Asymptotic stability for the sine-Gordon kink under odd perturbations via supersymmetry

Jonas Lührmann (Texas A&M University)
W. S. (Yale)

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Classical nonlinear scalar field theories on the line

For scalar fields $\phi\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ and scalar potentials $W\colon \mathbb{R} \to [0,\infty)$, we consider the Lagrangian action functional

$$\mathcal{L}[\phi] = \int_{\mathbb{R}} \int_{\mathbb{R}} \Big(-\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + W(\phi) \Big) \, \mathrm{d}x \, \mathrm{d}t.$$

Euler-Lagrange or equation of motion

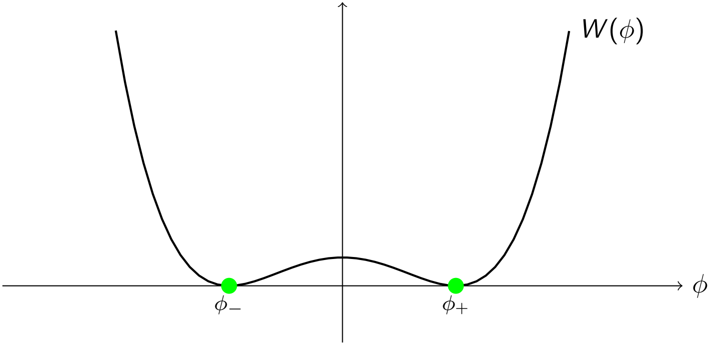
$$(\partial_t^2 - \partial_x^2)\phi = -W'(\phi), \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Hamiltonian or conserved energy functional

$$\mathcal{E}[\phi] = \int_{\mathbb{R}} \Big(\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + W(\phi) \Big) \, \mathrm{d}x.$$

Goal: Understand long-time dynamics of all solutions of finite energy. These do not need to decay at $\pm\infty$ or have finite $\|\phi(t)\|_{L^2_x}$.

Double-well scalar potentials



Classical examples:

- ϕ^4 model: $W(\phi) = \frac{1}{4}(1-\phi^2)^2 \quad \Rightarrow \phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0$
- sine-Gordon model: $W(\phi) = 1 - \cos(\phi)$ [completely integrable] $\Rightarrow \phi_{tt} - \phi_{xx} + \sin\phi = 0$

Topology of finite energy solutions: $\lim_{x\to\pm\infty} \phi(t,x) \in \{\phi_-, \phi_+\}$. For sine-Gordon all translations by $2\pi\mathbb{Z}$.

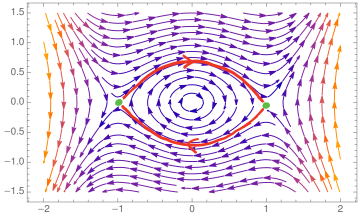
Vacuum solutions (for sine-Gordon add $2\pi\mathbb{Z}$):

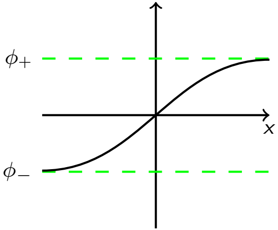
$$\phi(t,x) = \phi_{\pm} \quad \forall t \in \mathbb{R}.$$

These are **all constant** stationary solutions of finite energy, while all finite energy equilibria include kinks/anti-kinks.

Kink solutions as heteroclinic orbits in the phase portrait:

$$\begin{cases} \partial_x^2 K = W'(K), \quad x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} K(x) = \phi_{\pm}. \end{cases}$$





Classical examples (unique up to translation and $\mathbb{Z}_2$ symmetry):

- ϕ^4 model: $K(x) = \tanh(\frac{x}{\sqrt{2}})$
- sine-Gordon model: $K(x) = 4\arctan(e^x) \quad \mod 2\pi\mathbb{Z}$

Current and charge, topological sector, symmetries

Current $\mathbf{j} = (\phi_x, -\phi_t)$ satisfies continuity equation $\mathrm{div}_{t,x}\,\mathbf{j} = 0$ and the charge (for sine-Gordon normalize by $\frac{1}{2\pi}$)

$$Q(\phi) := \frac{1}{2}\int_{-\infty}^{\infty} j_0(t,x)\,dx = \frac{1}{2}(\phi(t,\infty) - \phi(t,-\infty))$$

is conserved:

$$\partial_t Q(\phi) = -\frac{1}{2}\int_{-\infty}^{\infty} \partial_x j_1(t,x)\,dx = -\frac{1}{2}(j_1(t,\infty) - j_1(t,-\infty)) = 0$$

For **vacua** $Q = 0$, while for kinks/anti-kinks $Q = \pm 1$. For ϕ^4 these are all Q. For s-G $Q \in \mathbb{Z}$ (multi-(anti)kinks, only stationary if $Q \in \{0, \pm 1\}$).

