High-Order Rogue Waves and Solitons and Solutions Interpolating Between Them

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Introduction

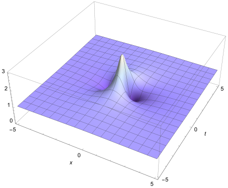
A *rogue wave* is a space-time localized burst of wave amplitude, conventionally exceeding the significant wave amplitude[1] by a factor of 2.2.

An asymptotic model for slowly-varying modulations of small-amplitude water waves is the focusing nonlinear Schrödinger equation

$$\mathrm{i}q_t + \frac{1}{2}q_{xx} + |q|^2 q = 0.$$

After some normalization, a periodic wavetrain (Stokes wave) is represented in this model by the exact solution $q(x,t) = \mathrm{e}^{\mathrm{i}t}$.

In this setting, the most basic model for a rogue wave is the *Peregrine solution*



$$q(x,t) = \mathrm{e}^{\mathrm{i}t}\left[1 - 4\frac{1+2\mathrm{i}t}{1+4x^2+4t^2}\right].$$

[1]four times the standard deviation of surface height for ocean waves

Frequent “Mad Dog Waves”

Never turn your back to the ocean. . .

Kaohsiung Harbor, Taiwan

Bar Harbor, Maine

Introduction

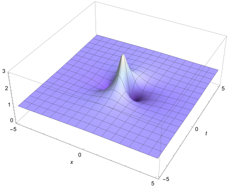
Using integrability, many generalizations of the Peregrine solution have been obtained:

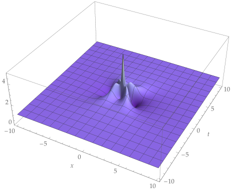
- Each of them retains the essential property of decaying to the background Stokes wave as $(x, t) \to \infty$.
- These solutions have typically been found by algebraic methods, leading to expressions in terms of determinants of arbitrary dimension — the *order* of the rogue wave.
- At each order, new parameters enter into the determinantal expressions. Distinguished limits in the parameter space are of particular interest:
    - In one limit, the rogue wave of order k resembles a triangular number of distant isolated copies of the Peregrine solution on the same background; upon rescaling, the peak locations converge to points determined by the roots of the Yablonskii-Vorob'ev polynomials (Yang-Yang, 2021).
    - In another limit, the peaks all combine at one point, forming a peak of large amplitude. This is the *fundamental rogue wave of order k*.

Introduction

In the interest of studying rogue waves with extreme amplitude, one should:

- Consider fundamental rogue waves, as they produce a single large-amplitude peak for each order k;
- Increase the order.





Considering the limit as $k \to \infty$ is difficult via $k \times k$ determinants. However, thanks to a 2×2 Riemann-Hilbert representation of these solutions in which k appears as an explicit parameter (Bilman-M, 2019), an effective large-k analysis has become possible.

Riemann-Hilbert representation

Let $\Sigma_\circ$ denote the circle $|\lambda| = \rho > 1$, let $\mathbf{G}$ be a 2×2 matrix with $\det(\mathbf{G}) = 1$ and $\mathbf{G}^* = \sigma_2 \mathbf{G} \sigma_2$, let $M \geq 0$ be real, and let $(x, t) \in \mathbb{R}^2$.

RH Problem

Seek a 2×2 matrix-valued function $\mathbf{P}(\lambda) = \mathbf{P}(\lambda; x, t, \mathbf{G}, M)$ with the following properties:

Analyticity: *$\mathbf{P}(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \Sigma_\circ$, and it takes continuous boundary values $\mathbf{P}_+/\mathbf{P}_-$ on $\Sigma_\circ$ from without/within;*

Jump condition: *The boundary values on $\Sigma_\circ$ are related by*

$$\mathbf{P}_+(\lambda) = \mathbf{P}_-(\lambda) \mathrm{e}^{-\mathrm{i}(\lambda x + \lambda^2 t)\sigma_3} \beta(\lambda)^{M\sigma_3} \mathbf{G} \beta(\lambda)^{-M\sigma_3} \mathrm{e}^{\mathrm{i}(\lambda x + \lambda^2 t)\sigma_3},$$

where $\beta(\lambda)$ denotes the Blaschke factor

$$\beta(\lambda) := \frac{\lambda - \mathrm{i}}{\lambda + \mathrm{i}};$$

Normalization: *$\mathbf{P}(\lambda) \to \mathbb{I}$ as $\lambda \to \infty$.*

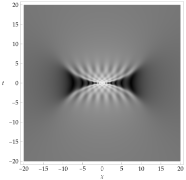
Rogue waves and more

A dressing argument shows that the function

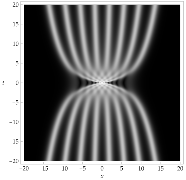
$$q(x,t;\mathbf{G},M) := 2\mathrm{i}\lim_{\lambda\to\infty} \lambda P_{12}(\lambda;x,t,\mathbf{G},M)$$

is a solution of $\mathrm{i}q_t + \frac{1}{2}q_{xx} + |q|^2 q = 0$.

