Some Recent Results On Wave Turbulence: Derivation, Analysis, Numerics and Physical Application

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OUTLINE OF THE TAL K

1. **Brief introduction to wave turbulence**
   - Wave Turbulence: The Physical History
   - Wave Turbulence: The Modern Mathematical Context
2. **Rigorous derivation of the wave kinetic equations**
3. **Analysis of wave kinetic equations**
4. **Numerics of wave kinetic equations**
5. **Physical aplication: Bose-Einstein Condensates**

BRIEF INTRODUCTION TO WAVE TURBULENCE

Wave Turbulence: The Physical History

Physical Literature

- Wave Turbulence is a non-equilibrium statistical system of many randomly interacting waves. Kinetic equations of Wave Turbulence describe evolution of the wave energy in Fourier space.
- Origin in the works of Peierls (1933) and Hasselmann (1962)
- Modern point of view Benney-Saffman-Newell (1966), Zakharov (1966)
- Recent developments Newell, Zakharov, L'vov, Nazarenko, Pomeau, Spohn,...
- Vast range of application:
    - inertial waves due to rotation
    - Alfvén wave turbulence in the solar wind
    - waves in plasmas of fusion devices
    - quantum physics: quantum Boltzmann equations are very similar with wave kinetic equations (Pomeau's work for BECs)

  and many others.

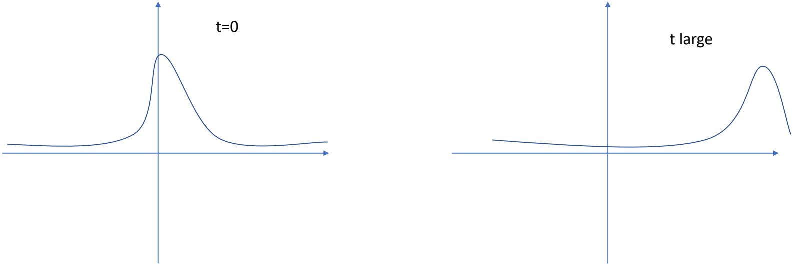
Wave Turbulence: The Modern Mathematical Context

We consider the KdV (KP) equation in d-dimension

$$\begin{aligned}\partial_t \phi(x,t) &= -\Delta \partial_{x_1} \phi(x,t) + \lambda \partial_{x_1} \left(\phi^2(x,t) \right)\\ \phi(x,0) &= \phi_0(x), x \in \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d.\end{aligned}$$

Energy Cascade Conjecture (Bourgain's 2000):

Give a solution $\phi(t,x)$ to a dispersive PDE on a compact manifold M, does a migration of energy occurs from low frequencies to high frequencies? Take $M = \mathbb{R}^d$ and $\hat{\phi}(t,k)$: the k-th Fourier coefficients.



Do we have migration of energy from low to high k?

Two different approaches

Given $\phi(t,x)$: solution of a dispersive equation.

Appr 1 We study

$$\sum_k |\hat{\phi}(t,k)|^2 \langle k \rangle^{2s} = \|\hat{u}(t)\|^2_{H^s}, \quad \lim_{t\to\infty} \|\hat{u}(t)\|^2_{H^s}$$

- PDE Approach: Bourgain, Kuksin, Staffilani, Sohinger, Hani, Deng-Germain, Colliander-Keel-Staffilani-Takaoka-Tao, Carles-Fau, Staffilani-Wilson, ...
- Computational Approach: Pan,...
- Dynamical System Approach: Haus-Procesi, Berti-Maspero, Hani...

Appr 2 Set $a_k = \hat{\phi}(t,k)$ and $|a_k(t)|^2 \to n(\tau,k)$. We arrive at a wave-kinetic equation

$$\partial_\tau n(\tau,k) = Q[n(\tau,k)]$$

in which Q is a non-local operator of kinetic type.

