Ground states of the energy super-critical Gross-Pitaevskii equation with harmonic potential

Dmitry E Pelinovsky
Joint work with Szymon Sobieszek (McMaster), Piotr Bizon and Filip Ficek (Krakow)
Department of Mathematics, McMaster University, Canada
http://dmpeli.math.mcmaster.ca
Gross–Pitaevskii equation

The Gross-Pitaevskii theory in $\mathbb{R}^d$ with harmonic potential,

$$i\partial_t w = -\Delta w + |x|^2 w - |w|^{2p} w,$$

admits two conserved quantities of mass and energy,

$$M(w) = \int_{\mathbb{R}^d} |w|^2 \, dx, \quad E(w) = \int_{\mathbb{R}^d} \left( |\nabla w|^2 + |x|^2 |w|^2 - \frac{1}{p+1} |w|^{2p+2} \right) \, dx.$$

In the absence of harmonic potential, we adopt the following classification based on the scaling transformation:

$$w(t, x) \mapsto w_L(t, x) = L^{\frac{1}{p}} w(L^2 t, Lx), \quad L > 0,$$

which yields $M(w_L) = L^{\frac{2}{p} - d} M(w)$ and $E(w_L) = L^{\frac{2}{p} + 2 - d} E(w)$. 
Gross–Pitaevskii equation

The Gross-Pitaevskii theory in $\mathbb{R}^d$ with harmonic potential,

$$i\partial_tw = -\Delta w + |x|^2w - |w|^{2p}w,$$

admits two conserved quantities of mass and energy,

$$M(w) = \int_{\mathbb{R}^d} |w|^2dx, \quad E(w) = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + |x|^2|w|^2 - \frac{1}{p+1}|w|^{2p+2}\right)dx.$$

- Mass-subcritical case ($dp < 2$): global existence in $H^1$
- Mass-critical case ($dp = 2$): global existence for small $L^2$ data and finite-time blow-up for large $L^2$
- Mass-supercritical case ($dp > 2$): global existence and scattering for $E(w) > 0$ and finite-time blow-up for $E(w) < 0$. 
Gross–Pitaevskii equation

The Gross-Pitaevskii theory in $\mathbb{R}^d$ with harmonic potential,

$$i\partial_t w = -\Delta w + |x|^2 w - |w|^{2p} w,$$

admits two conserved quantities of mass and energy,

$$M(w) = \int_{\mathbb{R}^d} |w|^2 \, dx, \quad E(w) = \int_{\mathbb{R}^d} \left( |\nabla w|^2 + |x|^2 |w|^2 - \frac{1}{p + 1} |w|^{2p+2} \right) \, dx.$$

- Energy-subcritical case: $(d - 2)p < 2$.
- Energy-critical case: $(d - 2)p = 2$, $d \geq 3$.
- Energy-supercritical case: $(d - 2)p > 2$, $d \geq 3$.

We only consider the case $p = 1$ to simplify technical details so that $d = 4$ is the energy-critical case.
Standing wave solutions (bound states)

Standing wave solutions \( w(t, x) = e^{-i\lambda t}u(x) \) satisfy the stationary Gross-Pitaevskii equation with harmonic potential:

\[
-\Delta u + |x|^2 u - |u|^2 u = \lambda u,
\]

Variationally, \( u \in \mathcal{E} := H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \cap L^4(\mathbb{R}^d) \) is a critical point of energy \( E(u) \) subject to fixed mass \( M(u) \), \( \lambda \) is Lagrange multiplier.

Among all bound states, we are only interested in the ground state with \( u(x) \) satisfying:

- real and positive on \( \mathbb{R}^d \);
- radially symmetric in \( |x| \);
- bounded and monotonically decreasing to zero.

Such solutions bifurcate from \( \lambda = d \) to \( \lambda \lesssim d \).
No ground state solutions exist for \( \lambda > d \).
Standing wave solutions (bound states)

Standing wave solutions \( w(t, x) = e^{-i\lambda t}u(x) \) satisfy the stationary Gross-Pitaevskii equation with harmonic potential:

\[
-\Delta u + |x|^2 u - |u|^2 u = \lambda u,
\]

Variationally, \( u \in \mathcal{E} := H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \cap L^4(\mathbb{R}^d) \) is a critical point of energy \( E(u) \) subject to fixed mass \( M(u) \), \( \lambda \) is Lagrange multiplier.

Energy-subcritical case \( d \leq 3 \):

- Existence for every \( \lambda < d \) follows from variational theory due to compactness of embedding of \( H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \) into \( L^4(\mathbb{R}^d) \) (Kavian & Weissler, 1994) (Fukuizumi, 2002)
- Uniqueness follows from ODE theory (Hirose & Ohta, 2002) (Hirose & Ohta, 2007)
Standing wave solutions (bound states)

Standing wave solutions \( w(t, x) = e^{-i\lambda t}u(x) \) satisfy the stationary Gross-Pitaevskii equation with harmonic potential:

\[
-\Delta u + |x|^2 u - |u|^2 u = \lambda u,
\]

Variationally, \( u \in \mathcal{E} := H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \cap L^4(\mathbb{R}^d) \) is a critical point of energy \( E(u) \) subject to fixed mass \( M(u) \), \( \lambda \) is Lagrange multiplier.