A fundamental rogue wave of order k is obtained by restricting to $M = \frac{1}{4} + \frac{1}{2}k$, $k \in \mathbb{Z}_{\geq 0}$. If $M = \frac{1}{4}$, this is the background Stokes wave $q(x,t) = \mathrm{e}^{\mathrm{i}t}$.



The same RHP also encodes multi-soliton solutions: when $M = \frac{1}{2}k$, $k \in \mathbb{Z}_{\geq 0}$, $q(x,t;\mathbf{G},M)$ is a kth order pole soliton solution with conjugate eigenvalues $\lambda = \pm\mathrm{i}$. If $M = 0$, this is the trivial solution $q(x,t) \equiv 0$.



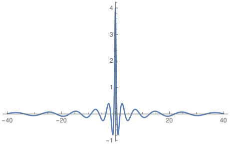
So rogue waves are "half-solitons" and solitons are "half-rogue waves". And there are many solutions continuously interpolating in between.

Common Features: The Near-Field Limit

Because they can be placed within the common analytic framework of the RHP for $\mathbf{P}$, the solutions $q(x, t; \mathbf{G}, M)$ have interesting common features in the limit $M \to \infty$, regardless of whether q is a soliton, a rogue wave, or something in between.

For instance, the following near-field limit exists:

$$\Psi(X, T; \mathbf{G}) = \lim_{M\to\infty} M^{-1} q(M^{-1}X, M^{-2}T; \mathbf{G}, M).$$

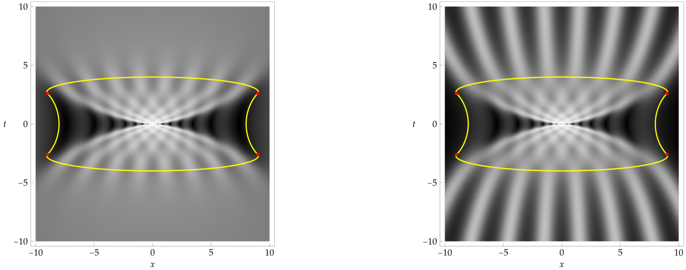


It is a solution of the NLS equation in the form $\mathrm{i}\Psi_T + \frac{1}{2}\Psi_{XX} + |\Psi|^2\Psi = 0$ that is "isomonodromic" in that it satisfies also ordinary differential equations in the Painlevé-III hierarchy with respect to X and T separately. See Bilman-Ling-M (2020) for the rogue-wave case and Bilman-Buckingham (2019) for the soliton case. The generalization to arbitrary M is given in Bilman-M (2021).

Common Features: The Far-Field Regime

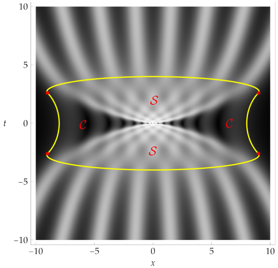
The same isomonodromic solution has recently been derived also in the semiclassical limit of focusing NLS as well as in boundary-layer theory for the Maxwell-Bloch system. See Suleimanov (2017), Li-M (2021), and Buckingham-Jenkins-M (in preparation).

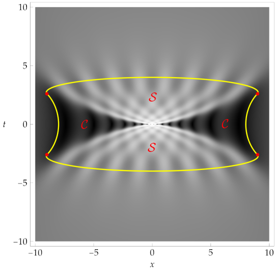
Other features common to all of the solutions as $M \to \infty$ occur in the *far-field regime*. There is a bounded but expanding region of the (x, t)-plane of size proportional to M on which $q(x, t; \mathbf{G}, M)$ exhibits behavior independent of the "type" of solution:



Common Features: The Far-Field Regime

The expanding region of common far-field behavior consists of two components: the "channels" $\mathcal{C}$, and the "shelves" $\mathcal{S}$.