From Dispersive Equations to Kinetic Equations

Two different approaches: Second Approach

Dispersive Equation $\phi(t,x)) \longrightarrow$ Kinetic equation $(n(t,k) = |\hat{\phi}(t,k)|^2)$.

$$\partial_t\phi(x,t) \;=\; -\Delta\partial_{x_1}\phi(x,t) \;+\; \lambda\partial_{x_1}\left(\phi^2(x,t)\right)$$

$|\hat{\phi}(t,k)|^2 \to n(\tau,k)$,

$$\partial_\tau n(\tau,k) \;=\; Q[n(\tau,k)],$$

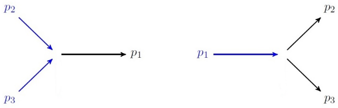
$$\begin{aligned}Q(n)(k_1) = &\iint \mathrm{d}k_2\mathrm{d}k_3|\mathcal{W}(k_1,k_2,k_3)|^2\delta(\omega(k_3)+\omega(k_2)-\omega(k_1))\\ &\times\delta(k_2+k_3-k_1)\Big(n_2n_3 - n_1n_2\mathrm{sign}(k_1^1)\mathrm{sign}(k_3^1) - n_1n_3\mathrm{sign}(k_1^1)\mathrm{sign}(k_2^1)\Big)\\ &-2\iint \mathrm{d}k_2\mathrm{d}k_3|\mathcal{W}(k_1,k_2,k_3)|^2\delta(\omega(k_1)+\omega(k_2)-\omega(k_3))\\ &\times\delta(k_1+k_2-k_3)\Big(n_2n_1 - n_3n_2\mathrm{sign}(k_3^1)\mathrm{sign}(k_1^1) - n_3n_1\mathrm{sign}(k_3^1)\mathrm{sign}(k_2^1)\Big),\end{aligned}$$

where $n_1(\tau) = n(\tau,k_1)$, $n_2(\tau) = n(\tau,k_2)$, $n_3(\tau) = n(\tau,k_3)$.

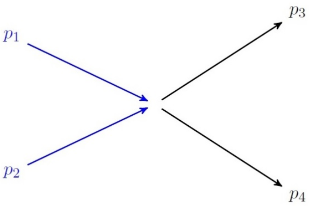
- $\omega = \widehat{\Delta\partial_{x_1}}$
- $\partial_{x_1}\left(\phi^2(x,t)\right)$ quadratic nonlinearity $\longrightarrow Q(n)(k_1)$ quadratic collision operator.

Recalling

Given a wave equation whose nonlinear is quadratic, we obtain a 3-wave kinetic equation in the Fourier space.



Given a wave equation whose nonlinear is cubic, we obtain a 4-wave kinetic equation in the Fourier space.



RIGOROUS DERIVATION OF THE WAVE KINETIC EQUATIONS

Homogeneous Problem

Set $a_k = \hat{u}(t,k)$ and $|a_k(t)|^2 \to n(\tau,k)$. Derive the wave-kinetic equation

$$\partial_\tau n(\tau,k) \;=\; Q[n(\tau,k)]$$

at kinetic limit

$$t = \tau\lambda^{-2} = \mathcal{O}(\lambda^{-2})$$

Mathematical Literature: Rigorous Derivations

- Erdos-Yau (CPAM 2020), Erdos-Salmhofer-Yau (Acta Math 2008) (based on the work of Spohn 1977): Random Linear Schrödinger $\to$ linear Boltzmann (kinetic limit) $\to$ heat equation (diffusive limit)
- Pioneering work: Lukkarinen-Spohn (Invent Math 2010): Random Cubic Nonlinear Schrödinger at equilibrium $\to$ homogeneous wave kinetic equation at (kinetic limit).

$\longrightarrow$ To derive the wave kinetic equation, randomizing the equation needs to be done (Spohn-ICM 2010).

Recent results

Random initial data

- Buckmaster-Germain-Hani-Shatah (CPAM 2019, Invent Math 2021) $\longrightarrow$ homogeneous wave kinetic equation at a little below kinetic time. The results triggered the recent whole field of research.
- Collot-Germain (2019, 2020), Deng-Hani (Forum Pi, 2021) $\longrightarrow$ homogeneous wave kinetic equation at a little below kinetic time.
- (2021) Homogeneous kinetic equation, at kinetic time NLS : Deng-Hani $\longrightarrow$ propagation of chaos (2021)
- (2021) Inhomogeneous kinetic equation, a little below kinetic time NLS : Ampatzoglou-Collot-Germain