Conserved momentum $P = \int \phi_t \phi_x\,dx$, Newton's law $\dot{P} = F$ defines **force** between (anti)kinks.

Poincaré group (Lorentz and translations) acts on (anti)kinks. Soliton resolution? For (sG) have IST, see Chen-Liu-Lu '20. But not for ϕ^4, so fairly uncharted territory. First step is to find perturbative PDE methods other than IST to analyze (sG).

Stability of 0 solution for sine-Gordon

Orbital stability: for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(\phi(0), \phi_t(0))\|_{H^1 \times L^2} < \delta$ implies $\|(\phi(t), \phi_t(t))\|_{H^1 \times L^2} < \varepsilon$ for all $t \geq 0$.
Energy is a Lyapunov functional

$$\mathcal{E}[\phi] = \frac{1}{2} \int_{\mathbb{R}} \Big((\partial_t \phi)^2 + (\partial_x \phi)^2 + \phi^2 (1 - O(\phi^2)) \Big) \, \mathrm{d}x.$$

Asymptotic stability: in addition, $\lim_{t \to \infty} \|(\phi, \phi_t)(t)\|_X = 0$, but $X = ?$

If $\|\phi\|_\infty \to 0$ then $\sin \phi = \phi(1 + O(\phi^2))$.
Free Klein-Gordon equation $u_{tt} - u_{xx} + u = 0$ has solution

$$\begin{pmatrix} u \\ \dot{u} \end{pmatrix}(t) = e^{tJH} \begin{pmatrix} u(0) \\ \dot{u}(0) \end{pmatrix} = \begin{bmatrix} \cos(t\omega) & \sin(t\omega)/\omega \\ -\omega \sin(t\omega) & \cos(t\omega) \end{bmatrix} \begin{pmatrix} u(0) \\ \dot{u}(0) \end{pmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\omega := \langle D \rangle, \ D = -i\partial_x$, $\widehat{\omega f}(\xi) = \langle \xi \rangle \hat{f}(\xi)$, $\langle \xi \rangle = \sqrt{1 + \xi^2}$.

Asymptotic stability of 0 solution for sine-Gordon

d'Alembert solution for $u_{tt} - u_{xx} = 0$ is $u(t,x) = w_r(x-t) + w_\ell(x+t)$.
No decay: characteristic variety $\tau^2 - \xi^2 = 0$ has no curvature, but for KG hyperbolas $\tau^2 - \langle\xi\rangle^2 = 0$ do have curvature.
By stationary phase and some harmonic analysis for all $t \geq 1$

$$\left\|(e^{it\omega} f)(x) - \frac{e^{i\pi/4}}{\sqrt{t}} e^{i\rho} \langle\xi_*\rangle^{\frac{3}{2}} \hat{f}(\xi_*) \mathbb{1}_{[|x|<t]}\right\|_\infty \leq \frac{C}{t^{\frac{2}{3}}} \|\langle x\rangle f\|_{H^2_x} \qquad (*)$$

where $\rho = \sqrt{t^2 - x^2}$, critical point of the phase

$$\partial_\xi (x\xi + t\langle\xi\rangle) = x + t\xi\langle\xi\rangle^{-1} = 0$$

$\xi = \xi_* = -\frac{x}{\rho}$. Curvature of the hyperbola: $\partial_\xi^2 \langle\xi\rangle = \langle\xi\rangle^{-3}$.

Theorem (Delort '01, Lindblad-Soffer '05, Hayashi-Naumkin '08)

If $\|\langle x\rangle(\phi(0), \phi_t(0))\|_{H^2_x \times H^1_x} \leq \varepsilon$, the solution to sine-Gordon satisfies

$$\|\phi(t,\cdot)\|_{L^\infty_x} \lesssim \frac{\varepsilon}{t^{\frac{1}{2}}} \qquad \forall t \geq 1$$

Logarithmic phase corrections $\log t$ *due to* ϕ^3 *term in* $\sin\phi$.

