Asymptotic stability for the sine-Gordon kink under odd perturbations via supersymmetry

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Classical nonlinear scalar field theories on the line

For scalar fields \( \phi: \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{R} \) and scalar potentials \( W: \mathbb{R} \rightarrow [0, \infty) \), we consider the Lagrangian action functional

\[
\mathcal{L}[\phi] = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( -\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + W(\phi) \right) \, dx \, dt.
\]

Euler-Lagrange or equation of motion

\[
(\partial_t^2 - \partial_x^2)\phi = -W'(\phi), \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\]

Hamiltonian or conserved energy functional

\[
\mathcal{E}[\phi] = \int_{\mathbb{R}} \left( \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + W(\phi) \right) \, dx.
\]

Goal: Understand long-time dynamics of all solutions of finite energy. These do not need to decay at \( \pm \infty \) or have finite \( \| \phi(t) \|_{L_x^2} \).
Double-well scalar potentials

Classical examples:
- $\phi^4$ model: $W(\phi) = \frac{1}{4}(1 - \phi^2)^2 \Rightarrow \phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0$
- sine-Gordon model: $W(\phi) = 1 - \cos(\phi)$ [completely integrable] \Rightarrow \phi_{tt} - \phi_{xx} + \sin \phi = 0

Topology of finite energy solutions: $\lim_{x \to \pm \infty} \phi(t, x) \in \{\phi_-, \phi_+\}$. For sine-Gordon all translations by $2\pi \mathbb{Z}$. 
Vacuum solutions (for sine-Gordon add $2\pi\mathbb{Z}$):

$$\phi(t, x) = \phi_{\pm} \quad \forall t \in \mathbb{R}.$$ 

These are all **constant** stationary solutions of finite energy, while all finite energy equilibria include kinks/anti-kinks.

Kink solutions as heteroclinic orbits in the phase portrait:

$$\begin{align*}
\partial_x^2 K &= W'(K), \quad x \in \mathbb{R}, \\
\lim_{x \to \pm \infty} K(x) &= \phi_{\pm}.
\end{align*}$$

Classical examples (unique up to translation and $\mathbb{Z}_2$ symmetry):

- $\phi^4$ model: $K(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$
- Sine-Gordon model: $K(x) = 4 \arctan(e^x) \mod 2\pi\mathbb{Z}$
Current \( \mathbf{j} = (\phi_x, -\phi_t) \) satisfies continuity equation \( \text{div}_{t,x} \mathbf{j} = 0 \) and the charge (for sine-Gordon normalize by \( \frac{1}{2\pi} \))

\[
Q(\phi) := \frac{1}{2} \int_{-\infty}^{\infty} j_0(t,x) \, dx = \frac{1}{2} (\phi(t,\infty) - \phi(t,-\infty))
\]

is conserved:

\[
\partial_t Q(\phi) = -\frac{1}{2} \int_{-\infty}^{\infty} \partial_x j_1(t,x) \, dx = -\frac{1}{2} (j_1(t,\infty) - j_1(t,-\infty)) = 0
\]

For \text{vacua} \( Q = 0 \), while for kinks/anti-kinks \( Q = \pm 1 \). For \( \phi^4 \) these are all \( Q \). For s-G \( Q \in \mathbb{Z} \) (multi-(anti)kinks, only stationary if \( Q \in \{0, \pm 1\} \)).

Conserved momentum \( P = \int \phi_t \phi_x \, dx \), Newton’s law \( \dot{P} = F \) defines \textbf{force} between (anti)kinks.

Poincaré group (Lorentz and translations) acts on (anti)kinks. \textbf{Soliton resolution?} For (sG) have IST, see Chen-Liu-Lu ’20. But not for \( \phi^4 \), so fairly uncharted territory. First step is to find perturbative PDE methods other than IST to analyze (sG).
Stability of 0 solution for sine-Gordon

Orbital stability: for all $\varepsilon > 0$ there exists $\delta > 0$ such that 
$\|(\phi(0), \phi_t(0))\|_{H^1 \times L^2} < \delta$ implies $\|(\phi(t), \phi_t(t))\|_{H^1 \times L^2} < \varepsilon$ for all $t \geq 0.$