The channels and shelves are fixed and bounded domains in the plane of the rescaled variables

$$(\chi, \tau) = (M^{-1}x, M^{-1}t).$$

Common Features: The Channels in the Far-Field

For $(\chi, \tau) \in \mathcal{C}$, there are two critical points $a(\chi, \tau) < b(\chi, \tau)$ of

$$\vartheta(\lambda; \chi, \tau) := \chi\lambda + \tau\lambda^2 + \mathrm{i}\log(\beta(\lambda)).$$

Theorem (Bilman-M 2021, Theorem 1.5)

Let $s = \pm 1$ be arbitrary. Then as $M \to +\infty$ through any sequence of values,

$$q(M\chi, M\tau; \mathbf{Q}^{-s}, M) = \mathfrak{L}_s^{[\mathcal{C}]}(\chi, \tau; M) + O(M^{-\frac{3}{2}}), \quad (\chi, \tau) \in \mathcal{C}, \quad \textit{where}$$

$$\mathfrak{L}_s^{[\mathcal{C}]}(\chi, \tau; M) := sM^{-\frac{1}{2}} \left[F_a^{[\mathcal{C}]}(\chi, \tau) \mathrm{e}^{\mathrm{i}\Theta_a^{[\mathcal{C}]}(\chi,\tau;M)} + F_b^{[\mathcal{C}]}(\chi, \tau) \mathrm{e}^{\mathrm{i}\Theta_b^{[\mathcal{C}]}(\chi,\tau;M)} \right],$$

$$\Theta_a^{[\mathcal{C}]}(\chi, \tau; M) := M\Phi_a^{[\mathcal{C}]}(\chi, \tau) - \ln(M)\frac{\ln(2)}{2\pi} + \eta_a^{[\mathcal{C}]}(\chi, \tau)$$

$$\Theta_b^{[\mathcal{C}]}(\chi, \tau; M) = M\Phi_b^{[\mathcal{C}]}(\chi, \tau) + \ln(M)\frac{\ln(2)}{2\pi} + \eta_b^{[\mathcal{C}]}(\chi, \tau),$$

and where the error term is uniform for (χ, τ) in any compact subset of $\mathcal{C}$.

For the soliton case $M \in \frac{1}{2}\mathbb{Z}$, see Bilman-Buckingham-Wang (2021).

Common Features: The Shelves in the Far-Field

Theorem (Bilman-M 2021, Theorem 1.8)

Let $s = \pm 1$ be arbitrary. Then as $M \to +\infty$ through any sequence of values,

$$q(M\chi, M\tau; \mathbf{Q}^{-s}; M) = \mathfrak{L}_s^{[\mathcal{S}]}(\chi, \tau; M) + \mathfrak{S}_s^{[\mathcal{S}]}(\chi, \tau; M) + O(M^{-1})$$

holds for $(\chi, \tau) \in \mathcal{S}$, where

$$\begin{aligned}\mathfrak{L}_s^{[\mathcal{S}]}(\chi, \tau; M) + \mathfrak{S}_s^{[\mathcal{S}]}(\chi, \tau; M) = s\mathrm{e}^{-2\mathrm{i}\phi(\chi,\tau;M)}\Big[-\mathrm{i}B(\chi, \tau) \\ + M^{-\frac{1}{2}}\Big(m_a^+(\chi,\tau)F_a^{[\mathcal{S}]}(\chi,\tau)\mathrm{e}^{\mathrm{i}\phi_a(\chi,\tau;M)} - m_a^-(\chi,\tau)F_a^{[\mathcal{S}]}(\chi,\tau)\mathrm{e}^{-\mathrm{i}\phi_a(\chi,\tau;M)} \\ + m_b^+(\chi,\tau)F_b^{[\mathcal{S}]}(\chi,\tau)\mathrm{e}^{\mathrm{i}\phi_b(\chi,\tau;M)} - m_b^-(\chi,\tau)F_b^{[\mathcal{S}]}(\chi,\tau)\mathrm{e}^{-\mathrm{i}\phi_b(\chi,\tau;M)}\Big)\Big],\end{aligned}$$

where the error term is uniform for (χ, τ) in any compact subset of $\mathcal{S}$.

The leading term was obtained for the soliton case ($M \in \frac{1}{2}\mathbb{Z}$) in Bilman-Buckingham-Wang (2021).

Common Features: The Shelves in the Far-Field

This result is more complicated because for $(\chi,\tau)\in\mathcal{S}$, the phase $\vartheta(\lambda;\chi,\tau)$ has to be modified by a genus-zero g-function. The modification $h(\lambda;\chi,\tau)$ has critical points $a(\chi,\tau)<b(\chi,\tau)$ and a branch cut with c.c. endpoints $\lambda_0=A(\chi,\tau)\pm\mathrm{i}B(\chi,\tau)$.