Detour through NLS

Consider small data asymptotic completeness problem for

$$iu_t + \frac{1}{2}u_{xx} = u|u|^2, \quad u(0) = u_0 \qquad (NLS)$$

Again long-range effect due to $|u|^2 \sim t^{-1}$.

Theorem (Hayashi-Naumkin '98, Lindblad-Soffer '06, Kato-Pusateri '11, Ifrim-Tataru '15)

If $\|\langle x\rangle u_0\|_{L^2_x} \leq \varepsilon$, the solution to (NLS) satisfies

$$\left\| u(t,x) - t^{-\frac{1}{2}} e^{i\frac{x^2}{2t}} W(x/t) e^{-i|W(x/t)|^2 \log t} \right\|_{L^\infty_x \cap L^2_x} \lesssim \varepsilon t^{-\frac{1}{2}-\sigma} \qquad \forall t \geq 1$$

for some $W \in H^{1-C\varepsilon^2}(\mathbb{R})$.

Wave packet: $\Psi_v(t,x) := e^{i\phi(t,x)}\chi((x-vt)/\sqrt{t})$, $\phi(t,x) = \frac{x^2}{2t}$, with $\int_{-\infty}^{\infty} \chi = 1$. Invariant under symmetries:

- Scaling $(t,x) \to (\lambda^2 t, \lambda x)$, $v \to v/\lambda$
- Galilei transforms: $(G_v(0)f)(0) := e^{ixv} f(x)$ and conjugacy $G_v(t) e^{\frac{i}{2}t\Delta} = e^{\frac{i}{2}t\Delta} G_v(0)$ imply $(G_v(t)f)(x) = e^{ixv} e^{-\frac{i}{2}tv^2} f(x-tv)$, and $G_v(t)\Psi_0(t) = \Psi_v(t)$.

NLS, wave packet dynamics

Generator $L := x + it\partial_x$ satisfies $[i\partial_t + \frac{1}{2}\partial_{xx}, L] = 0$, $e^{it\partial_x^2/2}x = Le^{it\partial_x^2/2}$
Wave packets are approximate solutions to NLS

$$(i\partial_t + \frac{1}{2}\partial_{xx})\Psi_v = t^{-1}e^{i\phi}\tilde{\chi}((x - vt)/\sqrt{t})$$

Define $\gamma(t, v) := \langle u(t), \Psi_v(t)\rangle$. Then any function u well approximated along rays by γ:

$$\|u(t, vt) - t^{-\frac{1}{2}}e^{i\phi(t,vt)}\gamma(t, v)\|_{L_v^\infty} \lesssim t^{-\frac{3}{4}}\|Lu(t)\|_2$$

Evolution equation produces logarithmic phase correction

$$\dot{\gamma}(t, v) = -it^{-1}|\gamma(t, v)|^2\gamma(t, v) - R(t, v)$$

with error bounds

$$\|R\|_{L_x^\infty} \lesssim t^{-\frac{1}{4}}\|Lu\|_{L_x^2}(t^{-1} + \|u\|_{L_x^\infty}^2), \quad \|R\|_{L_v^2} \lesssim t^{-\frac{1}{2}}\|Lu\|_{L_x^2}(t^{-1} + \|u\|_{L_x^\infty}^2)$$

Close by bootstrap: (i) slow energy growth (ii) pointwise decay

From asymptotic stability of vacuum to that of kinks

Linearizing sine-Gordon/ϕ^4 around kink ($\phi = K + u$) leads to

$$(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta(x)u^3 + O(u^4)$$

For long-term analysis only useful if K does not move, i.e., if u is odd.
Spectra: Linearized sine-Gordon has 0 eigenvalue, resonance at 1

$$H_{sG}(\text{sech}) = 0, \quad (H_{sG} - 1)(\tanh) = 0$$

Linearized ϕ^4 has 0 and $\frac{3}{2}$ as eigenvalues, and 2 as threshold resonance: with functions of the variable $x/\sqrt{2}$ we have

$$H_{\phi^4}(\text{sech}^2) = 0, \;\; (H_{\phi^4} - \frac{3}{2})(\text{sech}\tanh) = 0, \;\; (H_{\phi^4} - 2)(1 - \frac{3}{2}\,\text{sech}^2) = 0$$

Note: even ground states are $K' > 0$, come from translation symmetry. So odd perturbations keep kink stationary. Odd internal mode of H_{ϕ^4} produces harmonic oscillator at linear level.

