High-Order Rogue Waves and Solitons and Solutions Interpolating Between Them

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A rogue wave is a space-time localized burst of wave amplitude, conventionally exceeding the significant wave amplitude\(^1\) by a factor of 2.2.

An asymptotic model for slowly-varying modulations of small-amplitude water waves is the focusing nonlinear Schrödinger equation

\[
iq_t + \frac{1}{2}q_{xx} + |q|^2q = 0.
\]

After some normalization, a periodic wavetrain (Stokes wave) is represented in this model by the exact solution \(q(x, t) = e^{it}\).

In this setting, the most basic model for a rogue wave is the Peregrine solution

\[
q(x, t) = e^{it} \left[ 1 - 4 \frac{1 + 2it}{1 + 4x^2 + 4t^2} \right].
\]

\(^1\)four times the standard deviation of surface height for ocean waves
Frequent “Mad Dog Waves”
Never turn your back to the ocean…

Kaohsiung Harbor, Taiwan

Bar Harbor, Maine
Using integrability, many generalizations of the Peregrine solution have been obtained:

- Each of them retains the essential property of decaying to the background Stokes wave as \((x, t) \to \infty\).
- These solutions have typically been found by algebraic methods, leading to expressions in terms of determinants of arbitrary dimension — the \(\textit{order}\) of the rogue wave.
- At each order, new parameters enter into the determinantal expressions. Distinguished limits in the parameter space are of particular interest:
  
  - In one limit, the rogue wave of order \(k\) resembles a triangular number of distant isolated copies of the Peregrine solution on the same background; upon rescaling, the peak locations converge to points determined by the roots of the Yablonskii-Vorob’ev polynomials (Yang-Yang, 2021).
  
  - In another limit, the peaks all combine at one point, forming a peak of large amplitude. This is the \textit{fundamental rogue wave of order \(k\)}.  

In the interest of studying rogue waves with extreme amplitude, one should:

- Consider fundamental rogue waves, as they produce a single large-amplitude peak for each order $k$;
- Increase the order.

Considering the limit as $k \to \infty$ is difficult via $k \times k$ determinants. However, thanks to a $2 \times 2$ Riemann-Hilbert representation of these solutions in which $k$ appears as an explicit parameter (Bilman-M, 2019), an effective large-$k$ analysis has become possible.
Riemann-Hilbert representation

Let $\Sigma_\circ$ denote the circle $|\lambda| = \rho > 1$, let $\mathbf{G}$ be a $2 \times 2$ matrix with $\det(\mathbf{G}) = 1$ and $\mathbf{G}^* = \sigma_2 \mathbf{G} \sigma_2$, let $M \geq 0$ be real, and let $(x, t) \in \mathbb{R}^2$.

RH Problem

Seek a $2 \times 2$ matrix-valued function $\mathbf{P}(\lambda) = \mathbf{P}(\lambda; x, t, \mathbf{G}, M)$ with the following properties:

**Analyticity:** $\mathbf{P}(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \Sigma_\circ$, and it takes continuous boundary values $\mathbf{P}_+/\mathbf{P}_-$ on $\Sigma_\circ$ from without/within;

**Jump condition:** The boundary values on $\Sigma_\circ$ are related by

$$
\mathbf{P}_+(\lambda) = \mathbf{P}_-(\lambda)e^{-i(\lambda x + \lambda^2 t)\sigma_3}\beta(\lambda)^M\mathbf{G}\beta(\lambda)^{-M}\sigma_3 e^{i(\lambda x + \lambda^2 t)\sigma_3},
$$

where $\beta(\lambda)$ denotes the Blaschke factor

$$
\beta(\lambda) := \frac{\lambda - i}{\lambda + i};
$$

**Normalization:** $\mathbf{P}(\lambda) \to \mathbb{I}$ as $\lambda \to \infty$. 
Rogue waves and more

A dressing argument shows that the function

\[ q(x, t; G, M) := 2i \lim_{\lambda \to \infty} \lambda P_{12}(\lambda; x, t, G, M) \]

is a solution of \( i q_t + \frac{1}{2} q_{xx} + |q|^2 q = 0. \)

A fundamental rogue wave of order \( k \) is obtained by restricting to \( M = \frac{1}{4} + \frac{1}{2} k, k \in \mathbb{Z}_{\geq 0} \). If \( M = \frac{1}{4} \), this is the background Stokes wave \( q(x, t) = e^{it} \).