Stochastic PDEs

- Dymov, Kuksin and collaborators (2019-2021), Faou (CMP 2020).
- (2021) Homogeneous wave kinetic equation, at kinetic time stochastic KdV: Staffilani-MBT

ANALYSIS OF WAVE KINETIC EQUATIONS

3-wave turbulence kinetic equation

- The equation:

$$\begin{aligned} \partial_t f(t,k) &= C_{3W}[f](t,k), \\ f(0,k) &= f_0(k) \end{aligned}$$

$$C_{3W}[f](k) = \iint_{\mathbb{R}^{2N}} \Big[R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f] \Big] dk_1 dk_2$$

$$R_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)(f_1 f_2 - f f_1 - f f_2)$$

with the short-hand notation $f = f(t,k)$, $\omega = \omega(k)$ and $f_j = f(t,k_j)$, $\omega_j = \omega(k_j)$.
$\omega(k)$: the dispersion relation of the waves.

- In the isotropic ($\omega = |k|^\alpha$, $f(t,k) = f(t,|k|)$) case, we identify $f(t,k)$ with $f(t,\omega)$, the isotropic 3-wave kinetic equation takes the form

$$\begin{aligned} \partial_t f(t,\omega) &= Q[f](t,\omega), \quad \omega \in \mathbb{R}_+, \\ f(0,\omega) &= f_0(\omega), \\ Q[f](t,\omega) &= \int_0^\infty \int_0^\infty \big[R(\omega,\omega_1,\omega_2) - R(\omega_1,\omega,\omega_2) - R(\omega_2,\omega_1,\omega) \big] d\omega_1 d\omega_2, \\ R(\omega,\omega_1,\omega_2) &:= \delta(\omega - \omega_1 - \omega_2) \left[a(\omega_1,\omega_2) f_1 f_2 - a(\omega,\omega_1) f f_1 - a(\omega,\omega_2) f f_2 \right], \end{aligned}$$

where $a(\omega_1,\omega_2) = (\omega_1\omega_2)^{\gamma/2}$.

Energy Cascade Theorem for 3-wave (Soffer-MBT (CMP 2020))

- The equation has global weak solutions
- Define the energy of the solution as $g(t,\omega) = \omega f(t,\omega)$.
- g can be decomposed into two parts

$$g(t,\omega) = \bar{g}(t,\omega) + \tilde{g}(t)\delta_{\omega=\infty},$$

  where $\bar{g}(t,\omega) \geq 0$ is the regular part, which is a function, and $\tilde{g}(t)\delta_{\omega=\infty}$, is the singular part, which is a measure. The function $\tilde{g}(t)$ is non-negative.
- $\bar{g}(0,\omega) = g(0,\omega)$ and $\tilde{g}(0) = 0$.
- There exists a time t_0, such that for all time $t > t_0$, the function $\tilde{g}(t)$ is strictly positive.
- Starting from time t_0, the energy starts to transfer from the regular part $\bar{g}(t,\omega)$ to the singular part $\tilde{g}(t)\delta_{\omega=\infty}$, while the total energy of the two regular and singular parts is still conserved. In the limit that $t \to \infty$, all of the energy will be accumulated to the singular part.
- The cascade rate is bounded by $\mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \longrightarrow$ Is this optimal?
- Inspired by the result for 4-wave: Escobedo-Velazquez (Memoirs AMS 2015).

NUMERICS OF WAVE KINETIC EQUATIONS

Wave kinetic equation

Isotopic 3-wave kinetic equation

$$\partial_t f(t,\omega) \;=\; Q[f(t,\omega)]$$

$$\begin{aligned} Q[f](t,\omega) \;=\; & \int_0^\omega [a(\omega_1,\omega-\omega_1)f(\omega_1)f(\omega-\omega_1) - a(\omega,\omega_1)f(\omega)f(\omega_1) \\ & - a(\omega,\omega-\omega_1)f(\omega)f(\omega-\omega_1)]\mathrm{d}\omega_1 \; - 2\int_0^\infty [a(\omega,\omega_1)f(\omega)f(\omega_1) \\ & - a(\omega+\omega_1,\omega_1)f(\omega+\omega_1)f(\omega_1) - a(\omega_1+\omega,\omega)f(\omega)f(\omega_1+\omega)]\mathrm{d}\omega_1 \end{aligned}$$

where $a(\omega_1,\omega_2) \;=\; (\omega_1\omega_2)^{\gamma/2}$.