Pöschl-Teller operators

Explicitly: $H_{sG} = -\partial_x^2 - 2\,\mathrm{sech}^2(x) + 1$, $H_{\phi^4} = -\partial_x^2 - 3\,\mathrm{sech}^2(\frac{x}{\sqrt{2}}) + 2$
Jost solutions of $H_{sG} - 1$

$$f_+(x,\xi) = \frac{i\xi - \tanh(x)}{i\xi - 1} e^{ix\xi},$$
$$f_-(x,\xi) = \frac{-i\xi - \tanh(x)}{-i\xi + 1} e^{-ix\xi}.$$

The distorted Fourier basis associated with $H_{sG} - 1$

$$e(x,\xi) := \frac{1}{\sqrt{2\pi}} \begin{cases} \frac{i\xi - \tanh(x)}{i\xi - 1} e^{ix\xi} & \text{for } \xi \geq 0, \\ \frac{i\xi - \tanh(x)}{i\xi + 1} e^{ix\xi} & \text{for } \xi < 0 \end{cases}$$

Note: discontinuity at $\xi = 0$. Transmission coefficient $T(\xi) = -1$, reflection coefficients $R_\pm(\xi) = 0$
Distorted FT used by Germain-Pusateri '20, Chen-Pusateri '19, excluding threshold resonance.

Distorted Fourier transform

$H = -\partial_{xx} + V$, $\langle x \rangle V \in (L^1 \cap L^\infty)(\mathbb{R})$, V real-valued.
Jost solutions $Hf_\pm = \xi f_\pm$ with $f_\pm(x,\xi) \sim e^{\pm ix\xi}$ as $x \to \pm\infty$. Wronskian $W(\xi) = W[f_+(\cdot,\xi), f_-(\cdot,\xi)] \neq 0$ if $\xi \neq 0$. Reflection, transmission coefficients $R_\pm$, T

$$f_+(\cdot,-\xi) + R_+(\xi)f_+(\cdot,\xi) = T(\xi)f_-(x,\xi),$$
$$f_-(\cdot,-\xi) + R_-(\xi)f_-(\cdot,\xi) = T(\xi)f_+(x,\xi)$$

The following are equivalent:

- 0 is a resonance
- $W(0) = 0$
- $T(0) \neq 0$
- $\exists\, \varphi \neq 0$, bounded with $H\varphi = 0$ (then automatically $\varphi \notin L^2$)
- $(H - z^2)^{-1} = -z^{-1}\varphi \otimes \varphi + O(1)$ as $z \to 0$ (small divisor)

The distorted Fourier basis associated with H

$$e(x,\xi) := \frac{1}{\sqrt{2\pi}} \begin{cases} T(\xi)f_+(x,\xi), & \xi \geq 0, \\ T(-\xi)f_-(x,-\xi), & \xi < 0. \end{cases}$$

$\tilde{g}(\xi) = \langle g, e(\cdot,\xi) \rangle$, isometry on $L^2_{\mathrm{cont}}(\mathbb{R})$. $e(x,\xi) \otimes \overline{e(y,\xi)} = \delta_0(x-y)$

sine-Gordon kink asymp. stable under odd perturbations

Theorem (Lührmann-S. '21)

$\exists\, \varepsilon_0 \in (0,1)$ so that data $(K + u_0, u_1)$ with $\|\langle x\rangle (u_0,u_1)\|_{H^3_x \times H^2_x} = \varepsilon \le \varepsilon_0$ and odd lead to sine-Gordon field ϕ where $u(t,x) := \phi(t,x) - K(x)$ decays:

$$\|u(t,\cdot)\|_{L^\infty_x} \lesssim \varepsilon (1+t)^{-\frac{1}{2}}, \quad t \ge 0.$$

Moreover, $\exists\, \widehat{W} \in L^\infty$ and $0 < \delta \ll 1$ s.t.