Energy-critical case \( d = 4 \):

- No solution exists for \( \lambda < 0 \) due to Pohozaev’s identity
- Existence and uniqueness for some \( \lambda \in (0, d) \) has been shown (Selem, 2011)
- It is still open if the solution exists as \( \lambda \to 0 \)
Standing wave solutions (bound states)

Standing wave solutions $w(t, x) = e^{-i\lambda t}u(x)$ satisfy the stationary Gross-Pitaevskii equation with harmonic potential:

$$-\Delta u + |x|^2 u - |u|^2 u = \lambda u,$$

Variationally, $u \in \mathcal{E} := H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$ is a critical point of energy $E(u)$ subject to fixed mass $M(u)$, $\lambda$ is Lagrange multiplier.

Energy-supercritical case $d \geq 5$:

- No solution exists for $\lambda < 0$ due to Pohozaev’s identity
- The solution exists in a subset of $\lambda \in (0, d)$
  (Selem & Kikuchi, 2012)
- The solution branch is connected to an unbounded solution $u_\infty \in \mathcal{E}$, $u_\infty \notin L^\infty$ for some $\lambda_\infty \in (0, d)$
  (Selem & Kikuchi & Wei, 2013)
Shooting methods as a tool

The ground state is defined as a solution of the boundary-value problem for fixed $\lambda \in \mathbb{R}$:

$$
\begin{cases}
  u''(r) + \frac{d-1}{r} u'(r) - r^2 u(r) + \lambda u(r) + u(r)^3 = 0, & r > 0, \\
  u(r) > 0, & u'(r) < 0, \\
  \lim_{r \to 0} u(r) < \infty, & \lim_{r \to \infty} u(r) = 0.
\end{cases}
$$

Solutions $u$ may not exist or their number may depend on $\lambda$.

The shooting method (Joseph & Lundgren, 1973) allows to find solutions $u$ from the initial-value problem:

$$
\begin{cases}
  f_b'''(r) + \frac{d-1}{r} f_b'(r) - r^2 f_b(r) + \lambda f_b(r) + f_b(r)^3 = 0, & r > 0, \\
  f_b(0) = b, & f_b'(0) = 0,
\end{cases}
$$

where $b > 0$ is fixed parameter. If $f_b(r) > 0$, $f_b'(r) < 0$, and $f_b(r) \to 0$ as $r \to \infty$, then $u(r) = f_b(r)$ for some $\lambda$. 
First result: existence

Theorem (BFPS, 2021)

Fix $d \geq 4$. For every $b > 0$, there exists $\lambda \in (d - 4, d)$, labeled as $\lambda(b)$, such that the unique classical solution $f_b \in C^2(0, \infty)$ to the initial-value problem with $\lambda = \lambda(b)$ is a solution $u \in \mathcal{E} \cap L^\infty$ to the boundary-value problem.

- Uniqueness of $\lambda(b)$ is an open problem.
- This result holds both for critical and supercritical cases.
First result: existence

Figure 1: Graph of $\lambda$ as a function of $b$ for the ground state $u$ of the boundary-value problem for $d = 5$ (left) and $d = 13$ (right).
Ground state in the limit of $b \to \infty$?

The limiting singular solution $u_\infty \in \mathcal{E}$, $u_\infty \notin L^\infty$ is defined by

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} [1 + \mathcal{O}(r^2)] \quad \text{as} \quad r \to 0.$$
Ground state in the limit of $b \to \infty$?

The limiting singular solution $u_\infty \in \mathcal{E}$, $u_\infty \not\in L^\infty$ is defined by

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} \left[ 1 + \mathcal{O}(r^2) \right] \quad \text{as} \quad r \to 0.$$ 

Theorem (Selem–Kikuchi–Wei, 2013)

*Fix $d \geq 5$. There exists $\lambda \in (0, d)$, labeled as $\lambda_\infty$, such that the limiting singular solution $u_\infty \in \mathcal{E}$ exists so that $\lambda(b) \to \lambda_\infty$ and*

$$u(b) \to u_\infty \quad \text{in} \quad \mathcal{E} \quad \text{as} \quad b \to \infty.$$ 

- Uniqueness of $\lambda_\infty$ is an open problem.
- Details of convergence $\lambda(b) \to \lambda_\infty$ were not studied.
Second result: convergence

Theorem (BFPS, 2021)

Fix $d \geq 5$. Under some non-degeneracy assumptions, $\lambda(b)$ is uniquely defined near $\lambda_\infty$ for $b \gg 1$ and

- $\lambda(b) - \lambda_\infty \sim A_\infty b^{-\beta} \sin(\alpha \ln b + \delta_\infty)$ if $5 \leq d \leq 12$, for some $A_\infty > 0$, $\delta_\infty \in (0, 2\pi)$, $\alpha > 0$, and $\beta > 0$
- $\lambda(b) - \lambda_\infty \sim B_\infty b^{-\kappa}$ if $d \geq 13$, for some $B_\infty \neq 0$ and $\kappa > 0$.  

