Low regularity solutions for nonlinear waves

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joint work with Albert Ai and Mihaela Ifrim

Two nonlinear wave equations

Two nonlinear (quasilinear) wave equations in $\mathbb{R} \times \mathbb{R}^n$:

$$g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u = 0 \qquad (NLW)$$

$$g^{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u = 0 \qquad (DNLW)$$

Cauchy data:

$$u[0] := (u(t=0), \partial_t u(t=0)) = (u_0, u_1)$$

- By differentiation (DNLW) turns into (NLW)
- One may add a source $N(u, \partial u)$ on the right.

The local well-posedness question

[Enhanced] Hadamard local well-posedness in Sobolev spaces

$$u[0] \in \mathcal{H}^s := H^s \times H^{s-1}$$

- existence of solutions u in the class $C(0, T; \mathcal{H}^s)$
- uniqueness of solutions, either directly or as unique limits of smooth solutions
- continuous dependence in $\mathcal{H}^s$, i.e. continuity of the data to solution map

$$\mathcal{H}^s \ni u(0) \to u \in C(0, T; \mathcal{H}^s)$$

- weak Lipschitz dependence, i.e. for two $\mathcal{H}^s$ solutions u and v we have the difference bound

$$\|u - v\|_{C(0,T;\mathcal{H}^{s_0})} \lesssim \|u(0) - v(0)\|_{\mathcal{H}^{s_0}}$$

$s_0 = 1$

$s_0 \in [\frac{1}{2}, 1]$

Related question : long time behavior

- Extended lifespan of solutions for small data

Low regularity well-posedness: the scaling threshold

What is the lowest value of s for which an equation is Hadamard well-posed for initial data $u(0)$ in the Sobolev space $\mathcal{H}^s$?

Scaling symmetry:

$$(NLW): \qquad u(x,t) \to u(\lambda x, \lambda t)$$

$$(DNLW): \qquad u(x,t) \to \lambda^{-1} u(\lambda x, \lambda t)$$

Scaling threshold: Critical Sobolev space

$$s_c = \frac{n}{2} \; (NLW), \qquad s_c = \frac{n}{2} + 1 \; (DNLW)$$

Open question: Are nonlinear wave equations well-posed in $\mathcal{H}^s$ for all $s > s_c$?

Energy estimates and the classical threshold

Theorem (Hughes-Kato-Marsden ('76))

Nonlinear wave equations are locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + 1.$$

Classical energy estimates for (NLW):

$$\frac{d}{dt} E^\sigma(u) \lesssim \underbrace{\|\nabla u\|_{L^\infty}}_{\text{control parameter}} E^\sigma(u), \qquad E^\sigma(u[t]) \approx \|u[t]\|^2_{\mathcal{H}^\sigma} \qquad \sigma \geq 1.$$

or by Gronwall

$$\|u[t]\|^2_{\mathcal{H}^\sigma} \lesssim e^{\int_0^t \|\nabla u(t_1)\|_{L^\infty} dt_1} \|u[0]\|^2_{\mathcal{H}^\sigma}$$

- Sobolev embeddings for $s > s_c + 1$: $\|\nabla u(t)\|_{L^\infty} \lesssim \|u[t]\|_{\mathcal{H}^s}$.
- Continuation criteria: Solutions can be continued as long as

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt < \infty. \qquad \nabla u \in L^1_t L^\infty_x$$

Strichartz estimates and improved LWP

Strichartz estimates for $\Box u = 0$:

$$\|\nabla u\|_{L^2 L^\infty} \lesssim \|u[0]\|_{\mathcal{H}^{\frac{n}{2}+\frac{1}{2}}}, \qquad n \geq 3$$

$$\|\nabla u\|_{L^4 L^\infty} \lesssim \|u[0]\|_{\mathcal{H}^{\frac{n}{2}+\frac{3}{4}}}, \qquad n = 2$$

- If true for $\Box_g$, would give LWP for
  - $s > s_c + \frac{1}{2}$ $(n \geq 3)$
  - $s > s_c + \frac{3}{4}$ $(n \geq 2)$.
  - But we only have $\nabla g \in L^2 L^\infty$! Is this enough ?

Strichartz estimates for variable coefficient metrics:

- $g \in C^\infty$: Lindblad-Sogge, Kapitanskii
- $g \in C^2$: Smith $(n = 2, 3)$
- $\nabla^2 g \in L^1 L^\infty$: T. (all n)
- counterexamples below: Smith-T.

