Some Recent Results On Wave Turbulence: Derivation, Analysis, Numerics and Physical Application

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OUTLINE OF THE TALK

1. Brief introduction to wave turbulence
   - Wave Turbulence: The Physical History
   - Wave Turbulence: The Modern Mathematical Context

2. Rigorous derivation of the wave kinetic equations

3. Analysis of wave kinetic equations

4. Numerics of wave kinetic equations

5. Physical application: Bose-Einstein Condensates
BRIEF INTRODUCTION TO WAVE TURBULENCE
Wave Turbulence: The Physical History
Physical Literature

- Wave Turbulence is a non-equilibrium statistical system of many randomly interacting waves. Kinetic equations of Wave Turbulence describe evolution of the wave energy in Fourier space.
- Origin in the works of Peierls (1933) and Hasselmann (1962)
- Modern point of view Benney-Saffman-Newell (1966), Zakharov (1966)
- Recent developments Newell, Zakharov, L'vov, Nazarenko, Pomeau, Spohn,...
- Vast range of application:
  - inertial waves due to rotation
  - Alfvén wave turbulence in the solar wind
  - waves in plasmas of fusion devices
  - quantum physics: quantum Boltzmann equations are very similar with wave kinetic equations (Pomeau’s work for BECs)
  - and many others.
Wave Turbulence: The Modern Mathematical Context
We consider the KdV (KP) equation in $d$-dimension

\[
\partial_t \phi(x, t) = -\Delta \partial_{x_1} \phi(x, t) + \lambda \partial_{x_1} \left( \phi^2(x, t) \right)
\]

\[
\phi(x, 0) = \phi_0(x), \quad x \in \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d.
\]
Energy Cascade Conjecture (Bourgain’s 2000):
Given a solution $\phi(t, x)$ to a dispersive PDE on a compact manifold $M$, does a migration of energy occur from low frequencies to high frequencies? Take $M = \mathbb{R}^d$ and $\hat{\phi}(t, k)$: the $k$-th Fourier coefficients.

Do we have migration of energy from low to high $k$?
Two different approaches

Given $\phi(t, x)$: solution of a dispersive equation.

Appr 1 We study

$$\sum_{k} |\hat{\phi}(t, k)|^2 \langle k \rangle^{2s} = \|\hat{u}(t)\|^{2s}_{H^s}, \quad \lim_{t \to \infty} \|\hat{u}(t)\|^{2s}_{H^s}$$

- **PDE Approach:** Bourgain, Kuksin, Staffilani, Sohinger, Hani, Deng-Germain, Colliander-Keel-Staffilani-Takaoka-Tao, Carles-Fau, Staffilani-Wilson, ...
- **Computational Approach:** Pan,...
- **Dynamical System Approach:** Haus-Procesi, Berti-Maspero, Hani...

Appr 2 Set $a_k = \hat{\phi}(t, k)$ and $|a_k(t)|^2 \to n(\tau, k)$. We arrive at a wave-kinetic equation

$$\partial_\tau n(\tau, k) = Q[n(\tau, k)]$$

in which $Q$ is a non-local operator of kinetic type.
From Dispersive Equations to Kinetic Equations
Two different approaches: Second Approach

Dispersive Equation $\phi(t, x) \rightarrow$ Kinetic equation $(n(t, k) = |\hat{\phi}(t, k)|^2)$.

$$\partial_t \phi(x, t) = -\Delta \partial_{x_1} \phi(x, t) + \lambda \partial_{x_1} \left(\phi^2(x, t)\right)$$

$|\hat{\phi}(t, k)|^2 \rightarrow n(\tau, k)$,

$$\partial_\tau n(\tau, k) = Q[n(\tau, k)],$$

$$Q(n)(k_1) = \int \int dk_2 dk_3 |\mathcal{V}(k_1, k_2, k_3)|^2 \delta(\omega(k_3) + \omega(k_2) - \omega(k_1))$$

$$\times \delta(k_2 + k_3 - k_1) \left( n_2 n_3 - n_1 n_2 \text{sign}(k_1^1) \text{sign}(k_3^1) - n_1 n_3 \text{sign}(k_1^1) \text{sign}(k_2^1) \right)$$

$$- 2 \int \int dk_2 dk_3 |\mathcal{V}(k_1, k_2, k_3)|^2 \delta(\omega(k_1) + \omega(k_2) - \omega(k_3))$$

$$\times \delta(k_1 + k_2 - k_3) \left( n_2 n_1 - n_3 n_2 \text{sign}(k_3^1) \text{sign}(k_1^1) - n_3 n_1 \text{sign}(k_3^1) \text{sign}(k_2^1) \right),$$

where $n_1(\tau) = n(\tau, k_1), n_2(\tau) = n(\tau, k_2), n_3(\tau) = n(\tau, k_3).$