Smoluchowski coagulation equation

$$\partial_t f(t,\omega) \;=\; \mathbb{Q}[f(t,\omega)]$$

$$\mathbb{Q}[f](t,\omega) \;=\; \int_0^\omega a(\omega_1,\omega-\omega_1)f(\omega_1)f(\omega-\omega_1)\mathrm{d}\omega_1 \; - 2\int_0^\infty a(\omega,\omega_1)f(\omega)f(\omega_1)\mathrm{d}\omega_1$$

There is a huge amount of numerical schemes developed for the Smoluchowski coagulation equation $\longrightarrow$ rich resources for wave kinetic equations

Filbet-Laurencot's scheme (SIAM Sci. Comp. 2003 for the Smoluchowski coagulation equation)

$$\partial_t f(t,\omega) = \mathbb{Q}[f(t,\omega)]$$

$$\mathbb{Q}[f](t,\omega) = \int_0^\omega a(\omega_1,\omega-\omega_1)f(\omega_1)f(\omega-\omega_1)\mathrm{d}\omega_1 - 2\int_0^\infty a(\omega,\omega_1)f(\omega)f(\omega_1)\mathrm{d}\omega_1$$

where a satisfies $a(\omega_1,\omega_2) = (\omega_1\omega_2)^{\gamma/2}$.
Test function $\phi(\omega) = \omega\chi_{[0,c]}$

$$\int_0^c \partial_t f(t,\omega)\omega\mathrm{d}\omega = -2\int_0^c\int_{c-\omega}^\infty \omega a(\omega,\omega_1)f(\omega_1)f(\omega)\mathrm{d}\omega_1\mathrm{d}\omega$$

Taking the derivative

$$\partial_t f(t,c)c = -2\partial_c\int_0^c\int_{c-\omega}^\infty \omega a(\omega,\omega_1)f(\omega_1)f(\omega)\mathrm{d}\omega_1\mathrm{d}\omega$$

Truncating

$$\partial_t f(t,c)c = -2\partial_c\int_0^c\int_{c-\omega}^R \omega a(\omega,\omega_1)f(\omega_1)f(\omega)\mathrm{d}\omega_1\mathrm{d}\omega$$

After that, apply a Finite Volume Algorithm to solve the truncated problem.
$\longrightarrow$ Adapting this idea to 3-wave kinetic equations

Numerical Tests (Walton, PhD student)

Test 1 Here we choose initial condition

$$g_0(k) = 1.26157e^{-50(k-1.5)^2} \tag{1}$$

with $\Delta t = 0.05$ for $t \in [0, T]$, $T = 10000$ seconds, over a uniform grid, with $\Delta k = 0.5$, $\gamma = 2$, $R = 50, 100, 200$.