$$\left| u(t,x) + 2Re\left(\frac{e^{i\frac{\pi}{4}}}{t^{\frac{1}{2}}} \int_0^x \frac{\cosh(y)}{\cosh(x)} e^{i\rho} e^{-i\psi(\frac{y}{\rho})\log(t)} \widehat{W}\big(\tfrac{y}{\rho}\big) \mathbb{1}_{(-1,1)}\big(\tfrac{y}{t}\big)\, \mathrm{d}y \right) \right| \lesssim \frac{\varepsilon}{t^{\frac{2}{3}-\delta}},$$

where $\rho := \sqrt{t^2 - y^2}$ and

$$\psi(\xi) := \frac{1}{4} \langle \xi \rangle^{-7} \big(1 + 3\xi^2\big) \big|\widehat{W}(\xi)\big|^2.$$

- Asymp. stability false in local $H^1 \times L^2$, even with a weighted condition $\langle x\rangle^s, 0 < s < \frac{1}{2}$: wobbling kinks. ϕ^4 different.
- Orbital stability of kinks: Henry-Perez-Wreszinski ('82)

Further comments

- open problem: moving kinks and general data
- Alejo-Munoz-Palacios '20: Infinite co-dimensional smooth manifold in odd energy class near kink $\to$ asymptotically stable. Bäcklund transform, sine-Gordon as Gauss-Codazzi equations on pseudosphere.
- asymptotic stability of kink for odd data, and soliton resolution for generic data: Chen-Liu-Lu ('20) via inverse scattering; steepest descent, Riemann-Hilbert and ∂-bar. Genericity excludes kinks and breathers moving in parallel and other degeneracies.
- open problem: analogue for ϕ^4, internal mode leads to harmonic oscillator on the linear level, FGR coupling (which guarantees *resonance*) to continuous spectrum gives decay ? See Soffer-Weinstein '99.
- For ϕ^4, local asymptotic stability in energy class of kink, i.e., in odd $H^1 \times L^2(J)$, J compact: see Kowalczyk, Martel, Muñoz ('17), K., M., M., v.d.Bosch ('20). Essential to be perpendicular to threshold resonance for their virial/energy arguments to work.
- For ϕ^4 up to times ε^{-4+c} asymptotic stability of kink under odd perturbations proved by Delort-Masmoudi '20. Conjugate by intertwining wave operators to remove potential. They are perpendicular to threshold resonance.

Normal forms

A general model problem is of the form

$$(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta_0 u^3$$

Goal: Prove decay to zero of $u(t)$ for small initial data (ideally with sharp decay rates and asymptotics)

Try to remove the catastrophic quadratic term via normal forms, first for $V = 0$. Define new dependent variable $v := u + \gamma(x)u^2$. Then

$$v_{tt} - v_{xx} + v = (\alpha - \gamma - \gamma_{xx})v^2 + 2\gamma(v_t^2 - v_x^2) - 4\gamma_x v v_x + O(v^3)$$

We expect better local decay for v_x via linear estimate

$$\|\langle x\rangle^{-3}\partial_x e^{it\omega} f\|_2 \le C\langle t\rangle^{-\frac{3}{2}}\|\langle x\rangle^3 f\|_2$$

moreover, by asymptotics of free flow, see $(*)$:

$$v_t(t,0) = \pm i v(t,0) + O(t^{-\frac{3}{2}})$$

So heuristically at least we arrive at

$$\alpha - 3\gamma - \gamma_{xx} = 0, \qquad \hat{\alpha}(\xi) = (3 - \xi^2)\hat{\gamma}(\xi)$$

Resonant quadratic term: slowdown on two rays

- Nonresonant condition $\hat{\alpha}(\pm\sqrt{3}) = 0 \longrightarrow$ solve for $\gamma \in \mathcal{S}(\mathbb{R})$.
- dynamic formation of a resonant quadratic term

$$u_{lin}(t,0) \sim \frac{e^{\pm it}}{t^{\frac{1}{2}}} \quad \rightsquigarrow \quad \alpha(x)\frac{e^{2it}}{t}$$

- Lindblad-Lührmann-Soffer ('20) found logarithmic slow-down along rays $x = \pm\frac{\sqrt{3}}{2}t$. Duhamel integral exhibits time resonance

$$\int_1^t e^{i(t-s)\langle\xi\rangle}\widehat{\alpha}(\xi)\frac{e^{2is}}{s}\,\mathrm{d}s \qquad \text{small divisor } \; 2-\langle\xi\rangle = 0$$

  iff $\xi = \pm\sqrt{3}$ (note: $\frac{tv}{\sqrt{t^2-t^2v^2}} = \pm\sqrt{3}$, $v = \pm\sqrt{3}/2$)
- full nonlinear flow for $\alpha(x)u^2$ or $\alpha(x)u^2 + \beta_0 u^3$? The latter not understood if quadratic term resonant and $\beta_0 \neq 0$.
- Remarkably, the logarithmic slowdown $\frac{\log t}{\sqrt{t}}$ caused by threshold resonance, for free case by $-\partial_{xx}\, 1 = 0$.