Energy is a Lyapunov functional

$$\mathcal{E}[\phi] = \frac{1}{2} \int_{\mathbb{R}} \left( (\partial_t \phi)^2 + (\partial_x \phi)^2 + \phi^2(1-O(\phi^2)) \right) dx.$$ 

Asymptotic stability: in addition, $\lim_{t \to \infty} \|(\phi, \phi_t)(t)\|_X = 0,$ but $X = ?$

If $\|\phi\|_\infty \to 0$ then $\sin \phi = \phi (1 + O(\phi^2)).$

Free Klein-Gordon equation $u_{tt} - u_{xx} + u = 0$ has solution

$$\begin{pmatrix} u \\ \dot{u} \end{pmatrix}(t) = e^{tJH} \begin{pmatrix} u(0) \\ \dot{u}(0) \end{pmatrix} = \begin{bmatrix} \cos(t\omega) & \sin(t\omega)/\omega \\ -\omega \sin(t\omega) & \cos(t\omega) \end{bmatrix} \begin{pmatrix} u(0) \\ \dot{u}(0) \end{pmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\omega := \langle D \rangle, \quad D = -i\partial_x, \quad \hat{\omega}f(\xi) = \langle \xi \rangle \hat{f}(\xi), \quad \langle \xi \rangle = \sqrt{1 + \xi^2}.$$
Asymptotic stability of 0 solution for sine-Gordon

d’Alembert solution for \( u_{tt} - u_{xx} = 0 \) is \( u(t, x) = w_r(x - t) + w_\ell(x + t) \).

No decay: characteristic variety \( \tau^2 - \xi^2 = 0 \) has no curvature, but for KG hyperbolas \( \tau^2 - \langle \xi \rangle^2 = 0 \) do have curvature.

By stationary phase and some harmonic analysis for all \( t \geq 1 \)

\[
\left\| \left( e^{it\omega} f \right)(x) - \frac{e^{i\pi/4}}{\sqrt{t}} e^{i\rho} \langle \xi_* \rangle^{3/2} \hat{f}(\xi_*) \mathbb{1}_{|x| < t} \right\|_\infty \leq \frac{C}{t^{3/2}} \| \langle x \rangle f \|_{H_x^2} \quad \text{(*)}
\]

where \( \rho = \sqrt{t^2 - x^2} \), critical point of the phase

\[ \partial_\xi (x\xi + t\langle \xi \rangle) = x + t\xi \langle \xi \rangle^{-1} = 0 \]

\( \xi = \xi_* = -\frac{x}{\rho} \). Curvature of the hyperbola: \( \partial_\xi^2 \langle \xi \rangle = \langle \xi \rangle^{-3} \).

**Theorem** (Delort ’01, Lindblad-Soffer ’05, Hayashi-Naumkin ’08)

If \( \| \langle x \rangle (\phi(0), \phi_t(0)) \|_{H_x^2 \times H_x^1} \leq \varepsilon \), the solution to sine-Gordon satisfies

\[ \| \phi(t, \cdot) \|_{L_x^\infty} \lesssim \frac{\varepsilon}{t^{1/2}} \quad \forall t \geq 1 \]

Logarithmic phase corrections \( \log t \) due to \( \phi^3 \) term in \( \sin \phi \).
Detour through NLS

Consider small data asymptotic completeness problem for

\[ iu_t + \frac{1}{2} u_{xx} = u|u|^2, \quad u(0) = u_0 \]  \quad (NLS)

Again long-range effect due to \( |u|^2 \sim t^{-1} \).

**Theorem** (Hayashi-Naumkin '98, Lindblad-Soffer '06, Kato-Pusateri '11, Ifrim-Tataru '15)

If \( \| \langle x \rangle u_0 \|_{L_x^2} \leq \varepsilon \), the solution to (NLS) satisfies

\[
\left\| u(t, x) - t^{-\frac{1}{2}} e^{i \frac{x^2}{2t}} W(x/t) e^{-i |W(x/t)|^2 \log t} \right\|_{L_x^\infty \cap L_x^2} \lesssim \varepsilon t^{-\frac{1}{2} - \sigma} \quad \forall t \geq 1
\]

for some \( W \in H^{1-C\varepsilon^2}(\mathbb{R}) \).