The same RHP also encodes multi-soliton solutions: when \( M = \frac{1}{2} k, k \in \mathbb{Z}_{\geq 0} \), \( q(x, t; G, M) \) is a \( k \)th order pole soliton solution with conjugate eigenvalues \( \lambda = \pm i \). If \( M = 0 \), this is the trivial solution \( q(x, t) \equiv 0 \).

So rogue waves are “half-solitons” and solitons are “half-rogue waves”. And there are many solutions continuously interpolating in between.
Common Features: The Near-Field Limit

Because they can be placed within the common analytic framework of the RHP for $P$, the solutions $q(x, t; G, M)$ have interesting common features in the limit $M \to \infty$, regardless of whether $q$ is a soliton, a rogue wave, or something in between.

For instance, the following near-field limit exists:

$$\Psi(X, T; G) = \lim_{M \to \infty} M^{-1} q(M^{-1} X, M^{-2} T; G, M).$$

It is a solution of the NLS equation in the form

$$i\Psi_T + \frac{1}{2} \Psi_{XX} + |\Psi|^2 \Psi = 0$$

that is “isomonodromic” in that it satisfies also ordinary differential equations in the Painlevé-III hierarchy with respect to $X$ and $T$ separately. See Bilman-Ling-M (2020) for the rogue-wave case and Bilman-Buckingham (2019) for the soliton case. The generalization to arbitrary $M$ is given in Bilman-M (2021).
Common Features: The Far-Field Regime

The same isomonodromic solution has recently been derived also in the semiclassical limit of focusing NLS as well as in boundary-layer theory for the Maxwell-Bloch system. See Suleimanov (2017), Li-M (2021), and Buckingham-Jenkins-M (in preparation).

Other features common to all of the solutions as $M \to \infty$ occur in the far-field regime. There is a bounded but expanding region of the $(x, t)$-plane of size proportional to $M$ on which $q(x, t; G, M)$ exhibits behavior independent of the “type” of solution:
Common Features: The Far-Field Regime

The expanding region of common far-field behavior consists of two components: the “channels” $\mathcal{C}$, and the “shelves” $\mathcal{S}$.

The channels and shelves are fixed and bounded domains in the plane of the rescaled variables

$$(\chi, \tau) = (M^{-1}x, M^{-1}t).$$
Common Features: The Channels in the Far-Field

For \((\chi, \tau) \in \mathcal{C}\), there are two critical points \(a(\chi, \tau) < b(\chi, \tau)\) of

\[
\vartheta(\lambda; \chi, \tau) := \chi \lambda + \tau \lambda^2 + i \log(\beta(\lambda)).
\]

**Theorem (Bilman-M 2021, Theorem 1.5)**

Let \(s = \pm 1\) be arbitrary. Then as \(M \to +\infty\) through any sequence of values,

\[
q(M\chi, M\tau; Q^{-s}, M) = \mathcal{L}_s^{[\mathcal{C}]}(\chi, \tau; M) + O(M^{-\frac{3}{2}}), \quad (\chi, \tau) \in \mathcal{C},
\]

where

\[
\mathcal{L}_s^{[\mathcal{C}]}(\chi, \tau; M) := sM^{-\frac{1}{2}} \left[ F_a^{[\mathcal{C}]}(\chi, \tau) e^{i\Theta_a^{[\mathcal{C}]}(\chi, \tau; M)} + F_b^{[\mathcal{C}]}(\chi, \tau) e^{i\Theta_b^{[\mathcal{C}]}(\chi, \tau; M)} \right],
\]