Dmitry E. Pelinovsky
McMaster University
Ground states
Second result: convergence

**Theorem (BFPS, 2021)**

*Fix $d \geq 5$. Under some non-degeneracy assumptions, $\lambda(b)$ is uniquely defined near $\lambda_{\infty}$ for $b \gg 1$ and*

- $\lambda(b) - \lambda_{\infty} \sim A_{\infty} b^{-\beta} \sin(\alpha \ln b + \delta_{\infty})$ if $5 \leq d \leq 12$, for some $A_{\infty} > 0$, $\delta_{\infty} \in (0, 2\pi)$, $\alpha > 0$, and $\beta > 0$
- $\lambda(b) - \lambda_{\infty} \sim B_{\infty} b^{-\kappa}$ if $d \geq 13$ for some $B_{\infty} \neq 0$ and $\kappa > 0$.

The oscillatory behavior has been studied for the stationary NLS equation in a ball with dynamical system methods.

(Budd, Norbury, 1987), (Budd, 1989), (Merle & Peletier, 1991), (Dolbeault & Flores, 2007)
Linearization and Morse index

- Linearization around the ground state $u$:

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d - 1}{r} \frac{d}{dr} + r^2 - \lambda(b) - 3u^2(r).$$

- Linearization around the singular solution $u_\infty$:

$$\mathcal{L}_\infty := -\frac{d^2}{dr^2} - \frac{d - 1}{r} \frac{d}{dr} + r^2 - \lambda_\infty - 3u_\infty^2(r).$$
Linearization and Morse index

- Linearization around the ground state $u$:

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d - 1}{r} \frac{d}{dr} + r^2 - \lambda(b) - 3u^2(r).$$

- Linearization around the singular solution $u_\infty$:

$$\mathcal{L}_\infty := -\frac{d^2}{dr^2} - \frac{d - 1}{r} \frac{d}{dr} + r^2 - \lambda_\infty - 3u_\infty^2(r).$$

$\mathcal{L}_b$ is well-defined in the form domain $\mathcal{E} := H^1_r \cap L^2_r$. It is a self-adjoint Sturm–Liouville operator in $L^2_r$ with a purely point spectrum.
Linearization and Morse index

- Linearization around the ground state $u$:

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d - 1}{r} \frac{d}{dr} + r^2 - \lambda(b) - 3u^2(r).$$

- Linearization around the singular solution $u_\infty$:

$$\mathcal{L}_\infty := -\frac{d^2}{dr^2} - \frac{d - 1}{r} \frac{d}{dr} + r^2 - \lambda_\infty - 3u_\infty^2(r).$$

Stability of standing waves in the Gross–Pitaevskii equation:

- $u$ is orbitally stable if $\mathcal{L}_b$ has exactly one negative eigenvalue and the mapping $\lambda \mapsto \|u\|^2_{L^2}$ is decreasing.

- $u$ is orbitally unstable if $\mathcal{L}_b$ has two or more negative eigenvalues.

Note that $\langle \mathcal{L}_b u, u \rangle = -2\|u\|^4_{L^4_r} < 0$, hence $\mathcal{L}_b$ is not positive.
Oscillatory versus monotone convergence

Since

\[ \mathcal{L}_b \partial_b u = \lambda'(b) u, \quad \partial_b u \in \mathcal{E}_r, \]

the number of negative eigenvalues of \( \mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^* \) change for every \( b \) for which \( \lambda'(b) = 0 \).
Third result: stability

Theorem (P & Sobieszek, 2022)

For every $d \geq 13$, there exists $b_0 > 0$ such that the Morse index of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ is finite and is independent of $b$ for every $b \in (b_0, \infty)$. Moreover, it coincides with the Morse index of $\mathcal{L}_\infty : \mathcal{E} \mapsto \mathcal{E}^*$. 
Third result: stability

Theorem (P & Sobieszek, 2022)

For every $d \geq 13$, there exists $b_0 > 0$ such that the Morse index of $L_b : \mathcal{E} \mapsto \mathcal{E}^*$ is finite and is independent of $b$ for every $b \in (b_0, \infty)$. Moreover, it coincides with the Morse index of $L_\infty : \mathcal{E} \mapsto \mathcal{E}^*$.

These approximations of $L_b v = 0$ suggest that the Morse index is one.
Third result: stability

Theorem (P & Sobieszek, 2022)

For every \( d \geq 13 \), there exists \( b_0 > 0 \) such that the Morse index of \( \mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^* \) is finite and is independent of \( b \) for every \( b \in (b_0, \infty) \). Moreover, it coincides with the Morse index of \( \mathcal{L}_\infty : \mathcal{E} \mapsto \mathcal{E}^* \).