Wave packet: \( \psi_v(t, x) := e^{i \phi(t, x)} \chi((x - vt)/\sqrt{t}), \phi(t, x) = \frac{x^2}{2t} \), with \( \int_{-\infty}^{\infty} \chi = 1 \). Invariant under symmetries:

- **Scaling** \((t, x) \rightarrow (\lambda^2 t, \lambda x), \quad v \rightarrow v/\lambda\)
- **Galilei transforms**: \( (G_v(0)f)(0) := e^{ixv} f(x) \) and conjugacy \( G_v(t)e^{\frac{ix}{2} t \Delta} \) implies \( (G_v(t)f)(x) = e^{ixv} e^{-\frac{ix}{2} tv^2} f(x - tv) \), and \( G_v(t)\psi_0(t) = \psi_v(t) \).
NLS, wave packet dynamics

Generator $L := x + it \partial_x$ satisfies $[i \partial_t + \frac{1}{2} \partial_{xx}, L] = 0$, $e^{it \partial_x^2 / 2} x = L e^{it \partial_x^2 / 2}$

Wave packets are approximate solutions to NLS

$$(i \partial_t + \frac{1}{2} \partial_{xx}) \psi_v = t^{-1} e^{i \phi} \tilde{\chi}((x - vt)/\sqrt{t})$$

Define $\gamma(t, v) := \langle u(t), \psi_v(t) \rangle$. Then any function $u$ well approximated along rays by $\gamma$:

$$\| u(t, vt) - t^{-\frac{1}{2}} e^{i \phi(t, vt)} \gamma(t, v) \|_{L^\infty_x} \lesssim t^{-\frac{3}{4}} \| Lu(t) \|_2$$

Evolution equation produces logarithmic phase correction

$$\dot{\gamma}(t, v) = -it^{-1} |\gamma(t, v)|^2 \gamma(t, v) - R(t, v)$$

with error bounds

$$\| R \|_{L^\infty_x} \lesssim t^{-\frac{1}{4}} \| Lu \|_{L^2_x} (t^{-1} + \| u \|_{L^\infty_x}^2), \quad \| R \|_{L^2_v} \lesssim t^{-\frac{1}{2}} \| Lu \|_{L^2_x} (t^{-1} + \| u \|_{L^\infty_x}^2)$$

Close by bootstrap: (i) slow energy growth (ii) pointwise decay
From asymptotic stability of vacuum to that of kinks

Linearizing sine-Gordon/\(\phi^4\) around kink \((\phi = K + u)\) leads to

\[
(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta(x)u^3 + O(u^4)
\]

For long-term analysis only useful if \(K\) does not move, i.e., if \(u\) is odd.

**Spectra:** Linearized sine-Gordon has 0 eigenvalue, resonance at 1

\[
H_{\text{sg}}(\text{sech}) = 0, \quad (H_{\text{sg}} - 1)(\text{tanh}) = 0
\]

Linearized \(\phi^4\) has 0 and \(\frac{3}{2}\) as eigenvalues, and 2 as threshold resonance: with functions of the variable \(x/\sqrt{2}\) we have

\[
H_{\phi^4}(\text{sech}^2) = 0, \quad (H_{\phi^4} - \frac{3}{2})(\text{sech tanh}) = 0, \quad (H_{\phi^4} - 2)(1 - \frac{3}{2}\text{sech}^2) = 0
\]

Note: even ground states are \(K' > 0\), come from translation symmetry. So odd perturbations keep kink stationary. Odd internal mode of \(H_{\phi^4}\) produces harmonic oscillator at linear level.
Pöschl-Teller operators

Explicitly: $H_{sG} = -\partial_x^2 - 2 \text{sech}^2(x) + 1$, $H_{\phi^4} = -\partial_x^2 - 3 \text{sech}^2\left(\frac{x}{\sqrt{2}}\right) + 2$

