf{P}_{\neq 0} u_2\right) \right\|_{X^{0,-\frac{1}{2}+}_{p,2}} \lesssim \|u_1\|_{X^{0,b}_{p,2}} \|u_2\|_{X^{0,0}_{p',2}}$$

$$\langle n \rangle^s |n| \lesssim \frac{\langle n_1 \rangle^s |n_2 n_3 n_4|^{1/2}}{\langle n - n_1 \rangle^{1/2}}$$

$$\left\| \partial_x^s u_1 \cdot \partial_x^{-\frac{1}{2}} \left( \partial_x^{\frac{1}{2}} u_2 \cdot \partial_x^{\frac{1}{2}} u_3 \cdot \partial_x^{\frac{1}{2}} u_4 \prod_{j=5}^{k} u_j \right) \right\|_{X^{0,-\frac{1}{2}+}_{p,2}}$$

$$\overset{\text{bilinear est.}}{\lesssim} \|u_1\|_{X^{s,\frac{1}{2}}_{p,2}} \left\| \partial_x^{\frac{1}{2}} u_2 \cdot \partial_x^{\frac{1}{2}} u_3 \cdot \partial_x^{\frac{1}{2}} u_4 \prod_{j=5}^{k} u_j \right\|_{X^{-\frac{1}{2},0}_{p',2}}$$

$$\lesssim \|u_1\|_{X^{s,\frac{1}{2}}_{p,2}} \left\| \partial_x^{\frac{1}{2}} u_2 \cdot \partial_x^{\frac{1}{2}} u_3 \cdot \partial_x^{\frac{1}{2}} u_4 \right\|_{X^{0,0}_{p,2}} \prod_{j=5}^{k} \|u_j\|_{X^{0,0}_{1,1}}$$

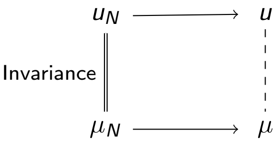
$$\overset{\substack{\text{trilinear} \\ \text{Strichartz est.}}}{\lesssim} \|u_1\|_{X^{s,\frac{1}{2}}_{p,2}} \|u_2\|_{X^{\frac{1}{2}+,\frac{1}{2}}_{p,2}} \|u_3\|_{X^{\frac{1}{2}+,\frac{1}{2}}_{2,2}} \|u_4\|_{X^{\frac{1}{2}+,\frac{1}{2}}_{2,2}} \prod_{j=5}^{k} \|u_j\|_{X^{0,0}_{1,1}}$$

□

# Invariance of the Gibbs measure for $\mathcal{G}$-gKdV

Bourgain's invariant measure argument:

- $\Phi(t)u_0 = u(t)$, $\forall t \in \mathbb{R}$
- $\mu\big(\Phi(-t)A\big) = \mu(A)$, $\forall t \in \mathbb{R}, A$ meas.




Study **truncated gauged gKdV equation:** $\big(\mathbf{P}(f) = f - k\mathbf{P}_0(f)\big)$

$$(\mathcal{G}\text{-gKdV}_N) \qquad \begin{cases} \partial_t u_N + \partial_x^3 u_N = \mathbf{P}_{\leq N}\partial_x\Big[\mathbf{P}\Big(\big(\mathbf{P}_{\leq N} u_N\big)^{k-1}\Big)\mathbf{P}_{\leq N} u_N\Big] \\ u_N\big|_{t=0} = u_0 \end{cases}$$

$$u_{\text{high}} = \mathbf{P}_{>N} u_N \to \text{linear solution}$$

$$u_{\text{low}} = \mathbf{P}_{\leq N} u_N \to \text{finite system of ODEs}$$

Gibbs measure $\mu_N = \tilde{\mu}_N \otimes \rho_N^{\perp}$ associated with $\mathcal{G}$-gKdV$_N$

$$d\tilde{\mu}_N = \tilde{Z}_N^{-1} \mathbb{1}_{\{\|\mathbf{P}_{\leq N} u_N\|_{L^2} \leq R\}} e^{\mp \frac{1}{k+1}\int_{\mathbb{T}} (\mathbf{P}_{\leq N} u_N)^{k+1}\, dx} \underbrace{e^{-\frac{1}{2}\int_{\mathbb{T}} (\mathbf{P}_{\leq N} u_N)^2\, dx} d(\mathbf{P}_{\leq N} u_N)}_{d\rho_N(u_N)}$$

is invariant under the flow!

# What about gKdV?

Recall the inverse "**gauge transform**"

$$\mathcal{G}^{-1}[u](t,x) = u\Big(t, x \pm k \int_0^t \mathbf{P}_0(u^{k-1}(t'))\, dt'\Big)$$

▷ **a.s. GWP of gKdV** follows from a.s. GWP of $\mathcal{G}$-gKdV:

$$u \text{ solves } \mathcal{G}\text{-gKdV} \implies \mathcal{G}^{-1}[u] \text{ solves gKdV}$$

▷ **Invariance** $\mu\big(\Psi(-t)A\big) = \mu(A)$

$$\Psi(t)u_0 = \Phi(t)u_0\Big(\cdot \mp k \int_0^t \int_{\mathbb{T}} \Phi(t')u_0^{k-1}\, dt'\Big) = \tau_{\beta(t,u_0)}\Phi(t)u_0$$

- **Problem:** cannot use invariance of $\mu$ under spatial translations
- **Solution:** average over all translations!

$$\mu\big(\Psi(-t)A\big) = \int_{\mathbb{T}} \mu\big(\tau_y \Psi(-t)A\big)\, dm(y) = \int_{\mathbb{T}} \int_{\Sigma} \mathbb{1}_A\big(\tau_{y+\beta(t,u_0)}\Phi(t)u_0\big)\, d\mu(u_0)\, dm(y)$$

**Thank you for your attention!**