Simple motion of stretch-limited elastic strings

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The **configuration** of a string of natural length L at time t is given by a curve

$$[0, L] \ni s \mapsto \boldsymbol{r}(s, t) \in \mathbb{R}^3.$$

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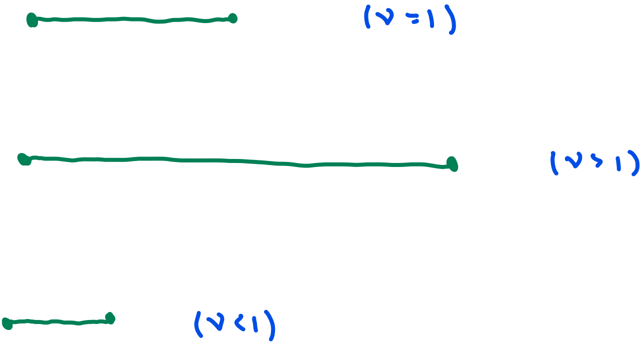
We always assume that the **stretch** $\nu(s,t) = |\boldsymbol{r}_s(s,t)| > 0$.

The conditions $\nu > 1$ and $\nu < 1$ are interpreted as the string be **elongated** and **compressed**, respectively.


($\nu = 1$)
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- **Contact force**: There exists a function $N(s,t)$, the **tension**, such that the resultant force exerted on a segment $[a,s]$ by the segments $[0,a]$ and $[s,L]$ is $\boldsymbol{n}(s,t) - \boldsymbol{n}(a,t)$ with

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- **Balance of linear momentum**: If $\rho(s)$ is the mass density of the string and there are no external forces then for all $[a,s] \subseteq [0,L]$

$$\frac{d}{dt}\int_a^s \rho(\sigma)\boldsymbol{r}_t(\sigma,t)d\sigma = \boldsymbol{n}(s,t) - \boldsymbol{n}(a,t).$$

Assuming appropriate smoothness we obtain the **equations of motion**:

$$\rho(s)\boldsymbol{r}_{tt}(s,t) = \boldsymbol{n}_s(s,t), \quad (s,t) \in [0,L] \times [0,T],$$

where as before $\boldsymbol{n}(s,t) = N(s,t)\boldsymbol{r}_s(s,t)/|\boldsymbol{r}_s(s,t)|$. To close this system of three equations in four unknowns, one must specify a relation between the **tension** N and **stretch** ν.

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For example, if $\nu - 1 = \frac{1}{E}N$ with $E > 0$ (a Hookean law), then the equations of motion are:

$$\rho(s)\boldsymbol{r}_{tt}(s,t) = E\Big(\frac{|\boldsymbol{r}_s(s,t)| - 1}{|\boldsymbol{r}_s(s,t)|}\boldsymbol{r}_s(s,t)\Big)_s, \quad (s,t) \in [0,L] \times [0,T],$$

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- **Extensible**: $N = \hat{N}(\nu)$ with $\hat{N} \in C^\infty(0,\infty)$, increasing,

$$\hat{N}(1) = 0, \quad \hat{N}(0^+) = -\infty, \quad \hat{N}(\infty) = \infty.$$

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- **Inextensible**: $\nu =$ const. regardless of the motion, and N is undetermined.

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Stretch-limited strings (R. 20'):

- The stretch is a function of the tension $\nu = \hat{\nu}(N)$ and there exist values of tension $N_0 < 0 < N_1$ so that

$$\nu - 1 = \begin{cases} \frac{1}{E} N_0 & \text{if } N \leq N_0, \\ \frac{1}{E} N & \text{if } N \in [N_0, N_1], \\ \frac{1}{E} N_1 & \text{if } N \geq N_1. \end{cases}$$

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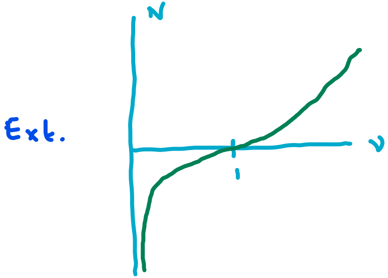
- In particular, we have

