Rigidity of the quintic, nonlinear Schrödinger equation

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Quintic nonlinear Schrödinger equation

Consider the nonlinear Schrödinger equation on a line,

$$iu_t + u_{xx} + |u|^4 u = 0, \qquad u(0,x) = u_0, \qquad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}.$$

Conserved mass

$$M(u(t)) = \int |u(t,x)|^2 = M(u(0)),$$

Conserved energy

$$E(u(t)) = \frac{1}{2} \int |u_x(t,x)|^2 - \frac{1}{6} \int |u(t,x)|^6.$$

Goal: Understand long time behavior of solutions with mass equal to the mass of the soliton.

Soliton

Let Q be the unique positive solution of the elliptic partial differential equation

$$Q_{xx} + Q^5 = Q, \qquad Q(x) = \frac{3^{1/4}}{\cosh^{1/2}(2x)}$$

See Berestycki, Lions, Peletier, Kwong, Strauss.

Then we have a solution to the Schrödinger equation,

$$u(t,x) = e^{it} Q(x).$$

We may also take the pseudoconformal transformation,

$$u(t,x) = \frac{1}{t^{1/2}} Q(\frac{x}{t}) e^{-i/t} e^{ix^2/4t}.$$

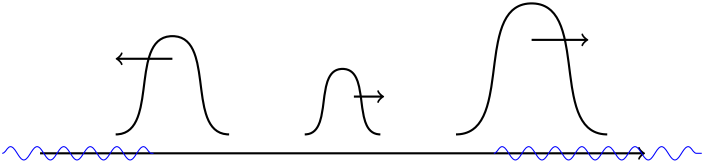
For the generalized KdV equation,

$$u_t + u_{xxx} + \partial_x(u^5) = 0, \qquad u(t,x) = Q(x-t),$$

is a soliton.

Ultimate goal: soliton resolution

Any generic global-in-time solution to the quintic nonlinear Schrödinger equation evolves asymptotically as a sum of decoupled solitons plus a radiative term.



Defocusing problem

The defocusing problem,

$$iu_t + u_{xx} = |u|^4 u, \qquad u(0,x) = u_0 \in L^2,$$

is known to scatter ([D.], 2016). Meanwhile, for the focusing equation below the ground state,

$$iu_t + u_{xx} + |u|^4 u = 0, \qquad \|u_0\|_{L^2} < \|Q\|_{L^2},$$

we also have scattering. ([D.], 2016).

Goal: Describe the long time behavior of solutions to the focusing problem when $\|u_0\|_{L^2} = \|Q\|_{L^2}$.

Gagliardo–Nirenberg inequality,

$$\|u\|_{L^6}^6 \leq 3(\frac{\|u_0\|_{L^2}^4}{\|Q\|_{L^2}^4})\|u_x\|_{L^2}^2.$$

New difficulty: For $\|u_0\|_{L^2} < \|Q\|_{L^2}$, $\|u_x\|_{L^2}^2 \lesssim E(u(t))$. When $\|u\|_{L^2} = \|Q\|_{L^2}$, we can only say that $E(u(t)) \geq 0$.

In fact, the Gagliardo–Nirenberg inequality is maximized if and only if u is a soliton under certain symmetries. Thus, if $\|u\|_{H^1} \to \infty$, then u approaches Q up to symmetries.

Theorem (Merle '93)

If $u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and u blows up in finite time, then u is a pseudoconformal transformation of the soliton.

Theorem (Killip, Li, Visan, Zhang '09)

If $d \geq 4$, u is radially symmetric, and u blows up in both time directions, then u is a soliton.

By blowup forward in time we mean that $\lim_{T \nearrow \sup(I)} \|u\|_{L^6_{t,x}([T_0,T))} = \infty$, where I is the maximal interval of existence and $T_0 \in I$ is fixed.

Result

Theorem (D. '21)

If $\|u_0\|_{L^2} = \|Q\|_{L^2}$ *and* u *fails to scatter, then* u *is either a soliton or a pseudoconformal transformation of a soliton.*

Step one

Theorem

The result reduces to proving the result for a solution that blows up forward in time, and for some $\eta_* > 0$ small,

$$\sup_{t>0} dist(u, \mathcal{M}) < \eta_*,$$

where $\mathcal{M}$ is the manifold of solitons.

The manifold $\mathcal{M}$ is generated by four symmetries:

1. Translation
2. Galilean translation - Schlag
3. Scaling - Ntekoume
4. Multiplication by $e^{i\theta}$ - Holmer

It's much better to sketch the proof on a blackboard than to try to put it on a slide.

The proof uses

Theorem (Fan, 2018, D. 2021)

If u is a solution at mass $\|Q\|_{L^2}$ that fails to scatter, say, forward in time, then there exists a sequence $t_n \nearrow \sup(I)$ such that $u(t_n)$ converges to $\mathcal{M}$ in L^2.

From now on, for simplicity, we will only discuss the case when u is a symmetric solution.