Jost solutions of $H_{sG} - 1$

$$f_+(x, \xi) = \frac{i\xi - \tanh(x)}{i\xi - 1} e^{ix\xi},$$

$$f_-(x, \xi) = \frac{-i\xi - \tanh(x)}{-i\xi + 1} e^{-ix\xi}.$$ 

The distorted Fourier basis associated with $H_{sG} - 1$

$$e(x, \xi) := \frac{1}{\sqrt{2\pi}} \begin{cases} 
\frac{i\xi - \tanh(x)}{i\xi - 1} e^{ix\xi} & \text{for } \xi \geq 0, \\
\frac{i\xi - \tanh(x)}{i\xi + 1} e^{ix\xi} & \text{for } \xi < 0
\end{cases}$$

Note: discontinuity at $\xi = 0$. Transmission coefficient $T(\xi) = -1$, reflection coefficients $R_{\pm}(\xi) = 0$

Distorted FT used by Germain-Pusateri ‘20, Chen-Pusateri ‘19, excluding threshold resonance.
Distorted Fourier transform

\[ H = -\partial_{xx} + V, \langle x \rangle V \in (L^1 \cap L^\infty)(\mathbb{R}), \]\[ V \text{ real-valued.} \]

Jost solutions \( Hf_\pm = \xi f_\pm \) with \( f_\pm(x, \xi) \sim e^{\pm ix\xi} \) as \( x \to \pm \infty \). Wronskian \( W(\xi) = W[f_+(\cdot, \xi), f_-(\cdot, \xi)] \neq 0 \) if \( \xi \neq 0 \). Reflection, transmission coefficients \( R_\pm, T \)

\[ f_+(\cdot, -\xi) + R_+(\xi)f_+(\cdot, \xi) = T(\xi)f_-(x, \xi), \]
\[ f_-(\cdot, -\xi) + R_-(\xi)f_-(\cdot, \xi) = T(\xi)f_+(x, \xi) \]

The following are equivalent:

- 0 is a resonance
- \( W(0) = 0 \)
- \( T(0) \neq 0 \)
- \( \exists \varphi \neq 0 \), bounded with \( H\varphi = 0 \) (then automatically \( \varphi \notin L^2 \))
- \( (H - z^2)^{-1} = -z^{-1}\varphi \otimes \varphi + O(1) \) as \( z \to 0 \) (small divisor)

The distorted Fourier basis associated with \( H \)

\[ e(x, \xi) := \frac{1}{\sqrt{2\pi}} \begin{cases} T(\xi)f_+(x, \xi), & \xi \geq 0, \\ T(-\xi)f_-(x, -\xi), & \xi < 0. \end{cases} \]

\( \tilde{g}(\xi) = \langle g, e(\cdot, \xi) \rangle \), isometry on \( L^2_{\text{cont}}(\mathbb{R}) \). \( e(x, \xi) \otimes e(y, \xi) = \delta_0(x - y) \)
sine-Gordon kink asymp. stable under odd perturbations

**Theorem (Lührmann-S. '21)**

∃ \( \varepsilon_0 \in (0,1) \) so that data \((K + u_0, u_1)\) with \( \|\langle x\rangle (u_0, u_1)\|_{H_x^3 \times H_x^2} = \varepsilon \leq \varepsilon_0 \) and odd lead to sine-Gordon field \( \phi \) where \( u(t, x) := \phi(t, x) - K(x) \) decays:

\[
\|u(t, \cdot)\|_{L_x^\infty} \lesssim \varepsilon (1 + t)^{-\frac{1}{2}}, \quad t \geq 0.
\]

Moreover, \( \exists \hat{W} \in L^\infty \) and \( 0 < \delta \ll 1 \) s.t.

\[
\left| u(t, x) + 2\text{Re} \left( \frac{e^{i \pi \frac{1}{4}}}{t^{\frac{1}{2}}} \int_0^x \frac{\cosh(y)}{\cosh(x)} e^{i \rho} e^{-i \psi(\frac{y}{\rho}) \log(t)} \hat{W}(\frac{y}{\rho}) \mathbb{1}_{(-1,1)}(\frac{y}{t}) \, dy \right) \right| \lesssim \frac{\varepsilon}{t^{\frac{2}{3}} - \delta},
\]

where \( \rho := \sqrt{t^2 - y^2} \) and

\[
\psi(\xi) := \frac{1}{4} \langle \xi \rangle^{-7} (1 + 3\xi^2) |\hat{W}(\xi)|^2.
\]

▶ Asymp. stability false in local \( H^1 \times L^2 \), even with a weighted condition \( \langle x \rangle^s, 0 < s < \frac{1}{2} \): wobbling kinks. \( \phi^4 \) different.