Numerical Tests

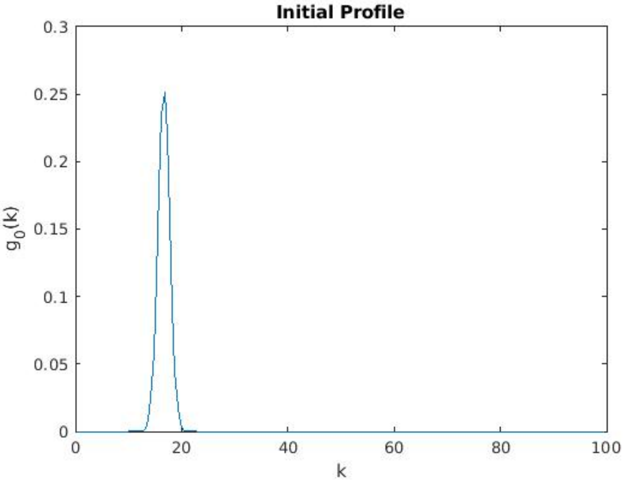


Figure: Initial Profile

Numerical Tests

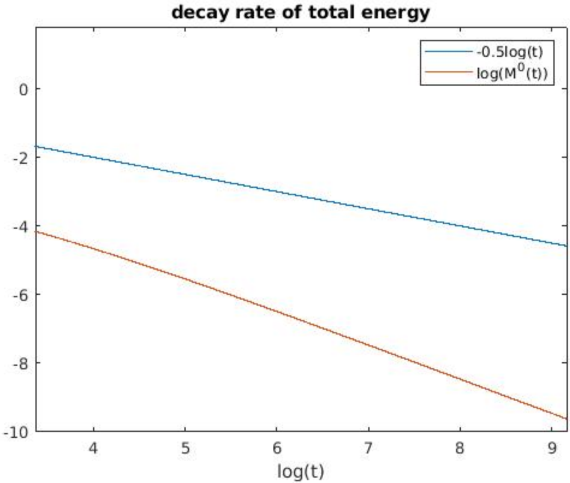


Figure: Log of the decay rate

Numerical Tests

Test 2 We consider the initial data given by

$$g_0(k) = \begin{cases} 1 & k \in \left[2n\pi, (2n+1)\pi\right] \\ 0 & k \in \left((2n+1)\pi, 2(n+1)\pi\right) \end{cases} \text{ for } n = 0, 1, 3, 5, \dots \tag{2}$$

and perform test for $t \in [0, T]$ for $T = 100$ and $\Delta t = 0.0004$, $R = 50$. The frequency step is $\Delta k = 0.1$ on the interval $[0, R]$.

Numerical Tests

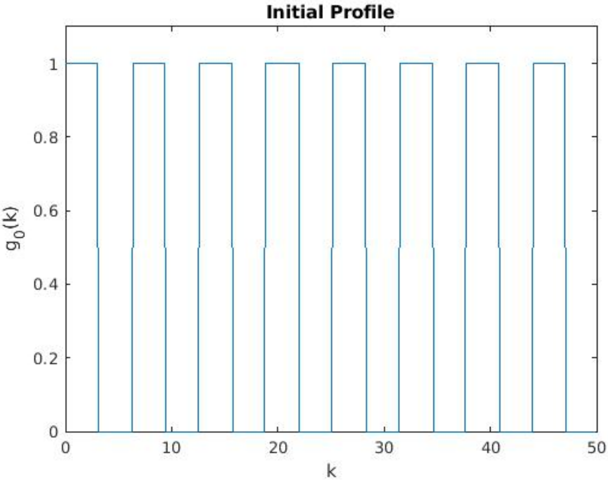


Figure: Initial Profile

Numerical Tests

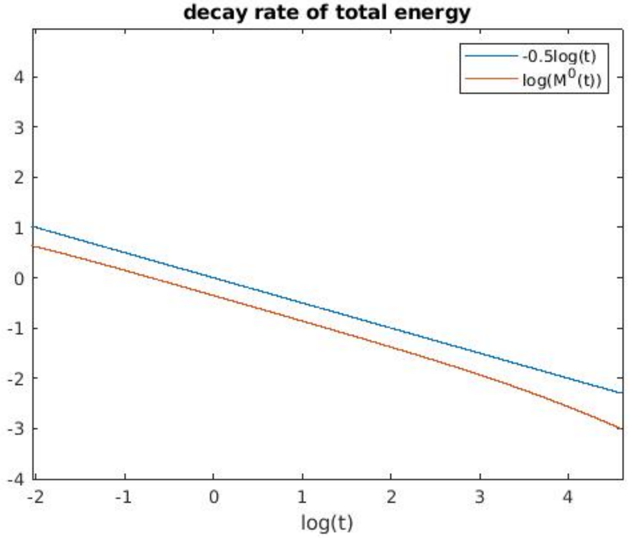


Figure: Log of the decay rate

Numerical Tests

Test 3 We consider the initial data given by

$$g_0(k) = \frac{k - 2n\pi}{2\pi} \quad k \in [2n\pi, 2(n+1)\pi), \tag{3}$$

for $n \in \mathbb{N}_0$. We set $\Delta k = 0.1$, $T = 100$ and $\Delta t = 0.0004$, $R = 50$.