This graph suggests that the mapping \( \lambda \mapsto \| u_b \|_{L^2}^2 \) is decreasing.

Conclusion: the standing waves are stable for \( d \geq 13 \).
Emden-Fowler transformation

The initial-value problem,

\[
\begin{align*}
    & f''_b(r) + \frac{d-1}{r} f'_b(r) - r^2 f_b(r) + \lambda f_b(r) + f_b(r)^3 = 0, \quad r > 0, \\
    & f_b(0) = b, \quad f'_b(0) = 0,
\end{align*}
\]

after the transformation

\[
r = e^t, \quad f(r) = \psi(t),
\]

becomes the invariant manifold problem:

\[
\begin{align*}
    & \psi'''(t) + (d - 2) \psi'(t) + e^{2t} \left( \lambda + \psi(t)^2 \right) \psi(t) - e^{4t} \psi(t) = 0, \quad t \in \mathbb{R}, \\
    & \psi(t) \to b, \quad t \to -\infty.
\end{align*}
\]
Emden-Fowler transformation

The initial-value problem,

\[
\begin{align*}
\left\{ \begin{array}{c}
f''(r) + \frac{d-1}{r} f'(r) - r^2 f(r) + \lambda f(r) + f(r)^3 = 0, \\
f(0) = b, \\
f'(0) = 0,
\end{array} \right. \\
\end{align*}
\]

after the transformation

\[
r = e^t, \quad f(r) = \psi(t),
\]

becomes the invariant manifold problem:

\[
\begin{align*}
\left\{ \begin{array}{c}
\psi''(t) + (d - 2) \psi'(t) + e^{2t} \left( \lambda + \psi(t)^2 \right) \psi(t) - e^{4t} \psi(t) = 0, \\
\psi(t) \to b, \\
t \to -\infty.
\end{array} \right.
\end{align*}
\]

The solution is a fixed point of the integral operator \( A(\psi) \) given by

\[
A(\psi)(t) := b + (d-2)^{-1} \int_{-\infty}^{t} [1 - e^{-(d-2)(t-t')}][e^{4t'} \psi - e^{2t'} (\lambda \psi + \psi^3)] dt'.
\]
Emden-Fowler transformation

The initial-value problem,

\[
\begin{cases}
    f''_b(r) + \frac{d-1}{r} f'_b(r) - r^2 f_b(r) + \lambda f_b(r) + f_b(r)^3 = 0, & r > 0, \\
    f_b(0) = b, & f'_b(0) = 0,
\end{cases}
\]

after the transformation

\[ r = e^t, \quad f(r) = \psi(t), \]

becomes the invariant manifold problem:

\[
\begin{cases}
    \psi''(t) + (d - 2) \psi'(t) + e^{2t} \left( \lambda + \psi(t)^2 \right) \psi(t) - e^{4t} \psi(t) = 0, & t \in \mathbb{R}, \\
    \psi(t) \to b, & t \to -\infty.
\end{cases}
\]

There exists a unique solution \( \psi \in C^2(\mathbb{R}) \) such that

\[ \psi_b(t) = b - (\lambda b + b^3)(2d)^{-1} e^{2t} + \mathcal{O}(e^{4t}), \quad \text{as } t \to -\infty. \]
Rigorously implemented shooting method

For the uniquely defined solution $\psi_b(t) = b + O(e^{2t})$, we define the partition of $\mathbb{R} = I_+ \cup I_0 \cup I_-$ for parameter $\lambda$:

$I_+ := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi(t_0) = 0 \text{, while } \psi(t) > 0, \psi'(t) < 0, t < t_0 \}$,

$I_- := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi'(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, t < t_0 \}$,

$I_0 := \{ \lambda \in \mathbb{R} : \psi(t) > 0, \psi'(t) < 0, t \in \mathbb{R} \}$.
Rigorously implemented shooting method

For the uniquely defined solution $\psi_b(t) = b + O(e^{2t})$, we define the partition of $\mathbb{R} = I_+ \cup I_0 \cup I_-$ for parameter $\lambda$:

$I_+ := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, \ t < t_0 \}$,

$I_- := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi'(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, \ t < t_0 \}$,

$I_0 := \{ \lambda \in \mathbb{R} : \psi(t) > 0, \psi'(t) < 0, \ t \in \mathbb{R} \}$.