The leading term $\mathfrak{L}_s^{[\mathcal{S}]}(\chi,\tau;M)$ is a modulated plane wave of amplitude $B(\chi,\tau)$. The subleading term $\mathfrak{S}_s^{[\mathcal{S}]}(\chi,\tau;M)$ is responsible for the interference pattern visible on plots:

Corollary (Bilman-M 2021, Corollary 1.10)

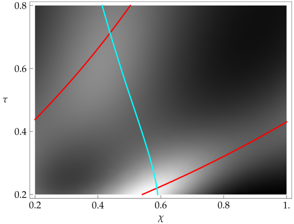
Let $s=\pm1$ be arbitrary. Then as $M\to+\infty$ through any sequence of values,

$$|q(M\chi,M\tau;\mathbf{Q}^{-s},M)|^2=B(\chi,\tau)^2-2M^{-\frac{1}{2}}B(\chi,\tau)$$
$$\times\left[F_a^{[\mathcal{S}]}(\chi,\tau)\sin(\phi_a(\chi,\tau;M))+F_b^{[\mathcal{S}]}(\chi,\tau)\sin(\phi_b(\chi,\tau;M))\right]+O(M^{-1})$$

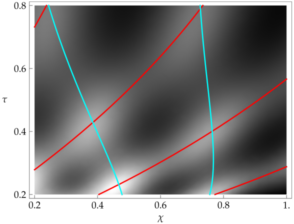
where the error term is uniform for (χ,τ) in compact subsets of $\mathcal{S}$.

Common Features: The Shelves in the Far-Field

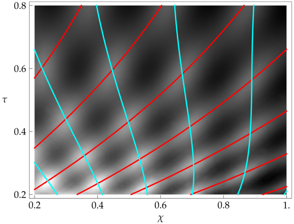
We plot with red/blue curves the maxima of the two sine functions in the correction term, over a common region in $\mathcal{S}$:



RW $k = 4$



RW $k = 8$



RW $k = 16$

Distinctive Features

The exterior of channels $\mathcal{C}$ and shelves $\mathcal{S}$ is a fixed unbounded domain in the (χ, τ)-plane we call $\mathcal{E}$. Multiple-pole solitons and rogue waves behave very differently for $(\chi, \tau) \in \mathcal{E}$ when M is large.

The multiple-pole soliton case ($M \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) was analyzed in Bilman-Buckingham (2019) and Bilman-Buckingham-Wang (2021). We review these results first.

The key exponent in the Riemann-Hilbert problem is

$$\vartheta(\lambda;\chi,\tau) := \chi\lambda + \tau\lambda^2 + \mathrm{i}\log(\beta(\lambda)).$$

When $(\chi,\tau) \in \mathcal{E}$, $\lambda \mapsto \vartheta(\lambda;\chi,\tau)$ has a complex-conjugate pair of critical points; letting Γ denote a Schwarz-symmetric contour joining the critical points and given upwards orientation, the condition

$$\mathrm{Re}\left(\int_\Gamma \mathrm{i}\vartheta'(\lambda;\chi,\tau)\,\mathrm{d}\lambda\right) = 0$$

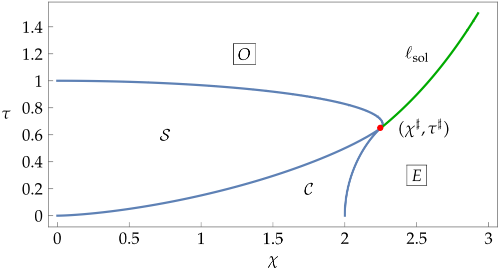
defines a curve ℓ_{sol} emanating from the special point

$$(\chi^\sharp,\tau^\sharp) := \left(\tfrac{9}{4}, \tfrac{3\sqrt{3}}{8}\right)$$

into the first quadrant, along with its reflections in the coordinate axes.

For multi-pole solitons, the exterior $\mathcal{E}$ in the first quadrant is subdivided along ℓ_{sol} into

- the *exponential decay region* E and
- the *oscillatory region* O.



The curve ℓ_{sol} behaves asymptotically like $\chi = \ln(\tau) + O(1)$ as $\tau \to +\infty$.