- $\omega = \Delta \partial_{x_1}$
- $\partial_{x_1} \left(\phi^2(x, t)\right)$ quadratic nonlinearity $\rightarrow Q(n)(k_1)$ quadratic collision operator.
Recalling

Given a wave equation whose nonlinear is quadratic, we obtain a 3-wave kinetic equation in the Fourier space.

Given a wave equation whose nonlinear is cubic, we obtain a 4-wave kinetic equation in the Fourier space.
RIGOROUS DERIVATION OF THE WAVE KINETIC EQUATIONS
**Homogeneous Problem**

Set \( a_k = \hat{u}(t, k) \) and \( |a_k(t)|^2 \rightarrow n(\tau, k) \). Derive the wave-kinetic equation

\[
\partial_\tau n(\tau, k) = Q[n(\tau, k)]
\]

at kinetic limit

\[
t = \tau \lambda^{-2} = \mathcal{O}(\lambda^{-2})
\]

**Mathematical Literature: Rigorous Derivations**


- **Pioneering work:** Lukkarinen-Spohn (Invent Math 2010): Random Cubic Nonlinear Schrödinger at equilibrium → homogeneous wave kinetic equation at (kinetic limit).

→ To derive the wave kinetic equation, randomizing the equation needs to be done (Spohn-ICM 2010).
Recent results

Random initial data

- Buckmaster-Germain-Hani-Shatah (CPAM 2019, Invent Math 2021) → homogeneous wave kinetic equation at a little below kinetic time. The results triggered the recent whole field of research.


- (2021) Inhomogeneous kinetic equation, a little below kinetic time NLS : Ampatzoglou-Collot-Germain

Stochastic PDEs


- (2021) Homogeneous wave kinetic equation, at kinetic time stochastic KdV: Staffilani-MBT
ANALYSIS OF WAVE KINETIC EQUATIONS
3-wave turbulence kinetic equation

The equation:
\[
\frac{\partial}{\partial t} f(t, k) = C_3W[f](t, k),
\]
\[
f(0, k) = f_0(k)
\]
\[
C_3W[f](k) = \int\int_{\mathbb{R}^{2N}} \left[ R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f] \right] dk_1 dk_2
\]
\[
R_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2) \delta(\omega - \omega_1 - \omega_2) (f_1 f_2 - ff_1 - ff_2)
\]

with the short-hand notation \( f = f(t, k), \omega = \omega(k) \) and \( f_j = f(t, k_j), \omega_j = \omega(k_j) \).

\( \omega(k) \): the dispersion relation of the waves.

In the isotropic (\( \omega = |k|^\alpha, f(t, k) = f(t, |k|) \)) case, we identify \( f(t, k) \) with \( f(t, \omega) \), the isotropic 3-wave kinetic equation takes the form

\[
\frac{\partial}{\partial t} f(t, \omega) = Q[f](t, \omega), \quad \omega \in \mathbb{R}_+,
\]
\[
f(0, \omega) = f_0(\omega),
\]
\[
Q[f](t, \omega) = \int_0^\infty \int_0^\infty \left[ R(\omega, \omega_1, \omega_2) - R(\omega_1, \omega, \omega_2) - R(\omega_2, \omega_1, \omega) \right] d\omega_1 d\omega_2,
\]
\[
R(\omega, \omega_1, \omega_2) := \delta(\omega - \omega_1 - \omega_2) [a(\omega_1, \omega_2)f_1 f_2 - a(\omega, \omega_1)ff_1 - a(\omega, \omega_2)ff_2],
\]

where \( a(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\gamma/2} \).
Energy Cascade Theorem for 3-wave (Soffer-MBT (CMP 2020))