Step two

Theorem (Martel and Merle)

Let

$$\epsilon(t,x) = \lambda_0(t)^{1/2} e^{i\theta_0(t)} u(t, \lambda_0(t)x) - Q(x), \qquad \|\epsilon\|_{L^2} \leq \eta_*.$$

Then there exists $\lambda(t)$ *and* $\theta(t)$ *such that*

$$\epsilon(t,x) = \lambda(t)^{1/2} e^{i\theta(t)} u(t, \lambda(t)x) - Q(x), \qquad \|\epsilon\|_{L^2} \leq 2\eta_*,$$

satisfies $(\epsilon, Q^3)_{L^2} = (\epsilon, iQ^3)_{L^2} = 0$. *Since* u *is symmetric* $(\epsilon, Q_x)_{L^2} = (\epsilon, iQ_x)_{L^2} = 0$.

The linearization of the soliton equation

$$\mathcal{L} = -\partial_{xx} - Q^4 + 1,$$

has one eigenvector with negative eigenvalue, Q^3 with eigenvalue -8, and one eigenvector with zero eigenvalue, Q_x. If $f \perp Q^3, Q_x$, then

$$(\mathcal{L}f, f)_{L^2} \geq \frac{1}{2}(f, f)_{L^2}.$$

This computation implies

$$E(Q + \epsilon) \gtrsim \|\epsilon\|_{H^1(\mathbb{R})}^2.$$

To simplify let $\epsilon = \epsilon_1 + i\epsilon_2$.

Step three Long time Strichartz estimates and the Morawetz estimate.

Theorem

If $J = [a, b]$ is an interval such that $\int_J \lambda(t)^{-2} dt = T$ and $1 \leq \lambda(t) \leq T^{1/100}$, for T large,

$$\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt$$

$$\leq 3(\epsilon_2(a), (\frac{1}{2}Q + xQ_x))_{L^2} - 3(\epsilon_2(b), (\frac{1}{2}Q + xQ_x))_{L^2} + O(T^{-9}).$$

Make a change of variables so that $ds = \lambda(t)^{-2} dt$. If u is a blowup solution then $s \in [0, \infty)$.

The proof uses a Morawetz potential $\psi(\frac{x}{R})$, where $\psi(x) = x$ for $|x| \leq 1$, and $\psi(x)$ is constant when $|x| \geq 2$. Take $R = T^{1/25}$.

Then compute,

$$\frac{d}{dt} \int \psi(\frac{x}{R}) Im[\bar{u} u_x] dx \gtrsim \lambda(t)^{-2} \|\epsilon(t)\|_{L^2}^2 - O(T^{-10}).$$

Meanwhile,

$$\int \psi(\frac{x}{R}) Im[\overline{P_{\leq N} u} \partial_x P_{\leq N} u] dx = -2(\epsilon, \frac{Q}{2} + xQ_x)_{L^2} + R\|\epsilon\|_{L^2} \|\nabla P_{\leq N} \epsilon\|_{L^2}.$$

$N = T^{1/3}$ is okay to use long time Strichartz estimates to handle the error. By almost conservation of energy,

$$\sup_{t \in J} \|\epsilon(t)\|_{L^2}^2 + \|\nabla P_{\leq N} \epsilon\|_{L^2}^2 \lesssim \frac{T^{1/100} N^2}{T} \int_J \lambda(t)^{-2} \|\epsilon(t)\|_{L^2}^2 + O(T^{-8}).$$

Step four

This is probably the key new step.

Theorem

For any $j \geq 0$, *let*

$$s_j = \inf\{s \in [0, \infty) : \|\epsilon(s)\|_{L^2} = 2^{-j}\eta_*\}.$$

By definition, $s_0 = 0$, *and by the sequential convergence result,* s_j *exists for any* $j \geq 0$.

Then,

$$\int_{s_j}^{\infty} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j}\eta_*,$$

with implicit constant independent of η_* *and* $j \geq 0$.

Corollary

For any $p > 1$,

$$\int_0^\infty \|\epsilon(s)\|_{L^2}^p ds < \infty.$$

For the pseudoconformal transformation of the soliton, $\lambda(t) = t$ and $\|\epsilon(t)\|_{L^2} = t$, so in that case we have L_s^p integrability if and only if $p > 1$. Therefore, this is as good as we could hope.

Step five

Using the virial identity from Merle–Raphael (2005), we can show that $\lambda(s)$ is monotone decreasing as $s \to \infty$.

$$\frac{d}{ds}(\epsilon, y^2 Q)_{L^2} + \frac{\lambda_s}{\lambda}\|yQ\|_{L^2}^2 + 4(\frac{Q}{2} + yQ_y, \epsilon_2)_{L^2} \sim O(\|\epsilon\|_{L^2}^2).$$

If $(\epsilon_2, \frac{Q}{2} + yQ_y)_{L^2}(a) < 0$ then $|\frac{\lambda_s}{\lambda}| \lesssim \|\epsilon\|_{L^2}$ implies that $\lambda(s) \sim \lambda(a)$ for all $s > a$.

If $(\epsilon_2, \frac{Q}{2} + yQ_y)_{L^2} \geq 0$ for all $s \geq 0$, then for any $0 < s_- < s_+ < \infty$,

$$\ln(\lambda(s_+)) - \ln(\lambda(s_-)) \lesssim \eta_*.$$

Step six Almost monotone $\lambda(t)$.

Theorem

If $\lambda(t)$ is almost monotone and $\sup(I) = \infty$, then u is a soliton solution.

Then if $\sup(I) < \infty$, then do a pseudoconformal transformation of u. After doing the pseudoconformal transformation, u is a soliton.

Thank you for your attention!