On the *exponential decay region*, we have the following:

Theorem (Bilman-Buckingham 2019, Theorem 1)

If $(\chi, \tau) \in \mathrm{E} \subset \mathcal{E}$*, then there exists* $d > 0$ *such that*

$$q(M\chi, M\tau; \mathbf{G}, M) = O(\mathrm{e}^{-dM}), \quad \tfrac{1}{2}\mathbb{Z}_{\geq 0} \ni M \to \infty.$$

The constant d *can be assumed fixed on any compact subset of* E*.*

On the *oscillatory region*, we have the following:

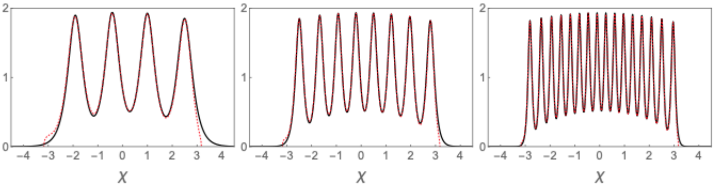
Theorem (Bilman-Buckingham-Wang 2021, Theorem 4)

If $(\chi, \tau) \in \mathrm{O} \subset \mathcal{E}$*, then for* $M \in \frac{1}{2}\mathbb{Z}_{\geq 0}$*,* $q(M\chi, M\tau; \mathbf{G}, M)$ *is approximated to order* $O(M^{-1})$ *by a modulated elliptic function with wavenumber and frequency proportional to* M *and elliptic modulus tending to* 1 *as* $(\chi, \tau) \to \ell_{\mathrm{sol}}$*.*

Multiple-Pole Solitons in the Exterior Domain $\mathcal{E}$

Exponential Decay and Oscillatory Regions

Comparing $|q|$ (dotted red curves) with its elliptic approximation (black curves) for $M = 2, 4, 8$ (multiple-pole soliton of order $k = 4, 8, 16$) for a certain choice of $\mathbf{G}$:



[Taken from Figure 6 of Bilman-Buckingham-Wang (2021).]

That's what happens in the exterior region $\mathcal{E}$ when $q(x, t; \mathbf{G}, M)$ is a multiple-pole soliton solution, i.e., if $M \in \frac{1}{2}\mathbb{Z}$. Next we consider what happens when it is a fundamental rogue wave solution, i.e., if $M \in \frac{1}{2}\mathbb{Z} + \frac{1}{4}$.

If $(\chi,\tau)\in\mathcal{E}\cup\mathcal{S}$, there is a unique real root $u=u(\chi,\tau)$ of odd multiplicity of the septic equation $P(u;\chi,\tau)=0$, where

$$\begin{aligned}P(u;\chi,\tau) := 81u^7 - 189\chi u^6 + (162\chi^2+72\tau^2)u^5 - (66\chi^2+120\tau^2)\chi u^4 \\ + (13\chi^4+56\chi^2\tau^2+16\tau^4+432\tau^2)u^3 \\ - (\chi^4+8\chi^2\tau^2+16\tau^4+432\tau^2)\chi u^2 \\ + 144\chi^2\tau^2 u - 16\chi^3\tau^2.\end{aligned}$$

Then set

$$v(\chi,\tau) := 2\tau\frac{2u(\chi,\tau)-\chi}{3u(\chi,\tau)-\chi}$$

and define $\lambda_0(\chi,\tau)=A(\chi,\tau)+\mathrm{i}B(\chi,\tau)$ with $B(\chi,\tau)>0$ by

$$A(\chi,\tau)=\frac{u(\chi,\tau)-\chi}{2\tau}$$

$$A(\chi,\tau)^2+B(\chi,\tau)^2=\frac{3u(\chi,\tau)^2}{4\tau^2}-\frac{v(\chi,\tau)}{\tau}+2-\frac{\chi u(\chi,\tau)}{\tau^2}+\frac{\chi^2}{4\tau^2}.$$

Let $R(\lambda;\chi,\tau)$ be a certain square root of $(\lambda-\lambda_0(\chi,\tau))(\lambda-\lambda_0(\chi,\tau)^*)$ and define a real phase by

$$\gamma(\chi,\tau):=\chi A(\chi,\tau)+\tau(A(\chi,\tau)^2-\tfrac{1}{2}B(\chi,\tau)^2)+2\mathrm{Re}\left(\mathrm{i}\int_{\lambda_0(\chi,\tau)}^{\mathrm{i}}\frac{\mathrm{d}\lambda}{R(\lambda;\chi,\tau)}\right).$$

For rogue waves it is better to consider $\psi_k(x,t)=\mathrm{e}^{-\mathrm{i}t}q(x,t;\mathbf{G},\frac{1}{4}+\frac{1}{2}k)$ and to choose a specific matrix for $\mathbf{G}$:

$$\mathbf{G}=\mathbf{Q}^{-s},\quad s:=(-1)^k,\quad \mathbf{Q}:=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

Then, the rogue wave has the most symmetry and concentration (making it *fundamental*) and $\psi_k(x,t)$ tends to the constant background $\psi_0(x,t)=1$ as $(x,t)\to\infty$.