Resonant quadratic term and threshold resonances

- Lindblad-Lührmann-S.-Soffer ('20): Decay and asymptotics for

$$(\partial_t^2 - \partial_x^2 + V(x) + 1)u = P_c(\alpha(\cdot)u^2)$$

for non-generic potentials $V(x)$ and spatially localized $\alpha(x)$.

  - $H\varphi = 0$, $\varphi(x) \to C_\pm \neq 0$ as $x \to \pm\infty$
  - logarithmic slow-down if $\widetilde{\mathcal{F}}[\alpha\varphi^2](\pm\sqrt{3}) \neq 0$
  - sine-Gordon model exhibits non-resonance property

$$\widetilde{\mathcal{F}}[\alpha\varphi^2](\pm\sqrt{3}) = 0$$

With $\langle \widetilde{D} \rangle P_{(0,\infty)}(H) = \sqrt{1+H}P_{(0,\infty)}(H)$ and all $t \geq 1$

$$\Big\| \langle x \rangle^{-4} \Big(e^{it\langle \widetilde{D} \rangle} \chi_0(H) P_c\, g - c_0 \frac{e^{i\frac{\pi}{4}} e^{it}}{t^{\frac{1}{2}}} (\varphi \otimes \varphi) g \Big) \Big\|_{L_x^\infty} \leq \frac{C}{t^{\frac{3}{2}}} \| \langle x \rangle^4 g \|_{L_x^1}$$

$\chi_0(H)$ a small energy cut-off. Stationary phase argument, using the resolvent rather than distorted Fourier transform (Stone formula). Tensor structure present only at 0 energy.

Main proof ideas: supersymmetry

Evolution equation for a perturbation of the static sine-Gordon kink

$$\big(\partial_t^2 - \partial_x^2 - 2\,\mathrm{sech}^2(x) + 1\big)\,u = -\,\mathrm{sech}(x)\,\tanh(x)u^2 + \tfrac{1}{6}\,u^3 + \dots$$

Step 1: Super-symmetric factorization and the transformed equation.

The linearized operator admits the factorization

$$\begin{aligned} \mathcal{D}\mathcal{D}^* &= -\partial_x^2 - 2\,\mathrm{sech}^2(x) + 1, \\ \mathcal{D}^*\mathcal{D} &= -\partial_x^2 + 1, \end{aligned}$$

where

$$\mathcal{D} = \partial_x - \tanh(x) = \cosh(x)\partial_x\,\mathrm{sech}(x), \quad \mathrm{sech} = \partial_x K$$

Differentiating the evolution equation for $u(t)$ by $\mathcal{D}^*$ gives

$$(\partial_t^2 - \partial_x^2 + 1)(\mathcal{D}^* u) = \mathcal{D}^*\big(\alpha(x)u^2 + \beta_0 u^3 + \dots\big).$$

Rodnianski-Sterbenz '10, Raphaël-Rodnianski '12, Krieger-Miao '18, Krieger-Miao-S. '20, Kowalczyk-Martel-Muñoz '19, Kowalczyk-Martel-Muñoz-Bosch '20, ...

Transformed PDE

We pass to the transformed nonlinear Klein-Gordon equation for the new (even) variable

$$w(t,x) := (\mathcal{D}^* u)(t,x).$$

Using that

$$u(t,x) = \mathcal{I}[w(t,\cdot)](x) := -\operatorname{sech}(x) \int_0^x \cosh(y) w(t,y)\,\mathrm{d}y,$$

we obtain

$$(\partial_t^2 - \partial_x^2 + 1)w = \mathcal{Q}(w) + \mathcal{C}(w) + \{\text{higher order}\},$$

with quadratic nonlinearities given by

$$\mathcal{Q}(w) = \left(-2\operatorname{sech}(x) + 3\operatorname{sech}^3(x)\right)\left(\mathcal{I}[w]\right)^2 - 2\operatorname{sech}(x)\tanh(x)\mathcal{I}[w]w.$$

Resonant quadratic term?

Step 2: Variable coefficient quadratic normal form.