Strichartz estimates with loss of derivatives

Work with paradifferential form of the equation:

$$g^{\alpha\beta}_{<\lambda^\gamma}\partial_\alpha\partial_\beta u_\lambda = \text{perturbative}, \qquad \frac{1}{2} \leq \gamma \leq 1.$$

- For each frequency λ, find "semiclassical" time $\delta t \approx \lambda^{-\delta}$ scales where loss-less Strichartz holds.
- Add these bounds to get Strichartz with derivative losses on unit time.

Multiple iterations:

- Bahouri-Chemin ('98-'99), $s_c + \frac{1}{2} + [\frac{1}{5}, \frac{1}{4}]$
- T. ('98-'99), $s_c + \frac{1}{2} + [\frac{1}{6}, \frac{1}{4}]$

Sharp LWP for generic (NLW)

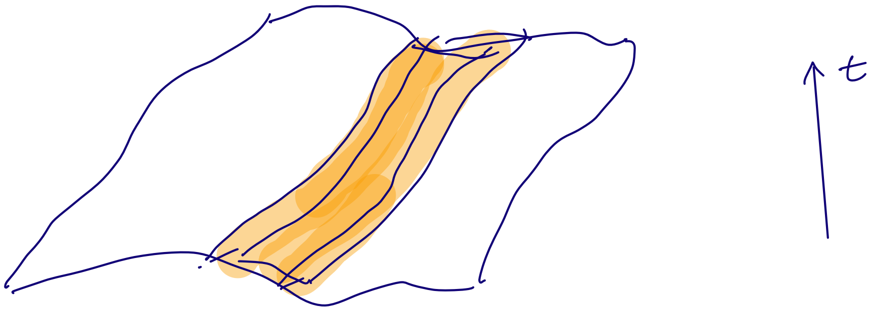
Theorem (Smith-T. ('01))

Nonlinear wave equations are locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + \frac{1}{2} \ (n \geq 3), \qquad s > s_c + \frac{3}{4} \ (n = 2)$$

- Classical energy + (nearly) loss-less Strichartz estimates
- intermediate step, Klainerman-Rodnianski: wave equation for g; then same result for Einstein equation.

This result is sharp (Lindblad (3+1-d)) ! Heuristic: self-interaction of wave packets.



The null condition conjecture (T., ICM '02)

$$g^{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u = 0 \qquad (DNLW)$$

Definition

(DNLW) satisfies the *nonlinear null condition* if

$$\frac{\partial g^{\alpha\beta}(p)}{\partial p_\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0 \qquad \text{in} \qquad g^{\alpha\beta}(p)\xi_\alpha\xi_\beta = 0.$$

- kills the self-interaction of wave packets along null geodesics

Conjecture

The LWP regularity threshold can be improved over the Smith-T. result for nonlinear wave equations satisfying the nonlinear null condition.

Minimal surfaces: nonlinear waves with null condition

- A time-like submanifold $\Sigma \subseteq \mathbb{R}^{n+2}$ of Minkowski space, critical point of

$$\int_\Sigma dA$$

- Euler-Lagrange equation:

$$-\frac{\partial}{\partial t}\left(\frac{u_t}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)=0$$

- Re-express using trace of Minkowski metric on Σ:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta u=0, \qquad g_{\alpha\beta}=m_{\alpha\beta}+\partial_\alpha u\partial_\beta u$$

- aka. Born-Infeld in electromag., aka. zero mean curvature flow, aka. relativistic membrane equation, aka. branes in string theory

LWP for nonlinear waves with null condition

Theorem (Albert Ai, Mihaela Ifrim, D.T. '21)

The time-like minimal surface equation is locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + \frac{1}{4}, \qquad n \geq 3$$

$$s > s_c + \frac{3}{8}, \qquad n = 2$$

- First result proving the null condition LWP conjecture
- improves Smith-T. by 1/4 derivatives if $n \geq 3$, and by 3/8 derivatives if $n = 2$.
- Prior ϵ-removal results by Klainerman-Rodnianski-Szeftel (GR) and Ettinger (minimal surface)

In a nutshell

Classical energy estimates + Loss-less Strichartz estimates → $s > s_{ST}$ for generic problem

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Cubic balanced energy estimates + Strichartz estimates with loss → $s > s_{AIT}$ for null problem

The long time existence problem: baseline

Objective: Lifespan bounds for small data

- Equations with quadratic nonlinearities,

$$u_t + Au = B(u,u)$$

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$, Gronwall provides lifespan $T_\epsilon \approx \epsilon^{-1}$

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- Equations with cubic nonlinearities:

$$u_t + Au = Q(u,u,u)$$

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$, Gronwall provides lifespan $T_\epsilon \approx \epsilon^{-2}$