In the exterior domain we then have the following result:

Theorem (Bilman-M 2021, Theorem 1.7)

The fundamental rogue wave $\psi_k(x,t)$ of order $k \in \mathbb{Z}_{>0}$ satisfies

$$\psi_k(M\chi, M\tau) = B(\chi,\tau)\mathrm{e}^{-\mathrm{i}M\tau}\mathrm{e}^{-2\mathrm{i}M\gamma(\chi,\tau)} + O(k^{-1}), \quad M = \tfrac{1}{2}k + \tfrac{1}{4},$$

where the error term is uniform for (χ,τ) in compact subsets of $\mathcal{E}$.

When $\tau = 0$, the domain $\mathcal{E}$ becomes $|\chi| > 2$, and this asymptotic formula becomes explicit:

$$\psi_k(M\chi, M\tau) = \sqrt{1 - \frac{4}{\chi^2}} + O(k^{-1}), \quad \mathbb{Z}_{>0} \ni k \to \infty, \quad |\chi| > 2.$$

More generally, one can show that $B(\chi,\tau) \to 1$ and $\gamma(\chi,\tau) + \frac{1}{2}\tau \to 0$ as $(\chi,\tau) \to \infty$, so the leading term reproduces the known decay to $\psi_0(x,t) = 1$ of the exact solution $\psi_k(x,t)$.

The approximating formula $B(\chi,\tau)\mathrm{e}^{-\mathrm{i}M(\tau+2\gamma(\chi,\tau))}$ is a modulated plane wave with slowly varying amplitude $B(\chi,\tau)$, scaled wavenumber $-2\gamma_\chi(\chi,\tau)$, and scaled frequency $1+2\gamma_\tau(\chi,\tau)$. One can check directly that the following holds:

Corollary (Bilman-M 2021, Corollary 1.13)

For $(\chi,\tau)\in\mathcal{E}$, the expressions

$$\rho(\chi,\tau)=B(\chi,\tau)^2 \quad \textit{and} \quad U(\chi,\tau)=-2\gamma_\chi(\chi,\tau)$$

satisfy the dispersionless NLS (hydrodynamic) system

$$\rho_\tau+(\rho U)_\chi=0 \quad \textit{and} \quad U_\tau+(\tfrac{1}{2}U^2-\rho)_\chi=0.$$

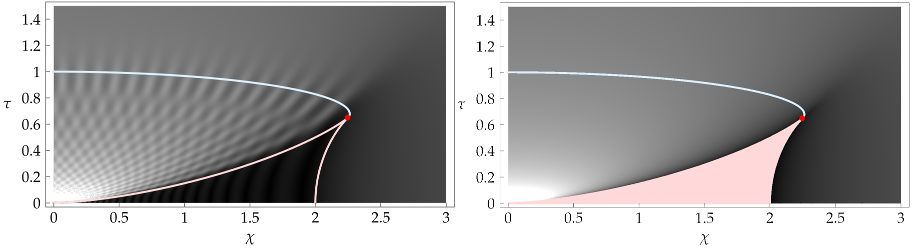
The proof relies on an identity: $\gamma_\chi(\chi,\tau)=A(\chi,\tau)=\mathrm{Re}(\lambda_0(\chi,\tau))$.

Fundamental Rogue Waves in the Exterior Domain $\mathcal{E}$

Modulated plane-wave asymptotics

This extends to the exterior domain $\mathcal{E}$ a result valid for the whole family of solutions (continuous M) on the shelves $\mathcal{S}$. Indeed, the leading term of $\psi_k(M\chi, M\tau)$ on $\mathcal{S}$ is the *same modulated plane-wave* as on $\mathcal{E}$, modified by a M-independent phase factor $\mathrm{e}^{-2\mathrm{i}\mu(\chi,\tau)}$ vanishing on the common boundary curve. However the interesting sub-leading term responsible for the interference pattern of waves visible on $\mathcal{S}$ is not present on $\mathcal{E}$.

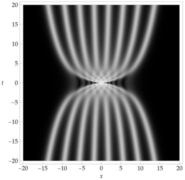
Left: Modulus $|\psi_{32}(M\chi, M\tau)|$. Right: Leading-order amplitude $B(\chi, \tau)$.