$$N \leq N_0 \implies \nu = \nu_0 = 1 + \frac{1}{E} N_0,$$
$$N \geq N_1 \implies \nu = \nu_1 = 1 + \frac{1}{E} N_1.$$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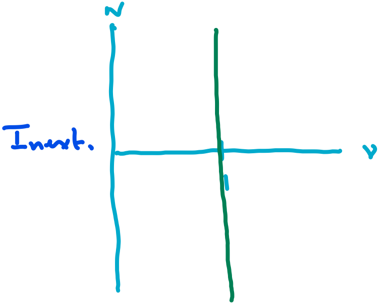
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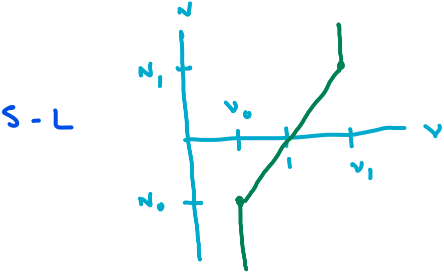
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The **extensible, inextensible** and **stretch-limited** relations model "elastic" behvavior in the sense that no mechanical energy is dissipated: for all motions with continuously differentiable $(\boldsymbol{r}_t, \boldsymbol{r}_s, N)$ and for all $[a, b] \subseteq [0, L]$

$$\frac{d}{dt}\int_a^b \frac{1}{2}\rho(s)|\boldsymbol{r}_t(s,t)|^2 ds + \frac{d}{dt}\int_a^b W(\nu(s,t), N(s,t))ds = \boldsymbol{n}(s,t)\cdot\boldsymbol{r}_t(s,t)|_a^b$$

where the stored energy $W(\nu, N) =$

$$\begin{cases} \int_1^\nu \hat{N}(\bar{\nu})d\bar{\nu} & \text{(extensible)}, \\ 0, & \text{(inextensible)}, \\ \chi_{[N_0 \le N \le N_1]}(N)\frac{E}{2}(\nu - 1)^2 + \chi_{[N \ge N_1]}(N)\frac{E}{2}(\nu_1 - 1)^2 & \text{(stretch-limited)}. \end{cases}$$

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- $\rho = E = 1$,
- $\boldsymbol{r}(0,t) = \mathbf{0}$, $\boldsymbol{n}(L,t) = (\zeta t + \tau)\boldsymbol{i}$, with $\zeta \geq 0$ and $\tau > 0$,
- the segment parameterized by $[0, \sigma(t)]$ is **inextensible** $(N(s,t) > N_1)$ and the segment parameterized by $[\sigma(t), L]$ is **extensible** $(0 < N(s,t) < N_1)$, and

$$\sigma'(t) > 0$$

Thus, $\sigma(t)$ is a moving shock front separating the growing inextensible segment from the shrinking extensible segment.

$$\underline{r}_{tt} = \left(\frac{|\underline{r}_s| - 1}{|\underline{r}_s|} \underline{r}_s \right)_s$$

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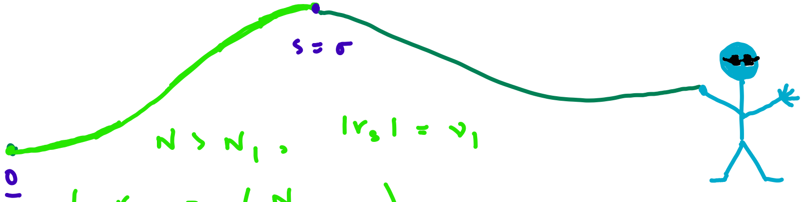

$$N > N_1, \quad |r_s| = \nu_1$$

$$\begin{cases} \underline{r}_{tt} = \left(\dfrac{N}{\nu_1} \underline{r}_s \right)_s \\ -N_{ss} + |\underline{r}_{ss}|^2 N = \nu_1 |\underline{r}_{st}|^2 \end{cases}$$

For $\boldsymbol{r}(s,t) = \chi(s,t)\boldsymbol{i}$, the equations of motion and Rankine-Hugoniot jump conditions are then equivalent to a novel shock-front problem:

- for $t \in [0,T], s \in [0,\sigma(t)]$,

$$\chi(s,t) = (1+N_1)s,$$
$$N(s,t) = N(\sigma^+,t) + (\sigma')^2(N_1 - N(\sigma^+,t)),$$

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$$\begin{aligned} &\chi_{tt}(s,t) = \chi_{ss}(s,t), \\ &\chi(\sigma^+, t) = (1 + N_1)\sigma, \quad \chi_s(L,t) = \zeta t + \tau + 1, \end{aligned}$$