▶ Orbital stability of kinks: Henry-Perez-Wrezinski ('82)
Further comments

- **open problem**: moving kinks and general data
- **Alejo-Munoz-Palacios '20**: Infinite co-dimensional smooth manifold in odd energy class near kink $\rightarrow$ asymptotically stable. Bäcklund transform, sine-Gordon as Gauss-Codazzi equations on pseudosphere.
- asymptotic stability of kink for odd data, and soliton resolution for generic data: Chen-Liu-Lu ('20) via inverse scattering; steepest descent, Riemann-Hilbert and $\partial$-bar. Genericity excludes kinks and breathers moving in parallel and other degeneracies.
- open problem: analogue for $\phi^4$, internal mode leads to harmonic oscillator on the linear level, FGR coupling (which guarantees resonance) to continuous spectrum gives decay? See Soffer-Weinstein '99.
- For $\phi^4$, local asymptotic stability in energy class of kink, i.e., in odd $H^1 \times L^2(J)$, $J$ compact: see Kowalczyk, Martel, Muñoz ('17), K., M., M., v.d.Bosch ('20). Essential to be perpendicular to threshold resonance for their virial/energy arguments to work.
- For $\phi^4$ up to times $\varepsilon^{-4+c}$ asymptotic stability of kink under odd perturbations proved by Delort-Masmoudi '20. Conjugate by intertwining wave operators to remove potential. They are perpendicular to threshold resonance.
Normal forms

A general model problem is of the form

$$(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta_0 u^3$$

**Goal:** Prove decay to zero of $u(t)$ for small initial data (ideally with sharp decay rates and asymptotics)

Try to remove the catastrophic quadratic term via normal forms, first for $V = 0$. Define new dependent variable $v := u + \gamma(x)u^2$. Then

$$v_{tt} - v_{xx} + v = (\alpha - \gamma - \gamma_{xx})v^2 + 2\gamma(v_t^2 - v_x^2) - 4\gamma_x vv_x + O(v^3)$$

We expect better local decay for $v_x$ via linear estimate

$$\|\langle x \rangle^{-3} \partial_x e^{it\omega} f \|_2 \leq C\langle t \rangle^{-\frac{3}{2}}\|\langle x \rangle^3 f \|_2$$

moreover, by asymptotics of free flow, see (*):

$$v_t(t, 0) = \pm iv(t, 0) + O(t^{-\frac{3}{2}})$$

So heuristically at least we arrive at

$$\alpha - 3\gamma - \gamma_{xx} = 0, \quad \hat{\alpha}(\xi) = (3 - \xi^2)\hat{\gamma}(\xi)$$
Resonant quadratic term: slowdown on two rays

- Nonresonant condition $\hat{\alpha}(\pm \sqrt{3}) = 0 \longrightarrow \text{solve for } \gamma \in S(\mathbb{R})$.
- Dynamic formation of a resonant quadratic term

$$u_{\text{lin}}(t,0) \sim \frac{e^{\pm it}}{t^{\frac{1}{2}}} \quad \sim \quad \alpha(x) \frac{e^{2it}}{t}$$

- Lindblad-Lührmann-Soffer (’20) found logarithmic slow-down along rays $x = \pm \frac{\sqrt{3}}{2} t$. Duhamel integral exhibits time resonance

$$\int_1^t e^{i(t-s)} \langle \xi \rangle \hat{\alpha}(\xi) \frac{e^{2is}}{s} \, ds$$

small divisor $2 - \langle \xi \rangle = 0$

iff $\xi = \pm \sqrt{3}$ (note: $\frac{tv}{\sqrt{t^2 - t^2 v^2}} = \pm \sqrt{3}$, $v = \pm \sqrt{3}/2$)

- Full nonlinear flow for $\alpha(x)u^2$ or $\alpha(x)u^2 + \beta_0 u^3$? The latter not understood if quadratic term resonant and $\beta_0 \neq 0$.