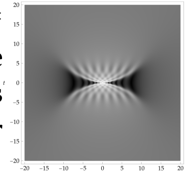
Summary: Solitons and Rogue Waves on $\mathcal{E}$

When $M = \frac{1}{2}k$ for $k \in \mathbb{Z}_{\geq 0}$, $q(x,t;\mathbf{G},M)$ is a k^{th}-order pole soliton solution that satisfies zero boundary conditions $q \to 0$ in x. Its behavior on the exterior domain $\mathcal{E}$ splits into two cases relative to the curve ℓ_{sol} and its axis reflections:

- On the sub-domain $\mathrm{E} \subset \mathcal{E}$, q decays exponentially, consistent with the zero boundary conditions.
- On the sub-domain $\mathrm{O} \subset \mathcal{E}$, q behaves like a modulated elliptic function, consistent with the slowly-diverging trajectories of the k soliton components propagating with common zero velocity.



When $M = \frac{1}{4} + \frac{1}{2}k$ for $k \in \mathbb{Z}_{\geq 0}$, the function $\psi_k(x,t) = \mathrm{e}^{-\mathrm{i}t}q(x,t;\mathbf{Q}^{(-1)^{k+1}},M)$ is the k^{th}-order fundamental rogue wave solution that satisfies nonzero boundary conditions $\psi_k \to 1$ in x. It behaves the same on the whole exterior domain $\mathcal{E}$, like a modulated plane-wave.



In general, we may write the "continuous order" M in modular form with quotient k and remainder r as

$$M = \tfrac{1}{2}k + r, \quad k \in \mathbb{Z}_{\geq 0}, \quad 0 \leq r < \tfrac{1}{2}.$$

Then the soliton case is $r = 0$ and the rogue wave case is $r = \frac{1}{4}$. If $r \neq 0, \frac{1}{4}$, then one might think that the boundary conditions at $x = \infty$ are somewhere "in between." Maybe $|q| \to c \in (0, 1)$ as $|x| \to \infty$? In fact, no:

Theorem (Bilman-M, 2021 (in preparation))

Suppose that $M = \frac{1}{2}k + r$ *with* $k \in \mathbb{Z}_{\geq 0}$ *and* $0 \leq r < \frac{1}{2}$. *If* $r \neq 0$, *then*

$$q(x, 0; \mathbf{Q}^{(-1)^{k+1}}, \tfrac{1}{2}k + r) = 1 + O(|x|^{-\frac{1}{2}}), \quad |x| \to \infty.$$

The error term is sharp unless $r = \frac{1}{4}$ *(rogue-wave case), in which case it becomes* $O(x^{-2})$.

So, *except for the soliton case of* $M \in \frac{1}{2}\mathbb{Z}$, the initial condition satisfies unit amplitude nonzero boundary conditions. In the rogue-wave case the decay to the background is fast enough for scattering theory ($L^1(\mathbb{R})$), but in general the decay is so slow that the difference does not even lie in $L^2(\mathbb{R})$.

Now the question arises: suppose that $r \neq 0, \frac{1}{4}$ is fixed and $k \in \mathbb{Z}_{\geq 0}$ tends to infinity. How might we expect the asymptotic behavior in the exterior domain $\mathcal{E}$ to vary with r in this limit?

Just as the large-x behavior is insensitive to $r \neq 0, \frac{1}{4}$, so is the large-k behavior. To describe it, the curve ℓ_{sol} takes a secondary role compared to another unbounded curve emanating from $(\chi^\sharp, \tau^\sharp)$ denoted by ℓ_{trig} and defined by the condition

$$u(\chi,\tau)^2 - 8\tau v(\chi,\tau) = 0.$$

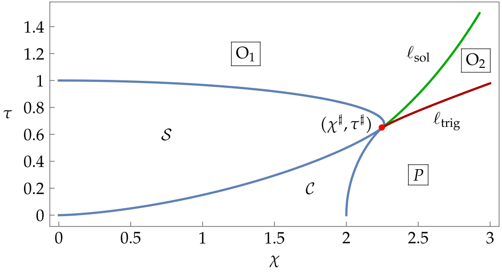
This unbounded curve is asymptotic to the line $\chi = \sqrt{8}\tau$ for large τ; it is another branch of the analytic curve separating $\mathcal{S}$ from $\mathcal{E}$.

Other Solutions on the Exterior Domain $\mathcal{E}$

Large-k behavior

For $r \neq 0, \frac{1}{4}$, the exterior domain $\mathcal{E}$ is divided along ℓ_{sol} and ℓ_{trig} into

- the *plane-wave region* P, and
- two *oscillatory regions* O_1 and O_2.