We have $I_- \cap I_+ = \emptyset$, $I_\pm \cap I_0 = \emptyset$, and furthermore,

- $[d, \infty) \subset I_+$ and $I_+$ is open;
Rigorously implemented shooting method

For the uniquely defined solution $\psi_b(t) = b + \mathcal{O}(e^{2t})$, we define the partition of $\mathbb{R} = I_+ \cup I_0 \cup I_-$ for parameter $\lambda$:

\[ I_+ := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, t < t_0 \}, \]
\[ I_- := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi'(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, t < t_0 \}, \]
\[ I_0 := \{ \lambda \in \mathbb{R} : \psi(t) > 0, \psi'(t) < 0, t \in \mathbb{R} \}. \]

We have $I_- \cap I_+ = \emptyset$, $I_\pm \cap I_0 = \emptyset$, and furthermore,

- $[d, \infty) \subset I_+$ and $I_+$ is open;
- $(-\infty, 0] \subset I_-$ and $I_-$ is open;
Rigorously implemented shooting method

For the uniquely defined solution \( \psi_b(t) = b + \mathcal{O}(e^{2t}) \), we define the partition of \( \mathbb{R} = I_+ \cup I_0 \cup I_- \) for parameter \( \lambda \):

\[
I_+ := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi(t_0) = 0, \text{ while } \psi(t) > 0, \; \psi'(t) < 0, \; t < t_0 \},
\]
\[
I_- := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi'(t_0) = 0, \text{ while } \psi(t) > 0, \; \psi'(t) < 0, \; t < t_0 \},
\]
\[
I_0 := \{ \lambda \in \mathbb{R} : \psi(t) > 0, \; \psi'(t) < 0, \; t \in \mathbb{R} \}.
\]

We have \( I_- \cap I_+ = \emptyset \), \( I_\pm \cap I_0 = \emptyset \), and furthermore,

- \([d, \infty) \subset I_+ \) and \( I_+ \) is open;
- \((-\infty, 0] \subset I_- \) and \( I_- \) is open;
- \( I_0 \subset (0, d) \) is closed and if \( \lambda(b) \in I_0 \), then \( \psi_b(t) \to 0 \) as \( t \to +\infty \) with the precise asymptotics:

\[
\psi_b(t) \sim c e^{\frac{\lambda-d}{2} t} e^{-\frac{1}{2} e^{2t}}, \quad \text{as} \quad t \to +\infty,
\]

for some \( c > 0 \).
Rigorously implemented shooting method

For the uniquely defined solution $\psi_b(t) = b + O(e^{2t})$, we define the partition of $\mathbb{R} = I_+ \cup I_0 \cup I_-$ for parameter $\lambda$:

$I_+ := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, \ t < t_0 \}$,

$I_- := \{ \lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi'(t_0) = 0, \text{ while } \psi(t) > 0, \psi'(t) < 0, \ t < t_0 \}$,

$I_0 := \{ \lambda \in \mathbb{R} : \psi(t) > 0, \psi'(t) < 0, \ t \in \mathbb{R} \}$.
Towards the proof of convergence as $b \to \infty$

Recall the limiting singular solution $u_\infty \in \mathcal{E}$, $u_\infty \notin L^\infty$ defined by

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} \left[ 1 + \mathcal{O}(r^2) \right] \quad \text{as} \quad r \to 0.$$ 

The solution can be represented by $u(r) = r^{-1}F(r)$ with bounded $F$. Using Emden-Fowler transformation and $\psi(t) = e^{-t}\Psi(t)$, we obtain

$$\Psi''(t) + (d-4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0.$$ 

Dmitry E. Pelinovsky 
McMaster University
Ground states
Towards the proof of convergence as $b \to \infty$

Recall the limiting singular solution $u_\infty \in \mathcal{E}$, $u_\infty \notin L^\infty$ defined by

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} \left[ 1 + \mathcal{O}(r^2) \right] \quad \text{as} \quad r \to 0.$$ 

The solution can be represented by $u(r) = r^{-1} F(r)$ with bounded $F$. Using Emden-Fowler transformation and $\psi(t) = e^{-t} \Psi(t)$, we obtain

$$\Psi''(t) + (d - 4) \Psi'(t) + (3 - d) \Psi(t) + \Psi(t)^3 + \lambda e^{2t} \Psi(t) - e^{4t} \Psi(t) = 0.$$ 

The limiting singular solution corresponds to the solution with

$$\Psi_\infty(t) = \sqrt{d-3} + \mathcal{O}(e^{2t}), \quad \text{as} \quad t \to -\infty \quad \text{and} \quad \Psi_\infty(t) \to 0, \quad \text{as} \quad t \to +\infty,$$

which exists for some $\lambda = \lambda_\infty$ (Selem–Kikuchi–Wei, 2013).
Towards the proof of convergence as $b \to \infty$

Recall the limiting singular solution $u_\infty \in \mathcal{E}$, $u_\infty \notin L^\infty$ defined by

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} \left[1 + \mathcal{O}(r^2)\right] \quad \text{as} \quad r \to 0.$$ 

The solution can be represented by $u(r) = r^{-1}F(r)$ with bounded $F$. Using Emden-Fowler transformation and $\psi(t) = e^{-t}\Psi(t)$, we obtain

$$\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0.$$
Two analytic family of solutions

Consider the differential equation

\[\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0.\]
Two analytic family of solutions

Consider the differential equation

\[ \Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0. \]

- The \( b \)-family \( \Psi_b(t) = e^t\psi_b(t) = be^t + \mathcal{O}(e^{3t}) \) as \( t \to -\infty \)
Two analytic family of solutions

Consider the differential equation

\[ 
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0. 
\]