Rewrite quadratic nonlinearities as

$$\mathcal{Q}(w) = \alpha_1(x) w^2 + \alpha_2(x) \widetilde{\mathcal{I}}[\partial_x w] w + \alpha_3(x) \big(\widetilde{\mathcal{I}}[\partial_x w] \big)^2,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{S}(\mathbb{R})$ are spatially localized and

$$\widetilde{\mathcal{I}}[\partial_x w(t)](x) := \operatorname{sech}(x) \int_0^x \sinh(y) (\partial_x w)(t, y) \, \mathrm{d}y.$$

Main enemy: The quadratic nonlinearity $\alpha_1(x) w^2$ could potentially lead to a logarithmic slowdown caused by the threshold resonance.

Key observation: The Fourier transform of $\alpha_1(x) = 3 \operatorname{sech}^3(x) \tanh^2(x)$ vanishes at the bad resonant frequencies ("null structure")

$$\widehat{\alpha}_1(\pm\sqrt{3}) = 0.$$

Normal form

Pass to the first-order formulation

$$v(t) := w(t) - i\langle D\rangle^{-1}(\partial_t w)(t)$$

use variable coefficient quadratic normal form (Lindblad-Lührmann-Soffer '20)

$$B(v,v)(t) := \alpha_{11}(x)v(t,0)^2 + \alpha_{12}(x)|v(t,0)|^2 + \alpha_{13}(x)\bar{v}(t,0)^2$$

with

$$\begin{aligned}
\widehat{\alpha}_{11}(\xi) &:= \frac{1}{2}\langle\xi\rangle^{-1}(2-\langle\xi\rangle)^{-1}\widehat{\alpha}_1(\xi),\\
\widehat{\alpha}_{12}(\xi) &:= -\langle\xi\rangle^{-2}\widehat{\alpha}_1(\xi),\\
\widehat{\alpha}_{13}(\xi) &:= -\frac{1}{2}\langle\xi\rangle^{-1}(2+\langle\xi\rangle)^{-1}\widehat{\alpha}_1(\xi),
\end{aligned}$$

to recast the problematic quadratic nonlinearity $\alpha_1(x)(v+\bar{v})^2$ into "variable coefficient cubic type".

PDE after normal form: cubic nonlinearities

Step 3: Analysis of long-range effects of non-local cubic nonlinearities.

We arrive at the following first-order nonlinear Klein-Gordon equation for the renormalized variable $v + B(v, v)$,

$$(\partial_t - i\langle D\rangle)\big(v + B(v, v)\big) = (2i)^{-1}\langle D\rangle^{-1}\big(\mathcal{Q}_{ren}(v, v) + \mathcal{C}(v + \bar{v}) + \ldots\big).$$

At this point we follow the so-called space-time resonances method to infer decay and asymptotics of $v(t, x)$, which we then transfer back to the original variable $u(t, x)$ via

$$u(t, x) = \mathcal{I}[v(t) + \bar{v}(t)](x).$$

Some key points:

- Fourier analysis of all nonlinearities can be carried out explicitly.
- Obtain slow energy growth estimates $\|Zv(t)\|_{L^2_x} \lesssim t^{\delta}\varepsilon$ for Lorentz boost $Z = t\partial_x + x\partial_t$ via delicate local decay argument and favorable structure of the non-local cubic nonlinearities.

Outlook

The hierarchy of Schrödinger operators with Pöschl-Teller potentials[1]

$$H_n = -\partial_x^2 - n(n+1)\,\mathrm{sech}^2(x), \quad n \in \mathbb{N}_0,$$

exhibits higher-order super-symmetric factorization properties and comes up in several classical 1D asymptotic stability problems.

Possible future applications:

- Asymptotic stability of the ϕ^4 kink (H_2): internal mode, FGR coupling to the continuous spectrum. Super-symmetric method does apply in principle for odd data ($\perp$ threshold resonance), internal mode produces resonant quadratic term.
- Conditional asymptotic stability of solitons of the 1D focusing cubic and quadratic Klein-Gordon equations (H_2 and H_3)

Major difficulties:
Interplay between resonant quadratic terms due to internal modes and/or threshold resonances, and the long-range effects of the non-localized cubic nonlinearity.

[1] Pöschl, G., Teller, E. Bemerkungen zur Quantenmechanik des anharmonischen Oszillators. Z. Physik 83, 143–151 (1933)