When $(\chi, \tau) \in \mathrm{P}$, the solution has the same asymptotic behavior as in the case $r = \frac{1}{4}$ (rogue-wave case), and when $(\chi, \tau) \in \mathrm{O}_1$, the solution has the same asymptotic behavior as in the case $r = 0$ (soliton case).

On the remaining domain O_2, the asymptotic behavior for large k is not like either the rogue-wave or soliton cases. It is governed by a leading-order approximation by a modulated elliptic function, with elliptic modulus varying from $m = 0$ on ℓ_{trig} to $m = 1$ on ℓ_{sol}.

Since ℓ_{sol} bends toward the vertical as $\tau \to \infty$ and ℓ_{trig} approaches the asymptote $\chi = \sqrt{8}\tau$, *when* (χ, τ) *are large, the large-M asymptotic of the solution is consistent with the universal long-time asymptotic obtained by Biondini-Mantzavinos (2017) for solutions of the focusing NLS equation with nonzero boundary conditions.* However, the solutions do not satisfy the full hypotheses of that result, which points toward a broader universality.

References

- D. Bilman and R. J. Buckingham, “Large-order asymptotics for multiple-pole solitons of the focusing nonlinear Schrödinger equation,” *J. Nonlinear Sci.* **29**, 2185–2229, 2019.
- D. Bilman, R. J. Buckingham, and D.-S. Wang, “Far-field asymptotics for multiple-pole solitons in the large-order limit,” *J. Diff. Eq.* **297**, 320–369, 2021.
- D. Bilman, L. Ling, and P. D. Miller, “Extreme superposition: rogue waves of infinite order and the Painlevé-III hierarchy,” *Duke Math. J.* **169**, 671–760, 2020.
- D. Bilman and P. D. Miller, “A robust inverse scattering transform for the focusing nonlinear Schrödinger equation,” *Comm. Pure Appl. Math.* **72**, 1722–1805, 2019.
- D. Bilman and P. D. Miller, “Extreme superposition: high-order fundamental rogue waves in the far-field regime,” `arXiv:2103.00337`, 2021.
- G. Biondini and D. Mantzavinos, “Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability,” *Comm. Pure Appl. Math.* **70**, 2300–2365, 2017.
- S. Li and P. D. Miller, “On the Maxwell-Bloch system in the sharp-line limit without solitons,” `arXiv:2105.13293`, 2021.
- B. I. Suleimanov, “Effect of a small dispersion on self-focusing in a spatially one-dimensional case,” *JETP Lett.* **106**, 400–405, 2017.
- B. Yang and J. Yang, “Rogue wave patterns in the nonlinear Schrödinger equation,” *Phys. D* **419**, 132850, 2021.

Thanks for listening! Questions?

Why?

When $(\chi,\tau)\in\mathcal{E}$, we should deform the original circular jump contour $\Sigma_\circ$ into a "dumbbell" shape consisting of a "neck" N oriented upwards and connecting two loops around $\lambda=\pm\mathrm{i}$. When $M=\frac{1}{2}k+r$ and $\mathbf{G}=\mathbf{Q}^{-s}$ with $s=\pm 1$ arbitrary, the jump of $\mathbf{P}(\lambda)=\mathbf{P}(\lambda;M\chi,M\tau,\mathbf{G},M)$ on the neck reads

$$\mathbf{P}_+(\lambda)=\mathbf{P}_-(\lambda)\mathrm{e}^{-\mathrm{i}M\vartheta_-(\lambda;\chi,\tau)\sigma_3}\mathbf{Z}\mathrm{e}^{\mathrm{i}M\vartheta_+(\lambda;\chi,\tau)\sigma_3},\quad \lambda\in N,$$

with "core" jump matrix

$$\mathbf{Z}:=\begin{bmatrix}(-1)^k\cos(2\pi r) & s(-1)^k\mathrm{i}\sin(2\pi r)\\ s(-1)^k\mathrm{i}\sin(2\pi r) & (-1)^k\cos(2\pi r)\end{bmatrix}.$$

The phase $\vartheta(\lambda;\chi,\tau)$ and its possible g-function modifications have nothing to do with r. But, the available algebraic factorizations of $\mathbf{Z}$ depend on which pivots are nonzero, and here one sees the key difference between $r=0$ ($\mathbf{Z}$ diagonal), $r=\frac{1}{4}$ ($\mathbf{Z}$ off-diagonal) and $r\neq 0,\frac{1}{4}$ ($\mathbf{Z}$ a full matrix).