- Remarkably, the logarithmic slowdown $\frac{\log t}{\sqrt{t}}$ caused by threshold resonance, for free case by $-\partial_{xx} 1 = 0$. 

Resonant quadratic term and threshold resonances

- Lindblad-Lührmann-S.-Soffer ('20): Decay and asymptotics for

\[(\partial_t^2 - \partial_x^2 + V(x) + 1)u = P_c(\alpha(\cdot)u^2)\]

for non-generic potentials \(V(x)\) and spatially localized \(\alpha(x)\).

- \(H\varphi = 0, \varphi(x) \to C_{\pm} \neq 0 \text{ as } x \to \pm\infty\)
- logarithmic slow-down if \(\tilde{F}[\alpha\varphi^2](\pm\sqrt{3}) \neq 0\)
- sine-Gordon model exhibits non-resonance property

\[\tilde{F}[\alpha\varphi^2](\pm\sqrt{3}) = 0\]

With \(\langle \tilde{D} \rangle P_{(0,\infty)}(H) = \sqrt{1 + HP_{(0,\infty)}(H)}\) and all \(t \geq 1\)

\[\left\| \langle x \rangle^{-4} \left( e^{i t \langle \tilde{D} \rangle} \chi_0(H)P_c g - c_0 \frac{e^{i \frac{\pi}{4}}}{t^{\frac{1}{2}}} (\varphi \otimes \varphi) g \right) \right\|_{L_x^\infty} \leq \frac{C}{t^{\frac{3}{2}}} \left\| \langle x \rangle^4 g \right\|_{L_x^1}\]

\(\chi_0(H)\) a small energy cut-off. Stationary phase argument, using the resolvent rather than distorted Fourier transform (Stone formula). Tensor structure present only at 0 energy.
Main proof ideas: supersymmetry
Evolution equation for a perturbation of the static sine-Gordon kink

\[
(\partial_t^2 - \partial_x^2 - 2 \text{sech}^2(x) + 1) u = -\text{sech}(x) \tanh(x) u^2 + \frac{1}{6} u^3 + \ldots
\]

Step 1: Super-symmetric factorization and the transformed equation.

The linearized operator admits the factorization

\[
\mathcal{D} \mathcal{D}^* = -\partial_x^2 - 2 \text{sech}^2(x) + 1, \\
\mathcal{D}^* \mathcal{D} = -\partial_x^2 + 1,
\]

where

\[
\mathcal{D} = \partial_x - \tanh(x) = \cosh(x) \partial_x \text{sech}(x), \quad \text{sech} = \partial_x K
\]

Differentiating the evolution equation for \( u(t) \) by \( \mathcal{D}^* \) gives

\[
(\partial_t^2 - \partial_x^2 + 1)(\mathcal{D}^* u) = \mathcal{D}^*(\alpha(x) u^2 + \beta_0 u^3 + \ldots).
\]

Rodnianski-Sterbenz '10, Raphaël-Rodnianski '12, Krieger-Miao '18, Krieger-Miao-S. '20, Kowalczyk-Martel-Muñoz '19, Kowalczyk-Martel-Muñoz-Bosch '20, ...
We pass to the transformed nonlinear Klein-Gordon equation for the new (even) variable

$$w(t, x) := (D^* u)(t, x).$$

Using that

$$u(t, x) = \mathcal{I}[w(t, \cdot)](x) := - \text{sech}(x) \int_0^x \cosh(y) w(t, y) \, dy,$$

we obtain

$$(\partial_t^2 - \partial_x^2 + 1)w = Q(w) + C(w) + \{\text{higher order}\},$$

with quadratic nonlinearities given by

$$Q(w) = (-2 \text{sech}(x) + 3 \text{sech}^3(x)) (\mathcal{I}[w])^2 - 2 \text{sech}(x) \tanh(x) \mathcal{I}[w] w.$$
Resonant quadratic term?