Numerical Tests

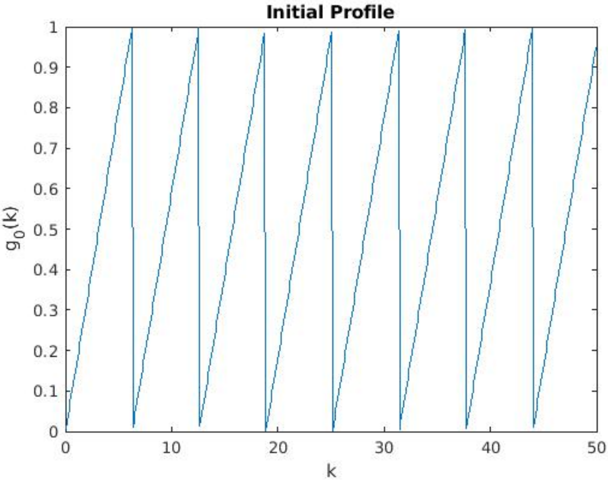


Figure: Initial Profile

Numerical Tests

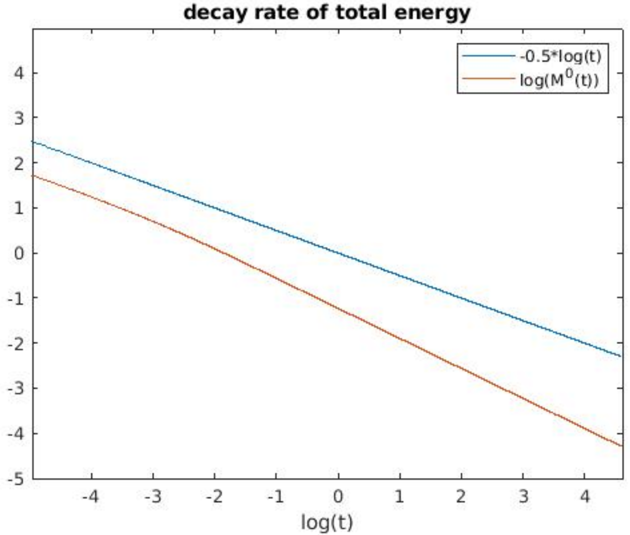


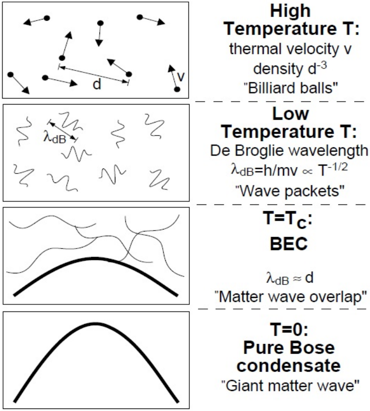
Figure: Log of the decay rate

APPLICATIONS IN BOSE-EINSTEIN CONDENSATES: WHAT'S NEW?

Phase transitions

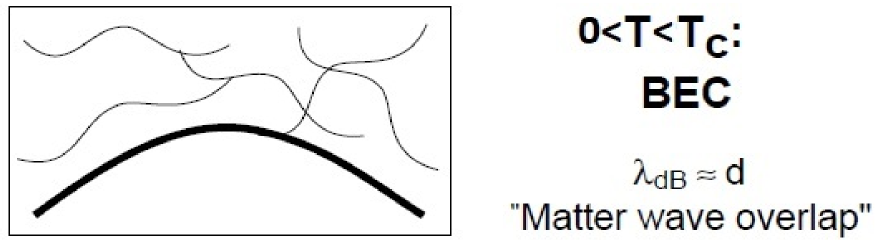
A Bose-Einstein condensate is a state of matter in which extremely cold atoms clump together and act as if they were a single atom. This state was first predicted, generally, in 1924-25 by Satyendra Nath Bose and Albert Einstein.
On June 5, 1995, the first gaseous condensate was produced by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST-JILA lab, in a gas of rubidium atoms cooled to 170 nanokelvins (nK). Shortly thereafter, Wolfgang Ketterle at MIT realized a BEC in a gas of sodium atoms. For their achievements Cornell, Wieman, and Ketterle received the 2001 Nobel Prize in Physics.