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- the shock front satisfies $\sigma' > 1$ (Lax condition) and

$$\sigma' = \frac{\chi_t(\sigma^+,t)}{N_1 - (\chi_s(\sigma^+,t) - 1)}.$$

We say a triple of real-valued functions

$$(\chi_0, \chi_1, N_0) \in C([0, L]) \times L^\infty([0, L]) \times L^\infty([0, L]).$$

is a set of classical **shock front initial data** if there exists a unique $\sigma_0 \in (0, L)$ (the initial shock front) such that

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$$0 < N_0(s) < N_1,$$

- $\chi_0'(L) = \tau + 1, \quad \chi_1'(L) = \zeta, \quad \chi_0(\sigma_0) = (1 + N_1)\sigma_0,$

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- for all $s \in [0, \sigma_0]$,
$$\begin{aligned} \chi_0(s) &= (1 + N_1)s, \\ \chi_1(s) &= 0, \\ N_0(s) &= N_0(\sigma_0^+) + (\sigma_1)^2(N_1 - N_0(\sigma_0^+)) > N_1. \end{aligned}$$

Since the shock solution is determined by $\sigma(t)$ and the restriction of χ to $\{(s,t) : t \in [0,T], s \in [\sigma, L]\}$, by using finite speed of propagation arguments and the fact that the shock speed $\sigma' > 1$ (the speed of propagation for the wave equation), we obtain the following.

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Theorem (R. 21')

Suppose (χ_0, χ_1, N_0) *is a set of classical shock front initial data with initial shock front* $\sigma_0 \in (0, \infty)$*. Then there exist a unique classical shock solution* (χ, N) *with* $(\chi, \partial_t \chi, N)|_{t=0} = (\chi_0, \chi_1, N_0)$*, defined on a maximal time interval* $[0, T_+)$ *of existence with*

$$T_+ \leq T(\zeta) = \begin{cases} \infty & \text{if } \zeta = 0, \\ \frac{N_1 - \tau}{\zeta} & \text{if } \zeta > 0. \end{cases}$$

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$$T_+ \leq T(\zeta) = \begin{cases} \infty & \text{if } \zeta = 0, \\ \frac{N_1 - \tau}{\zeta} & \text{if } \zeta > 0. \end{cases}$$

$T(\zeta)$ is precisely the time when the tension at the end $s = L$ exceeds the threshold for inextensibility, $N(L, T(\zeta)) = N_1$.

The simplest solutions are those with **piecewise constant stretch**:

$$\chi_s(s,t) = \begin{cases} 1 + N_1 & s \in [0,\sigma], \\ 1 + \zeta t + \tau & s \in [\sigma, L]. \end{cases}$$

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and

$$T_+ = T(\zeta)\Big(1 + \frac{\sigma_1 T(\zeta)}{L - \sigma_0}\Big)^{-1} \leq T(\zeta),$$

with $\sigma(T_+; \sigma_0, \sigma_1) = L$.

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and the string becomes **fully inextensible** at $T(\zeta)$. Moreover, if $\zeta > 0$ then for all $s \in (0, \infty)$

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Question: If $L = \infty$, is this two-parameter family stable?

Theorem (R. 21')

Assume $\zeta \geq 0$. Let $(\sigma_0, \sigma_1) \in (0, \infty) \times (1, \infty)$. There exist $\epsilon_0 > 0$ and $C > 0$ such that for all $\epsilon < \epsilon_0$, the following is true. Suppose (χ_0, χ_1, N_0) is a set of initial data with initial shock front σ_0 such that there exists $r > 0$ such that

$$B := \sup_{s \in [\sigma_0, \infty)} \Big[|\chi_0'(s) - \tau - 1| + s^{r+2} |\chi_0''(s)| + |\chi_1(s) - \sigma_1(N_1 - \tau)| + s^{r+2} |\chi_1'(s)| \Big] < \epsilon.$$

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Then the unique solution (χ, N) to the equations of motion with initial data (χ_0, χ_1, N_0) satisfies $T_+ = T(\zeta)$. Moreover, the state variables (χ_s, χ_t, N) and shock speed remain close to and asymptotically approach those of a piece-wise constant stretched motion as $t \to T(\zeta)$ in a quantitative way.

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The argument is more delicate when $\zeta > 0$ since one must show that the stretch grows like $\zeta t + \tau + 1$ to leading order (otherwise a second inextensible segment can form).

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- extension and investigation of strain-limiting behavior for rods.

Thank you for your attention.