- The \( b \)-family \( \Psi_b(t) = e^t\psi_b(t) = be^t + \mathcal{O}(e^{3t}) \) as \( t \to -\infty \)
- The \( c \)-family \( \Psi_c(t) \to 0 \) as \( t \to +\infty \) with

\[ \Psi_c(t) \sim ce^{\frac{\lambda - d + 2}{2}t}e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \to +\infty. \]
Two analytic family of solutions

Consider the differential equation

\[ \Psi''(t) + (d - 4) \Psi'(t) + (3 - d) \Psi(t) + \Psi(t)^3 + \lambda e^{2t} \Psi(t) - e^{4t} \Psi(t) = 0. \]

- The \( b \)-family \( \Psi_b(t) = e^t \psi_b(t) = be^t + \mathcal{O}(e^{3t}) \) as \( t \to -\infty \)
- The \( c \)-family \( \Psi_c(t) \to 0 \) as \( t \to +\infty \) with
  \[ \Psi_c(t) \sim ce^{\frac{\lambda - d + 2}{2} t} e^{-\frac{1}{2} e^{2t}}, \quad \text{as} \ t \to +\infty. \]
- Their intersection for some \( \lambda = \lambda(b) \) and \( c = c(b) \):
  \[ \Psi_b(t) = \Psi_{c(b)}(t). \]
  We want to prove: \( \lambda(b) \to \lambda_\infty \) with some \( c(b) \to c_\infty \) as \( b \to +\infty \).
Two analytic family of solutions

Consider the differential equation

$$\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0.$$ 

$$d = 5, \quad b = 14000:$$
Two analytic family of solutions

Consider the differential equation

\[ \Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t} \Psi(t) - e^{4t} \Psi(t) = 0. \]

\[ d = 13, \quad b = 14000 : \]

\[ \psi(t) \]

\[ \Psi(t) \]
The $b$-family of solutions

Consider the differential equation

$$
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
$$

for the solution $\Psi_b(t) = be^t + O(e^{3t})$ as $t \to -\infty$. 
The $b$-family of solutions

Consider the differential equation

$$\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t} \Psi(t) - e^{4t} \Psi(t) = 0,$$

for the solution $\Psi_b(t) = be^t + O(e^{3t})$ as $t \to -\infty$.

Formal truncation gives

$$\Theta''(t) + (d - 4)\Theta'(t) + (3 - d)\Theta(t) + \Theta(t)^3 = 0$$

with uniquely defined $\Theta(t) = e^t + O(e^{3t})$ as $t \to -\infty$. 
The $b$-family of solutions

Consider the differential equation

$$
Ψ''(t) + (d - 4)Ψ'(t) + (3 - d)Ψ(t) + Ψ(t)^3 + λe^{2t}Ψ(t) - e^{4t}Ψ(t) = 0,
$$

for the solution $Ψ_b(t) = be^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

Formal truncation gives

$$
Θ''(t) + (d - 4)Θ'(t) + (3 - d)Θ(t) + Θ(t)^3 = 0
$$

with uniquely defined $Θ(t) = e^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

Easy result for all large $b$:

$$
\sup_{t \in (-\infty, 0]} |Ψ_b(t - \log b) - Θ(t)| \leq C_0 b^{-2}.
$$
The \(b\)-family of solutions

Consider the differential equation

\[
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
\]

for the solution \(\Psi_b(t) = be^t + O(e^{3t})\) as \(t \to -\infty\).

Formal truncation gives

\[
\Theta''(t) + (d - 4)\Theta'(t) + (3 - d)\Theta(t) + \Theta(t)^3 = 0
\]

with uniquely defined \(\Theta(t) = e^t + O(e^{3t})\) as \(t \to -\infty\).

Harder result for every \(T > 0\) and \(a \in (0, 1)\):

\[
\sup_{t \in [0, T + a \log b]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_{T,a} b^{-2(1-a)}
\]
The $b$-family of solutions

Consider the differential equation

$$\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,$$

for the solution $\Psi_b(t) = be^t + O(e^{3t})$ as $t \to -\infty$.

Formal truncation gives

$$\Theta''(t) + (d - 4)\Theta'(t) + (3 - d)\Theta(t) + \Theta(t)^3 = 0$$

with uniquely defined $\Theta(t) = e^t + O(e^{3t})$ as $t \to -\infty$.

$\Theta(t) \to \sqrt{d - 3}$ as $t \to +\infty$ since

- $(\sqrt{d - 3}, 0)$ is a stable spiral point for $5 \leq d \leq 12$
- $(\sqrt{d - 3}, 0)$ is a stable nodal point for $d \geq 13$. 
The $b$-family of solutions

Consider the differential equation

$$
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
$$

for the solution $\Psi_b(t) = be^t + O(e^{3t})$ as $t \to -\infty$.

Formal truncation gives

$$
\Theta''(t) + (d - 4)\Theta'(t) + (3 - d)\Theta(t) + \Theta(t)^3 = 0
$$

with uniquely defined $\Theta(t) = e^t + O(e^{3t})$ as $t \to -\infty$.