Bose-Einstein condensate



Criterion for Bose-Einstein condensation. At high temperatures, a weakly interacting gas can be treated as a system of "billiard balls." In a simplified quantum description, the atoms can be regarded as wavepackets with an extension λ_{dB}. At the BEC transition temperature, λ_{dB} becomes comparable to the distance between atoms, and a Bose condensate forms. As the temperature approaches zero, the thermal cloud disappears leaving a pure Bose condensate.

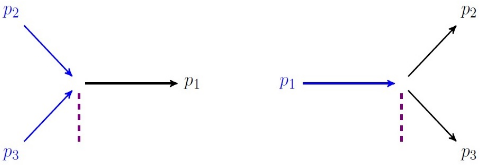
Finite Temperature BEC



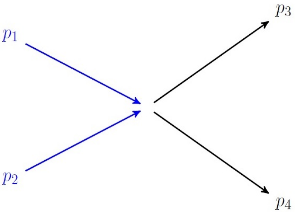
- $f(t, r, p)$ density of the non-condensate: Kinetic equation, similar to 3 and 4-wave kinetic equations
- $\Phi(t, r)$ wave function of the condensate: Gross-Pitaevski equation
- Kirkpatrick-Dorfman'85, Zaremba-Nikuni-Griffin'99, Pomeau-Métens-Brachet-Rica'99

C_{12} (3-wave) and C_{22} (4-wave)

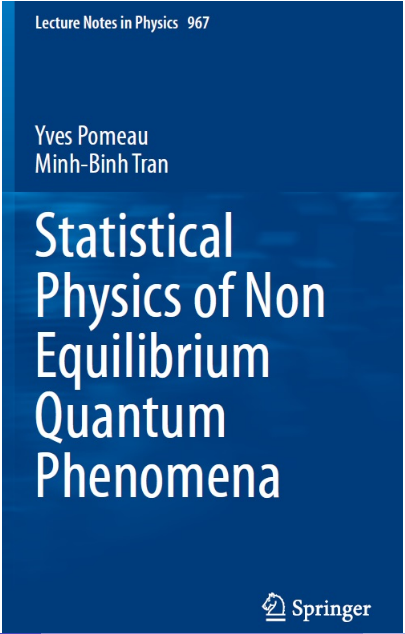
C_{12}



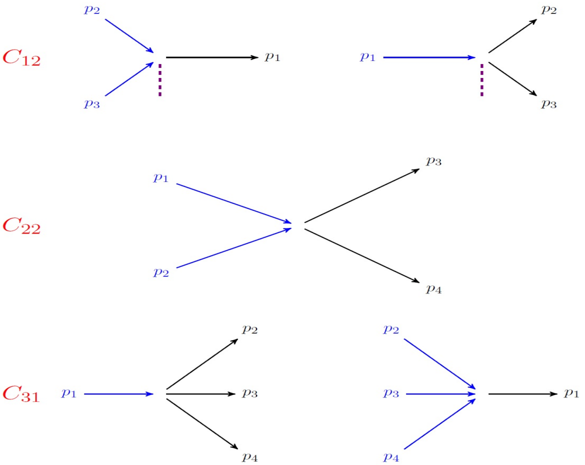
C_{22}



Bose-Einstein Condensates



C_{31}: The missing collision operator (Reichl-Gust'12, MBT-Pomeau'20'21)



C_{31}: The missing collision operator (Reichl-Gust'12, MBT-Pomeau'20'21)

- C_{31} : besides the $2 \leftrightarrow 2$ interaction, there should be another $3 \leftrightarrow 1$ one.
- The new kinetic operator should be

$$C_{22}[f] + C_{31}[f] + C_{12}[f].$$

- However, the formal derivation of the new collision operator C_{31} is very complicated, since the process generates around 40000 individual terms and one will need to do a combinatorics problem for all of them. Checking C_{31} is a challenging problem. The paper (Reichl-Gust'12) cannot be published.
- MBT-Pomeau (Physical Review E 2020, EPJP 2021): **The computations of C_{31} reduce from 40000 to only around 30 terms,** providing a full confirmation of C_{31}.

THANK YOU SO MUCH!