The long-time existence problem: normal forms

Objective: Improved lifespan for small data

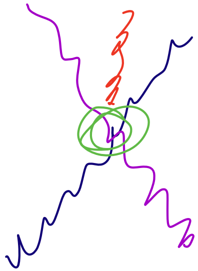
- Normal form method (Shatah '85): Transform equation with quadratic nonlinearities into one with cubic nonlinearities via

$$u \mapsto v = u + B(u, u)$$

New, cubic equation:

$$v_t + Av = Q(u, u, u)$$

  - Requirement: Nonresonant or null resonant quadratic interactions
  - Difficulty: Not invertible for quasilinear problems
- Space-time resonances (Germain-Masmoudi-Shatah '09)



The long-time existence problem: normal forms +

Objective: Improved lifespan for small data, quasilinear problems

- Paradiagonalization (Alazard-Delort '13) Combines a partial normal form with a paradifferential symmetrization

$$v_t + Av + T_{B(u)}v = Q(u,u,u)$$

  - microlocal based approach

- Modified energy method (Hunter-Ifrim-T. '12-'14) Modify the energy functional rather than the unknown:

$$E^s(u) \approx \|u\|_{H^s}^2$$

$$\frac{d}{dt}E^s(u) \lesssim \|u\|^2 E^s(u)$$

  - Modified energy easily computed from normal form transform
  - Work on Burgers-Hilbert, gravity waves, capillary waves, and several other water wave models

Cubic energy estimates for gravity waves

- Alazard-Burq-Zuily '11-15 baseline quadratic estimates:

$$\frac{d}{dt}E^s(u) \lesssim \|u\|_{C^{\frac{1}{2}}} E^s(u)$$

→ C^0 = scaling

- Hunter-Ifrim-T. '14 modified energy, cubic estimates:

$$\frac{d}{dt}E^{s,3}(u) \lesssim A_0 A_{1/2} E^{s,3}(u), \qquad A_\sigma = \|u\|_{BMO^\sigma}$$

→ $\sigma = 0$ = scaling

scale invariant

- Ai-Ifrim-T. '19 modified energy, balanced cubic estimates:

$$\frac{d}{dt}E^{s,3}_{bal}(u) \lesssim A^2_{1/4} E^{s,3}_{bal}(u)$$

  - (variable coeff.) normal form for balanced frequency interactions
  - modified energy for the paradifferential problem

Bony's paradifferential formalism (expanded)

Original nonlinear equation:

$$u_t + N(u) = 0$$

Linearized equation:

$$v_t + DN(u)v = 0$$

Linear paradifferential equation:

$$w_t + T_{DN(u)}w = 0$$

ℓ h

Original equation in paradifferential formulation

$$u_t + T_{DN(u)}u = R(u)$$

Linearized equation in paradifferential formulation

$$v_t + T_{DN(u)}v = R_{lin}(u)v$$

Balanced normal form analysis

$$U_t + F(U) = 0 \qquad \Longleftrightarrow \qquad (\partial_t + \underline{T_{DF(U)})U} = N(U)$$

Terms in $N(U)$:

- Quadratic $Q_2(U,U)$
  - Low-high $Q_2(U_{lo}, U_{hi})$, belongs into the paradiff. part.
  - $Q_2(U_{hi}, U_{hi})$, apply quadratic NFT, turns to cubic.
- Cubic $Q_3(U,U,U)$
  - Low–low-high $Q_3(U_{lo}, U_{lo}, U_{hi})$, goes into the paradiff. part.
  - Low-high-high $Q_3(U_{lo}, U_{hi}, U_{hi})$, apply quadratic NFT with coeff.
  - High-high-high $Q_3(U_{hi}, U_{hi}, U_{hi})$, perturbative.

Further difficulties:

- Also needed for the linearized equation: symmetry loss e.g. in $Q_3(u_{lo}, U_{med}, U_{hi})$, go to quartic order (WW) and/or Strichartz (MS).

Progression of LWP results: gravity waves 2D

		$s - s_c$
Wu ('97)	Energy	4
Alazard-Burq-Zuily ('12)	Energy	$\frac{1}{2} + \epsilon$
Hunter-Ifrim-Tataru ('14)	Cubic energy	$\frac{1}{2}$
Alazard-Burq-Zuily ('14)	Energy + Strichartz w. loss	$\frac{1}{2} - \frac{1}{24} + \epsilon$
Ai ('20) 18	Energy + sharp Strichartz	$\frac{3}{8} + \epsilon$
Ai-Ifrim-Tataru ('20) 19	Balanced cubic energy	$\frac{1}{4}$
(in progress) ('21-22)	Balanced cubic energy + Strichartz	$\frac{1}{8}$ (?)