\[
\Theta_a^{[\mathcal{C}]}(\chi, \tau; M) := M\Phi_a^{[\mathcal{C}]}(\chi, \tau) - \ln(M) \frac{\ln(2)}{2\pi} + \eta_a^{[\mathcal{C}]}(\chi, \tau)
\]

\[
\Theta_b^{[\mathcal{C}]}(\chi, \tau; M) = M\Phi_b^{[\mathcal{C}]}(\chi, \tau) + \ln(M) \frac{\ln(2)}{2\pi} + \eta_b^{[\mathcal{C}]}(\chi, \tau),
\]

and where the error term is uniform for \((\chi, \tau)\) in any compact subset of \(\mathcal{C}\).

For the soliton case \(M \in \frac{1}{2} \mathbb{Z}\), see Bilman-Buckingham-Wang (2021).
Theorem (Bilman-M 2021, Theorem 1.8)

Let $s = \pm 1$ be arbitrary. Then as $M \to +\infty$ through any sequence of values,

$$q(M\chi, M\tau; Q^{-s}; M) = \mathcal{L}_s^{[S]}(\chi, \tau; M) + \mathcal{G}_s^{[S]}(\chi, \tau; M) + O(M^{-1})$$

holds for $(\chi, \tau) \in S$, where

$$\mathcal{L}_s^{[S]}(\chi, \tau; M) + \mathcal{G}_s^{[S]}(\chi, \tau; M) = se^{-2i\phi(\chi, \tau; M)} \left[ -iB(\chi, \tau) 
+ M^{-\frac{1}{2}} \left( m_a^+(\chi, \tau)F_a^{[S]}(\chi, \tau)e^{i\phi_a(\chi, \tau; M)} - m_a^-(\chi, \tau)F_a^{[S]}(\chi, \tau)e^{-i\phi_a(\chi, \tau; M)} 
+ m_b^+(\chi, \tau)F_b^{[S]}(\chi, \tau)e^{i\phi_b(\chi, \tau; M)} - m_b^-(\chi, \tau)F_b^{[S]}(\chi, \tau)e^{-i\phi_b(\chi, \tau; M)} \right) \right],$$

where the error term is uniform for $(\chi, \tau)$ in any compact subset of $S$.

The leading term was obtained for the soliton case ($M \in \frac{1}{2}\mathbb{Z}$) in Bilman-Buckingham-Wang (2021).
Common Features: The Shelves in the Far-Field

This result is more complicated because for \((\chi, \tau) \in S\), the phase \(\vartheta(\lambda; \chi, \tau)\) has to be modified by a genus-zero \(g\)-function. The modification \(h(\lambda; \chi, \tau)\) has critical points \(a(\chi, \tau) < b(\chi, \tau)\) and a branch cut with c.c. endpoints \(\lambda_0 = A(\chi, \tau) \pm iB(\chi, \tau)\).

The leading term \(S_{s}^\big[S\big](\chi, \tau; M)\) is a modulated plane wave of amplitude \(B(\chi, \tau)\). The subleading term \(\mathcal{S}_{s}^\big[S\big](\chi, \tau; M)\) is responsible for the interference pattern visible on plots:

**Corollary (Bilman-M 2021, Corollary 1.10)**

Let \(s = \pm 1\) be arbitrary. Then as \(M \rightarrow +\infty\) through any sequence of values,

\[
|q(M\chi, M\tau; \mathbf{Q}^{-s}, M)|^2 = B(\chi, \tau)^2 - 2M^{-\frac{1}{2}}B(\chi, \tau)
\]

\[
\times \left[ F_{a}^\big[S\big](\chi, \tau) \sin(\phi_{a}(\chi, \tau; M)) + F_{b}^\big[S\big](\chi, \tau) \sin(\phi_{b}(\chi, \tau; M)) \right] + O(M^{-1})
\]

where the error term is uniform for \((\chi, \tau)\) in compact subsets of \(S\).
Common Features: The Shelves in the Far-Field

We plot with red/blue curves the maxima of the two sine functions in the correction term, over a common region in $S$:

- $RW_k = 4$
- $RW_k = 8$
- $RW_k = 16$
Distinctive Features

The exterior of channels $C$ and shelves $S$ is a fixed unbounded domain in the $(\chi, \tau)$-plane we call $E$. Multiple-pole solitons and rogue waves behave very differently for $(\chi, \tau) \in E$ when $M$ is large.