Non-degeneracy assumption ($5 \leq d \leq 12$):

$$
\Theta(t) = \sqrt{d - 3} + A_0 e^{-\beta t} \sin(\alpha t + \delta_0) + O(e^{-2\beta t}) \quad \text{as} \quad t \to +\infty,
$$

where $A_0 \neq 0$. 
The \( b \)-family of solutions

Consider the differential equation

\[
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
\]

for the solution \( \Psi_b(t) = be^t + \mathcal{O}(e^{3t}) \) as \( t \to -\infty \).
The \( c \)-family of solutions

Consider the differential equation

\[
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
\]

for the solution \( \Psi_c(t) \sim ce^{\frac{\lambda-d+2}{2}t}e^{-\frac{1}{2}e^{2t}} \) as \( t \to +\infty \).
The $c$-family of solutions

Consider the differential equation

$$
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
$$

for the solution $\Psi_c(t) \sim ce^\frac{\lambda-d+2}{2}te^{-\frac{1}{2}e^{2t}}$ as $t \to +\infty$.

Recall the limiting solution $\Psi_\infty(t) \to \sqrt{d-3}$ as $t \to -\infty$, which exists for $(\lambda, c) = (\lambda_\infty, c_\infty)$ and write

$$
\Psi_c = \Psi_\infty + (\lambda - \lambda_\infty)\Psi_1 + (c - c_\infty)\Psi_2 + \Sigma,
$$

for $(\lambda, c)$ near $(\lambda_\infty, c_\infty)$. 

Dmitry E. Pelinovsky
McMaster University

Ground states
The $c$-family of solutions

Consider the differential equation

$$
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
$$

for the solution $\Psi_c(t) \sim ce^{\frac{\lambda-d+2}{2}t}e^{-\frac{1}{2}e^{2t}}$ as $t \to +\infty$.

Recall the limiting solution $\Psi_\infty(t) \to \sqrt{d - 3}$ as $t \to -\infty$, which exists for $(\lambda, c) = (\lambda_\infty, c_\infty)$ and write

$$
\Psi_c = \Psi_\infty + (\lambda - \lambda_\infty)\Psi_1 + (c - c_\infty)\Psi_2 + \Sigma,
$$

for $(\lambda, c)$ near $(\lambda_\infty, c_\infty)$.

Easy result for every $t \in (-\infty, (a - 1)\log b + T]$:

$$
|\Psi_{1,2}(t) - A_{1,2}e^{-\beta t}\sin(\alpha t + \delta_{1,2})| \leq C_{T,a}b^{-2(1-a)}e^{-\beta t},
$$

where $A_1, A_2 \neq 0$ (non-degeneracy assumption).
The $c$-family of solutions

Consider the differential equation

$$
\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,
$$

for the solution $\Psi_c(t) \sim ce^{\frac{\lambda - d + 2}{2}t}e^{-\frac{1}{2}e^{2t}}$ as $t \to +\infty$.

Recall the limiting solution $\Psi_\infty(t) \to \sqrt{d - 3}$ as $t \to -\infty$, which exists for $(\lambda, c) = (\lambda_\infty, c_\infty)$ and write

$$
\Psi_c = \Psi_\infty + (\lambda - \lambda_\infty)\Psi_1 + (c - c_\infty)\Psi_2 + \Sigma,
$$

for $(\lambda, c)$ near $(\lambda_\infty, c_\infty)$.

Harder result for the remainder term for every $t \in [(a - 1) \log b, 0]$: 

$$
|\Sigma(t)| \leq CT, a \epsilon^2,
$$

as long as $(\lambda - \lambda_\infty)^2 + (c - c_\infty)^2 \leq \epsilon^2 b^{-2\beta(1-a)}$ with small $\epsilon > 0$. 

Dmitry E. Pelinovsky
McMaster University
Ground states
The $c$-family of solutions

Consider the differential equation

$$\Psi''(t) + (d - 4)\Psi'(t) + (3 - d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,$$

for the solution $\Psi_c(t) \sim ce^{\frac{\lambda - d + 2}{2}t}e^{-\frac{1}{2}e^{2t}}$ as $t \to +\infty$. 
Intersection of the $b$-family and the $c$-family

We define $\lambda = \lambda(b)$ and $c = c(b)$ from

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}.$$ 

We can use the two asymptotic representations for every $t \in [(a - 1) \log b, (a - 1) \log b + T]$ with arbitrary $T > 0$. 