Balanced energy estimates for minimal surface equation

- Classical energy estimates for (DNLW):

$$\frac{d}{dt}E^s(u) \lesssim \|\partial^2 u\|_{L^\infty} E^s(u)$$

- Cubic energy estimates for minimal surface:

$$\frac{d}{dt}E^s(u) \lesssim \|\partial u\|_{L^\infty} \|\partial^2 u\|_{L^\infty} E^s(u)$$

- Ai-Ifrim-T. '20: Balanced cubic energy estimates,

$$\frac{d}{dt}E^s(u) \lesssim \|\partial u\|^2_{B^{\frac{1}{2}}_{\infty,2}} E^s(u)$$

- Difficulties:
  - Normal form structure is weaker than for water waves
  - Need similar balanced estimates for the linearized equations, further weakening structure

$\mathcal{H}^1$ energy estimates via multiplier method

Covariant wave equation:

$$\Box_g u = 0$$

Energy momentum tensor,

$$T^{\alpha\beta} = \partial^\alpha u \partial^\beta u - \frac{1}{2} g^{\alpha\beta} \partial^\gamma u \partial_\gamma u, \qquad \nabla_\alpha T^{\alpha\beta} = 0$$

Contract with a forward time-like vector field X:

$$\partial_\alpha (T^{\alpha\beta} X_\beta) = T^{\alpha\beta} \nabla_\alpha X_\beta$$

Integrate in x over a time slice:

$$\frac{d}{dt} E_X(u) = \int T^{\alpha\beta} \nabla_\alpha X_\beta \, dx, \qquad E_X(u) = \int T(\partial_t, X) \, dx$$

- we would like to do this paradifferentially.
- we need a good vector field X (or ΨDO, for $\mathcal{H}^s$).

Good vector fields and paracontrolled distributions

Principal symbol for energy flux:

$$c_X(x,\xi) = \{X^\alpha \xi_\alpha, g^{\alpha\beta}\xi_\alpha\xi_\beta\} \quad \simeq g\,\partial X + X\,\partial g$$

$g = g(\partial u)$

Potential choices for X:

- $X = X(\partial u)$, too narrow
- X with same regularity as ∂u, too broad (l.o.t.)
- $X \lll \partial u$ (X paracontrolled by ∂u), optimal choice

Definition

X is paracontrolled by ∂u if it admits a representation

$$X = T_a \partial u + r$$

where a satisfies the same bounds as ∂u, and r is balanced.

- **Key argument:** construction of X, inductively with respect to dyadic frequency scales, conceptually similar to the wave map renormalization in work of Tao, T.

Strichartz estimates ($n \geq 3$)

a) Without losses '01:

$$\|\nabla^2 u\|_{L^2 L^\infty} \lesssim \|u\|_{L^\infty \mathcal{H}^s}, \qquad s > s_{ST}$$

b) With losses, but at lower regularity '21:

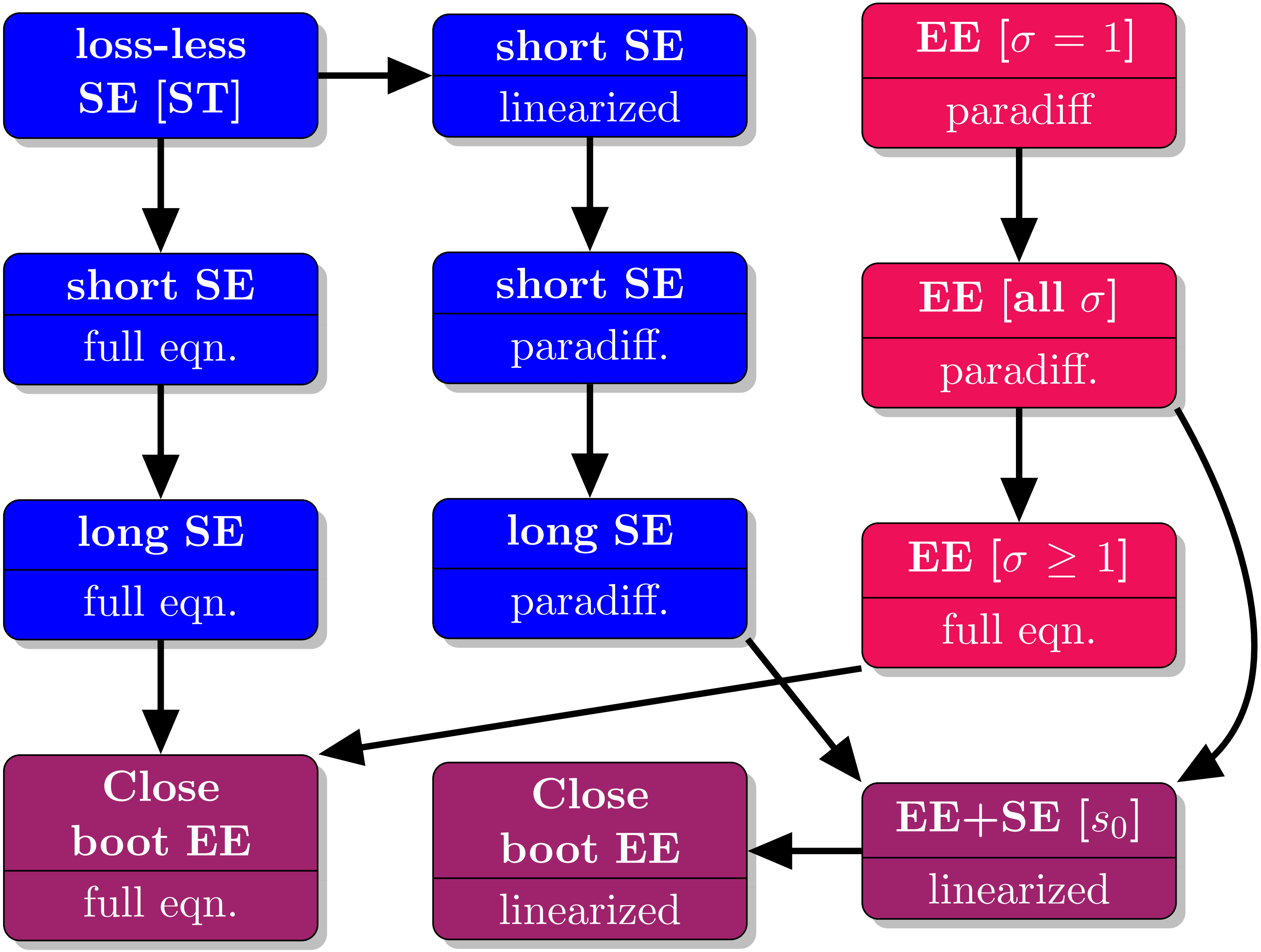
$$\|\nabla^{\frac{3}{2}} u\|_{L^2 L^\infty} \lesssim \|u\|_{L^\infty \mathcal{H}^s}, \qquad s > s_{AIT}$$

- Also for the linear paradifferential equation and for the linearized equation
- Heuristically (b) follows from (a) at the paradifferential level by rescaling to "semiclassical" scales and time interval summation.
- In practice, not as easy ...

Rough solutions as limits of smooth solutions

- Regularize initial data $u^h[0] = P_{<h} u[0]$.
- Consider associated solutions u^h, and increments $v^h = \frac{d}{dh} u^h$ which solve the associated linearized equation.
- Bootstrap argument for the energy bounds for both u^h (in $\mathcal{H}^s$) and v^h (in $\mathcal{H}^{s_0}$).
    - Use the Strichartz estimates of [ST] for u^h and v^h on short time scales, add up to unit scale taking losses.
    - Use this to close our unit time energy estimates for u^h.
    - Combine to obtain unit time Strichartz estimates for linearized equation.
    - Use this to close the energy estimates for v^h.
    - Obtain solution $u = \lim_{h \to \infty} u^h$, convergence in strong topology.
- Wrap the entire argument in frequency envelopes [see primer on quasilinear LWP via frequency envelopes, Ifrim-T '20]

A full diagram



Progression of results: Nonlinear wave equation 3D

		$s - s_c$
Hughes-Kato-Marsden ('76)	Energy	$1 + \epsilon$
Bahouri-Chemin ('98-'99)	Energy + Strichartz w. loss	$\frac{3}{4}$ to $\frac{7}{10}$
Tataru ('98-'99)	Energy + Strichartz w. loss	$\frac{3}{4}$ to $\frac{2}{3}$
Klainerman-Rodnianski ('00)	Energy + Strichartz w. loss	$\frac{3-\sqrt{3}}{2}$
Smith-Tataru ('01) → also K-R [GR]	Energy + sharp Strichartz	$\frac{1}{2} + \epsilon$
Kl.-Rod.-Szeftel ('15) $[GR]$ also Ettinger [MS]	Energy + sharp Strichartz	$\frac{1}{2}$
Ai-Ifrim-Tataru ('21) $[null]$	Balanced cubic energy + Strichartz w. loss	$\frac{1}{4} + \epsilon$

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Thank you!