The multiple-pole soliton case ($M \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) was analyzed in Bilman-Buckingham (2019) and Bilman-Buckingham-Wang (2021). We review these results first.
Multiple-Pole Solitons in the Exterior Domain $\mathcal{E}$

The boundary curve $\ell_{\text{sol}} \subset \mathcal{E}$

The key exponent in the Riemann-Hilbert problem is

$$\vartheta(\lambda; \chi, \tau) := \chi \lambda + \tau \lambda^2 + i \log(\beta(\lambda)).$$

When $(\chi, \tau) \in \mathcal{E}$, $\lambda \mapsto \vartheta(\lambda; \chi, \tau)$ has a complex-conjugate pair of critical points; letting $\Gamma$ denote a Schwarz-symmetric contour joining the critical points and given upwards orientation, the condition

$$\operatorname{Re} \left( \int_{\Gamma} i\vartheta'(\lambda; \chi, \tau) \, d\lambda \right) = 0$$

defines a curve $\ell_{\text{sol}}$ emanating from the special point

$$(\chi^\#, \tau^#) := \left(\frac{9}{4}, \frac{3\sqrt{3}}{8}\right)$$

into the first quadrant, along with its reflections in the coordinate axes.
For multi-pole solitons, the exterior $\mathcal{E}$ in the first quadrant is subdivided along $\ell_{\text{sol}}$ into

- the *exponential decay region* $E$ and
- the *oscillatory region* $O$.

The curve $\ell_{\text{sol}}$ behaves asymptotically like $\chi = \ln(\tau) + O(1)$ as $\tau \to +\infty$. 
On the **exponential decay region**, we have the following:

**Theorem (Bilman-Buckingham 2019, Theorem 1)**

If \((\chi, \tau) \in E \subset \mathcal{E}\), then there exists \(d > 0\) such that

\[
q(M\chi, M\tau; G, M) = O(e^{-dM}), \quad \frac{1}{2}\mathbb{Z}_{\geq 0} \ni M \to \infty.
\]

The constant \(d\) can be assumed fixed on any compact subset of \(E\).

On the **oscillatory region**, we have the following:

**Theorem (Bilman-Buckingham-Wang 2021, Theorem 4)**

If \((\chi, \tau) \in O \subset \mathcal{E}\), then for \(M \in \frac{1}{2}\mathbb{Z}_{\geq 0}\), \(q(M\chi, M\tau; G, M)\) is approximated to order \(O(M^{-1})\) by a modulated elliptic function with wavenumber and frequency proportional to \(M\) and elliptic modulus tending to 1 as \((\chi, \tau) \to \ell_{\text{sol}}\).
That’s what happens in the exterior region $\mathcal{E}$ when $q(x, t; G, M)$ is a multiple-pole soliton solution, i.e., if $M \in \frac{1}{2} \mathbb{Z}$. Next we consider what happens when it is a fundamental rogue wave solution, i.e., if $M \in \frac{1}{2} \mathbb{Z} + \frac{1}{4}$. 

[Taken from Figure 6 of Bilman-Buckingham-Wang (2021).]
The septic polynomial $P(u; \chi, \tau)$