Step 2: Variable coefficient quadratic normal form.
Rewrite quadratic nonlinearities as

\[ Q(w) = \alpha_1(x)w^2 + \alpha_2(x)\tilde{I}[\partial_x w]w + \alpha_3(x)(\tilde{I}[\partial_x w])^2, \]

where \( \alpha_1, \alpha_2, \alpha_3 \in \mathcal{S}(\mathbb{R}) \) are spatially localized and

\[ \tilde{I}[\partial_x w(t)](x) := \text{sech}(x) \int_0^x \text{sinh}(y)(\partial_x w)(t, y) \, dy. \]

Main enemy: The quadratic nonlinearity \( \alpha_1(x)w^2 \) could potentially lead to a logarithmic slowdown caused by the threshold resonance.

Key observation: The Fourier transform of \( \alpha_1(x) = 3 \text{sech}^3(x) \tanh^2(x) \) vanishes at the bad resonant frequencies ("null structure")

\[ \hat{\alpha}_1(\pm \sqrt{3}) = 0. \]
Normal form

Pass to the first-order formulation

\[ \nu(t) := w(t) - i\langle D \rangle^{-1}(\partial_t w)(t) \]

use variable coefficient quadratic normal form (Lindblad-Lührmann-Soffer '20)

\[ B(\nu, \nu)(t) := \alpha_{11}(x)\nu(t, 0)^2 + \alpha_{12}(x)|\nu(t, 0)|^2 + \alpha_{13}(x)\bar{\nu}(t, 0)^2 \]

with

\[ \hat{\alpha}_{11}(\xi) := \frac{1}{2}\langle \xi \rangle^{-1}(2 - \langle \xi \rangle)^{-1}\hat{\alpha}_1(\xi), \]

\[ \hat{\alpha}_{12}(\xi) := -\langle \xi \rangle^{-2}\hat{\alpha}_1(\xi), \]

\[ \hat{\alpha}_{13}(\xi) := -\frac{1}{2}\langle \xi \rangle^{-1}(2 + \langle \xi \rangle)^{-1}\hat{\alpha}_1(\xi), \]

to recast the problematic quadratic nonlinearity \( \alpha_1(x)(\nu + \bar{\nu})^2 \) into “variable coefficient cubic type”.
Step 3: Analysis of long-range effects of non-local cubic nonlinearities.

We arrive at the following first-order nonlinear Klein-Gordon equation for the renormalized variable $v + B(v, v)$,

$$(\partial_t - i\langle D\rangle)(v + B(v, v)) = (2i)^{-1}\langle D\rangle^{-1}(Q_{\text{ren}}(v, v) + C(v + \bar{v}) + \ldots).$$

At this point we follow the so-called space-time resonances method to infer decay and asymptotics of $v(t, x)$, which we then transfer back to the original variable $u(t, x)$ via

$$u(t, x) = \mathcal{I}[v(t) + \bar{v}(t)](x).$$

Some key points:

- Fourier analysis of all nonlinearities can be carried out explicitly.
- Obtain slow energy growth estimates $\|Zv(t)\|_{L_x^2} \lesssim t^\delta \varepsilon$ for Lorentz boost $Z = t\partial_x + x\partial_t$ via delicate local decay argument and favorable structure of the non-local cubic nonlinearities.
Outlook

The hierarchy of Schrödinger operators with Pöschl-Teller potentials

\[ H_n = -\partial_x^2 - n(n + 1) \text{sech}^2(x), \quad n \in \mathbb{N}_0, \]

exhibits higher-order super-symmetric factorization properties and comes up in several classical 1D asymptotic stability problems.

Possible future applications:

- Asymptotic stability of the $\phi^4$ kink ($H_2$): internal mode, FGR coupling to the continuous spectrum. Super-symmetric method does apply in principle for odd data ($\perp$ threshold resonance), internal mode produces resonant quadratic term.

- Conditional asymptotic stability of solitons of the 1D focusing cubic and quadratic Klein-Gordon equations ($H_2$ and $H_3$)

Major difficulties:
Interplay between resonant quadratic terms due to internal modes and/or threshold resonances, and the long-range effects of the non-localized cubic nonlinearity.

\[^1\] Pöschl, G., Teller, E. Bemerkungen zur Quantenmechanik des anharmonischen Oszillators. Z. Physik 83, 143–151 (1933)