Dmitry E. Pelinovsky
McMaster University

Ground states
Intersection of the $b$-family and the $c$-family

We define $\lambda = \lambda(b)$ and $c = c(b)$ from

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}.$$ 

We can use the two asymptotic representations for every $t \in [(a - 1) \log b, (a - 1) \log b + T]$ with arbitrary $T > 0$.

$$\Psi_b(T + (a - 1) \log b) = \Theta(T + a \log b) + \text{error}$$

$$= \sqrt{d - 3} + A_0 b^{-a\beta} e^{-\beta T} \sin(\alpha T + \delta_0) + \text{error}$$
Intersection of the $b$-family and the $c$-family

We define $\lambda = \lambda(b)$ and $c = c(b)$ from

$$
\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}.
$$

We can use the two asymptotic representations for every $t \in [(a - 1) \log b, (a - 1) \log b + T]$ with arbitrary $T > 0$.

$$
\Psi_b(T + (a - 1) \log b) = \Theta(T + a \log b) + \text{error}
$$

$$
= \sqrt{d - 3} + A_0 b^{-a\beta} e^{-\beta T} \sin(\alpha T + \delta_0) + \text{error}
$$

$$
\Psi_c(T + (a - 1) \log b) = \Psi_\infty(T + (a - 1) \log b) + \text{linear terms}
$$

$$
= \sqrt{d - 3} + A_1 (\lambda - \lambda_\infty) b^{(1-a)\beta} e^{-\beta T} \sin(\alpha T + \delta_1)
$$

$$
+ A_2 (c - c_\infty) b^{(1-a)\beta} e^{-\beta T} \sin(\alpha T + \delta_1) + \text{error}
$$
Intersection of the $b$-family and the $c$-family

We define $\lambda = \lambda(b)$ and $c = c(b)$ from

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}.$$ 

We can use the two asymptotic representations for every $t \in [(a - 1) \log b, (a - 1) \log b + T]$ with arbitrary $T > 0$.

Under the non-degeneracy assumption that $A_0, A_1, A_2 \neq 0$ we obtain with the implicit function theorem,

$$\lambda(b) - \lambda_\infty = A_\infty b^{-\beta} \sin(\alpha \log b + \delta_\infty) + \text{error},$$

inside $|\lambda - \lambda_\infty| \leq \epsilon b^{-\beta(1-a)}$. 

Intersection of the $b$-family and the $c$-family

We define $\lambda = \lambda(b)$ and $c = c(b)$ from

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}.$$  

We can use the two asymptotic representations for every $t \in [(a - 1) \log b, (a - 1) \log b + T]$ with arbitrary $T > 0$. 

Dmitry E. Pelinovsky
McMaster University
Ground states
Remarks

- Similar results but with monotone decay are obtained for $d \geq 13$. 
Remarks

- Similar results but with monotone decay are obtained for \( d \geq 13 \).

- Derivative \( \partial_b \Psi_b(t) \) is a solution of the linearized equation satisfying \( \partial_b \Psi_b(t) \to 0 \) as \( t \to -\infty \).
Remarks

- Similar results but with monotone decay are obtained for $d \geq 13$.

- Derivative $\partial_t \Psi_b(t)$ is a solution of the linearized equation satisfying $\partial_t \Psi_b(t) \to 0$ as $t \to -\infty$.

- Derivative $\partial_c \Psi_c(t)$ is a solution of the linearized equation satisfying $\partial_c \Psi_c(t) \to 0$ as $t \to +\infty$. 
Remarks

- Similar results but with monotone decay are obtained for \( d \geq 13 \).

- Derivative \( \partial_b \Psi_b(t) \) is a solution of the linearized equation satisfying \( \partial_b \Psi_b(t) \rightarrow 0 \) as \( t \rightarrow -\infty \).

- Derivative \( \partial_c \Psi_c(t) \) is a solution of the linearized equation satisfying \( \partial_c \Psi_c(t) \rightarrow 0 \) as \( t \rightarrow +\infty \).

- In the monotone case \( d \geq 13 \), under the non-degeneracy assumptions, we can show that if for \( \lambda = \lambda(b) \),

\[
\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R},
\]

then there exists no \( C \in \mathbb{R} \) such that

\[
\partial_b \Psi_b(t) = C \partial_c \Psi_{c(b)}(t), \quad t \in \mathbb{R}.
\]

Hence the linearized operator \( \mathcal{L}_b \) at \( u_b \) has no zero eigenvalues.
Future goals

- We have shown existence of $\lambda(b)$ and $\lambda_\infty$ but not uniqueness.
- No proof that if $L_b$ has a zero eigenvalue in $L^2_b$, then $\lambda'(b) = 0$.
- In the oscillatory case, the Morse index is expected to increase by one every time $\lambda(b)$ passes through the extremal point.
- The existence of $\lambda(b)$ has been shown in the energy critical case $d = 4$ but we should prove that $\lambda(b) \to 0$ as $b \to \infty$ with the limiting singular solution being the algebraic soliton.
Future goals

- We have shown existence of $\lambda(b)$ and $\lambda_\infty$ but not uniqueness.

- No proof that if $L_b$ has a zero eigenvalue in $L^2_b$, then $\lambda'(b) = 0$.

- In the oscillatory case, the Morse index is expected to increase by one every time $\lambda(b)$ passes through the extremal point.

- The existence of $\lambda(b)$ has been shown in the energy critical case $d = 4$ but we should prove that $\lambda(b) \to 0$ as $b \to \infty$ with the limiting singular solution being the algebraic soliton.

Thank you! Questions???