If $(\chi, \tau) \in \mathcal{E} \cup S$, there is a unique real root $u = u(\chi, \tau)$ of odd multiplicity of the septic equation $P(u; \chi, \tau) = 0$, where

$$P(u; \chi, \tau) := 81u^7 - 189\chi u^6 + (162\chi^2 + 72\tau^2)u^5 - (66\chi^2 + 120\tau^2)\chi u^4$$

$$+ (13\chi^4 + 56\chi^2\tau^2 + 16\tau^4 + 432\tau^2)u^3$$

$$- (\chi^4 + 8\chi^2\tau^2 + 16\tau^4 + 432\tau^2)\chi u^2$$

$$+ 144\chi^2\tau^2u - 16\chi^3\tau^2.$$ 

Then set

$$\nu(\chi, \tau) := 2\tau \frac{2u(\chi, \tau) - \chi}{3u(\chi, \tau) - \chi}$$

and define $\lambda_0(\chi, \tau) = A(\chi, \tau) + iB(\chi, \tau)$ with $B(\chi, \tau) > 0$ by

$$A(\chi, \tau) = \frac{u(\chi, \tau) - \chi}{2\tau}$$

$$A(\chi, \tau)^2 + B(\chi, \tau)^2 = \frac{3u(\chi, \tau)^2}{4\tau^2} - \frac{\nu(\chi, \tau)}{\tau} + 2 - \frac{\chi u(\chi, \tau)}{\tau^2} + \frac{\chi^2}{4\tau^2}.$$
Let $R(\lambda; \chi, \tau)$ be a certain square root of $(\lambda - \lambda_0(\chi, \tau))(\lambda - \lambda_0(\chi, \tau)^* )$ and define a real phase by

$$
\gamma(\chi, \tau) := \chi A(\chi, \tau) + \tau(A(\chi, \tau)^2 - \frac{1}{2} B(\chi, \tau)^2) + 2\text{Re}\left( i \int_{\lambda_0(\chi, \tau)}^{i} \frac{d\lambda}{R(\lambda; \chi, \tau)} \right).
$$

For rogue waves it is better to consider $\psi_k(x, t) = e^{-it} q(x, t; G, \frac{1}{4} + \frac{1}{2} k)$ and to choose a specific matrix for $G$:

$$
G = Q^{-s}, \quad s := (-1)^k, \quad Q := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
$$

Then, the rogue wave has the most symmetry and concentration (making it fundamental) and $\psi_k(x, t)$ tends to the constant background $\psi_0(x, t) = 1$ as $(x, t) \to \infty$. 

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**Fundamental Rogue Waves in the Exterior Domain $E$**

Phase $\gamma$ and fundamental rogue waves
In the exterior domain we then have the following result:

**Theorem (Bilman-M 2021, Theorem 1.7)**

The fundamental rogue wave $\psi_k(x,t)$ of order $k \in \mathbb{Z}_{>0}$ satisfies

$$
\psi_k(M\chi, M\tau) = B(\chi, \tau) e^{-iM\tau} e^{-2iM\gamma(\chi,\tau)} + O(k^{-1}), \quad M = \frac{1}{2}k + \frac{1}{4},
$$

where the error term is uniform for $(\chi, \tau)$ in compact subsets of $\mathcal{E}$.

When $\tau = 0$, the domain $\mathcal{E}$ becomes $|\chi| > 2$, and this asymptotic formula becomes explicit:

$$
\psi_k(M\chi, M\tau) = \sqrt{1 - \frac{4}{\chi^2}} + O(k^{-1}), \quad \mathbb{Z}_{>0} \ni k \to \infty, \quad |\chi| > 2.
$$

More generally, one can show that $B(\chi, \tau) \to 1$ and $\gamma(\chi, \tau) + \frac{1}{2} \tau \to 0$ as $(\chi, \tau) \to \infty$, so the leading term reproduces the known decay to $\psi_0(x,t) = 1$ of the exact solution $\psi_k(x,t)$. 
The approximating formula $B(\chi, \tau)e^{-iM(\tau+2\gamma(\chi, \tau))}$ is a modulated plane wave with slowly varying amplitude $B(\chi, \tau)$, scaled wavenumber $-2\gamma\chi(\chi, \tau)$, and scaled frequency $1 + 2\gamma\tau(\chi, \tau)$. One can check directly that the following holds:

**Corollary (Bilman-M 2021, Corollary 1.13)**

For $(\chi, \tau) \in \mathcal{E}$, the expressions

$$\rho(\chi, \tau) = B(\chi, \tau)^2 \quad \text{and} \quad U(\chi, \tau) = -2\gamma\chi(\chi, \tau)$$

satisfy the dispersionless NLS (hydrodynamic) system

$$\rho_\tau + (\rho U)_\chi = 0 \quad \text{and} \quad U_\tau + \left(\frac{1}{2}U^2 - \rho\right)_\chi = 0.$$

The proof relies on an identity: $\gamma\chi(\chi, \tau) = A(\chi, \tau) = \text{Re}(\lambda_0(\chi, \tau))$. 
This extends to the exterior domain $\mathcal{E}$ a result valid for the whole family of solutions (continuous $M$) on the shelves $S$. Indeed, the leading term of $\psi_k(M\chi, M\tau)$ on $S$ is the same modulated plane-wave as on $\mathcal{E}$, modified by a $M$-independent phase factor $e^{-2i\mu(\chi, \tau)}$ vanishing on the common boundary curve. However the interesting sub-leading term responsible for the interference pattern of waves visible on $S$ is not present on $\mathcal{E}$.

Left: Modulus $|\psi_{32}(M\chi, M\tau)|$. Right: Leading-order amplitude $B(\chi, \tau)$.
Summary: Solitons and Rogue Waves on $E$

When $M = \frac{1}{2}k$ for $k \in \mathbb{Z}_{\geq 0}$, $q(x, t; G, M)$ is a $k^{th}$-order pole soliton solution that satisfies zero boundary conditions $q \rightarrow 0$ in $x$. Its behavior on the exterior domain $E$ splits into two cases relative to the curve $\ell_{\text{sol}}$ and its axis reflections:

- On the sub-domain $E \subset E$, $q$ decays exponentially, consistent with the zero boundary conditions.
- On the sub-domain $O \subset E$, $q$ behaves like a modulated elliptic function, consistent with the slowly-diverging trajectories of the $k$ soliton components propagating with common zero velocity.

When $M = \frac{1}{4} \frac{1}{2}k$ for $k \in \mathbb{Z}_{\geq 0}$, the function $\psi_k(x, t) = e^{-it}q(x, t; Q(-1)^{k+1}, M)$ is the $k^{th}$-order fundamental rogue wave solution that satisfies nonzero boundary conditions $\psi_k \rightarrow 1$ in $x$. It behaves the same on the whole exterior domain $E$, like a modulated plane-wave.
Other Solutions on the Exterior Domain $\mathcal{E}$

Boundary conditions

In general, we may write the “continuous order” $M$ in modular form with quotient $k$ and remainder $r$ as

$$M = \frac{1}{2} k + r, \quad k \in \mathbb{Z}_{\geq 0}, \quad 0 \leq r < \frac{1}{2}.$$  

Then the soliton case is $r = 0$ and the rogue wave case is $r = \frac{1}{4}$. If $r \neq 0, \frac{1}{4}$, then one might think that the boundary conditions at $x = \infty$ are somewhere “in between.” Maybe $|q| \to c \in (0, 1)$ as $|x| \to \infty$? In fact, no:

**Theorem (Bilman-M, 2021 (in preparation))**

Suppose that $M = \frac{1}{2} k + r$ with $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq r < \frac{1}{2}$. If $r \neq 0$, then

$$q(x, 0; Q^{(-1)^{k+1}}, \frac{1}{2} k + r) = 1 + O(|x|^{-\frac{1}{2}}), \quad |x| \to \infty.$$

The error term is sharp unless $r = \frac{1}{4}$ (rogue-wave case), in which case it becomes $O(x^{-2})$. 

Other Solutions on the Exterior Domain $\mathcal{E}$

Large-$k$ behavior

So, except for the soliton case of $M \in \frac{1}{2}\mathbb{Z}$, the initial condition satisfies unit amplitude nonzero boundary conditions. In the rogue-wave case the decay to the background is fast enough for scattering theory ($L^1(\mathbb{R})$), but in general the decay is so slow that the difference does not even lie in $L^2(\mathbb{R})$.

Now the question arises: suppose that $r \neq 0, \frac{1}{4}$ is fixed and $k \in \mathbb{Z}_{\geq 0}$ tends to infinity. How might we expect the asymptotic behavior in the exterior domain $\mathcal{E}$ to vary with $r$ in this limit?

Just as the large-$x$ behavior is insensitive to $r \neq 0, \frac{1}{4}$, so is the large-$k$ behavior. To describe it, the curve $\ell_{\text{sol}}$ takes a secondary role compared to another unbounded curve emanating from $(\chi^#, \tau^#)$ denoted by $\ell_{\text{trig}}$ and defined by the condition

$$u(\chi, \tau)^2 - 8\tau v(\chi, \tau) = 0.$$  

This unbounded curve is asymptotic to the line $\chi = \sqrt{8\tau}$ for large $\tau$; it is another branch of the analytic curve separating $S$ from $\mathcal{E}$. 
Other Solutions on the Exterior Domain $\mathcal{E}$

Large-$k$ behavior

For $r \neq 0, \frac{1}{4}$, the exterior domain $\mathcal{E}$ is divided along $\ell_{\text{sol}}$ and $\ell_{\text{trig}}$ into

- the plane-wave region $P$, and
- two oscillatory regions $O_1$ and $O_2$.

When $(\chi, \tau) \in P$, the solution has the same asymptotic behavior as in the case $r = \frac{1}{4}$ (rogue-wave case), and when $(\chi, \tau) \in O_1$, the solution has the same asymptotic behavior as in the case $r = 0$ (soliton case).
On the remaining domain $O_2$, the asymptotic behavior for large $k$ is not like either the rogue-wave or soliton cases. It is governed by a leading-order approximation by a modulated elliptic function, with elliptic modulus varying from $m = 0$ on $l_{\text{trig}}$ to $m = 1$ on $l_{\text{sol}}$.

Since $l_{\text{sol}}$ bends toward the vertical as $\tau \to \infty$ and $l_{\text{trig}}$ approaches the asymptote $\chi = \sqrt{8}\tau$, when $(\chi, \tau)$ are large, the large-$M$ asymptotic of the solution is consistent with the universal long-time asymptotic obtained by Biondini-Mantzavinos (2017) for solutions of the focusing NLS equation with nonzero boundary conditions. However, the solutions do not satisfy the full hypotheses of that result, which points toward a broader universality.
References


Thanks for listening! Questions?
Why?

When \((\chi, \tau) \in \mathcal{E}\), we should deform the original circular jump contour \(\Sigma_0\) into a “dumbbell” shape consisting of a “neck” \(N\) oriented upwards and connecting two loops around \(\lambda = \pm i\). When \(M = \frac{1}{2}k + r\) and \(G = Q^{-s}\) with \(s = \pm 1\) arbitrary, the jump of \(P(\lambda) = P(\lambda; M\chi, M\tau, G, M)\) on the neck reads

\[
P_+(\lambda) = P_-(\lambda) e^{-iM\vartheta-(\lambda;\chi,\tau)\sigma_3} Ze^{iM\vartheta+(\lambda;\chi,\tau)\sigma_3}, \quad \lambda \in N,
\]

with “core” jump matrix

\[
Z := \begin{bmatrix}
(-1)^k \cos(2\pi r) & s(-1)^k i \sin(2\pi r) \\
-s(-1)^k i \sin(2\pi r) & (-1)^k \cos(2\pi r)
\end{bmatrix}.
\]

The phase \(\vartheta(\lambda; \chi, \tau)\) and its possible \(g\)-function modifications have nothing to do with \(r\). But, the available algebraic factorizations of \(Z\) depend on which pivots are nonzero, and here one sees the key difference between \(r = 0\) (\(Z\) diagonal), \(r = \frac{1}{4}\) (\(Z\) off-diagonal) and \(r \neq 0, \frac{1}{4}\) (\(Z\) a full matrix).