- The equation has global weak solutions
- Define the energy of the solution as \( g(t, \omega) = \omega f(t, \omega) \).
- \( g \) can be decomposed into two parts
  \[
  g(t, \omega) = \bar{g}(t, \omega) + \tilde{g}(t) \delta_{\omega=\infty},
  \]
  where \( \bar{g}(t, \omega) \geq 0 \) is the regular part, which is a function, and \( \tilde{g}(t) \delta_{\omega=\infty} \), is the singular part, which is a measure. The function \( \tilde{g}(t) \) is non-negative.
- \( \bar{g}(0, \omega) = g(0, \omega) \) and \( \tilde{g}(0) = 0 \).
- There exists a time \( t_0 \), such that for all time \( t > t_0 \), the function \( \tilde{g}(t) \) is strictly positive.
- Starting from time \( t_0 \), the energy starts to transfer from the regular part \( \bar{g}(t, \omega) \) to the singular part \( \tilde{g}(t) \delta_{\omega=\infty} \), while the total energy of the two regular and singular parts is still conserved. In the limit that \( t \to \infty \), all of the energy will be accumulated to the singular part.
- The cascade rate is bounded by \( O\left(\frac{1}{\sqrt{t}}\right) \) \( \longrightarrow \) Is this optimal?
- Inspired by the result for 4-wave: Escobedo-Velazquez (Memoirs AMS 2015).
Wave kinetic equation

Isotopic 3-wave kinetic equation

\[ \partial_t f(t, \omega) = Q[f(t, \omega)] \]

\[ Q[f](t, \omega) = \int_0^\omega [a(\omega_1, \omega - \omega_1)f(\omega_1)f(\omega - \omega_1) - a(\omega, \omega_1)f(\omega)f(\omega_1) \]
\[ - a(\omega, \omega - \omega_1)f(\omega)f(\omega - \omega_1)]d\omega_1 - 2 \int_0^\infty [a(\omega, \omega_1)f(\omega)f(\omega_1) \]
\[ - a(\omega + \omega_1, \omega_1)f(\omega + \omega_1)f(\omega_1) - a(\omega_1 + \omega, \omega)f(\omega)f(\omega_1 + \omega)]d\omega_1 \]

where \( a(\omega_1, \omega_2) = (\omega_1\omega_2)^{\gamma/2} \).

Smoluchowski coagulation equation

\[ \partial_t f(t, \omega) = Q[f(t, \omega)] \]

\[ Q[f](t, \omega) = \int_0^\omega a(\omega_1, \omega - \omega_1)f(\omega_1)f(\omega - \omega_1)d\omega_1 - 2 \int_0^\infty a(\omega, \omega_1)f(\omega)f(\omega_1)d\omega_1 \]

There is a huge amount of numerical schemes developed for the Smoluchowski coagulation equation \( \rightarrow \) rich resources for wave kinetic equations.
Filbet-Laurencot’s scheme (SIAM Sci. Comp. 2003 for the Smoluchowski coagulation equation)

\[ \partial_t f(t, \omega) = \mathbb{Q}[f(t, \omega)] \]

\[ \mathbb{Q}[f](t, \omega) = \int_0^\omega a(\omega_1, \omega - \omega_1)f(\omega_1)f(\omega - \omega_1)d\omega_1 - 2\int_0^\infty a(\omega, \omega_1)f(\omega)f(\omega_1)d\omega_1 \]

where \( a \) satisfies \( a(\omega_1, \omega_2) = (\omega_1\omega_2)^{\gamma/2} \).

Test function \( \phi(\omega) = \omega \chi_{[0,c]} \)

\[ \int_0^c \int_{c-\omega}^\infty \partial_t f(t, \omega) \omega d\omega d\omega = -2 \int_0^c \int_{c-\omega}^\infty \omega a(\omega, \omega_1)f(\omega_1)f(\omega)d\omega_1 d\omega \]

Taking the derivative

\[ \partial_t f(t, c)c = -2\partial_c \int_0^c \int_{c-\omega}^\infty \omega a(\omega, \omega_1)f(\omega_1)f(\omega)d\omega_1 d\omega \]

Truncating

\[ \partial_t f(t, c)c = -2\partial_c \int_0^c \int_{c-\omega}^R \omega a(\omega, \omega_1)f(\omega_1)f(\omega)d\omega_1 d\omega \]

After that, apply a Finite Volume Algorithm to solve the truncated problem.

\[ \rightarrow \] Adapting this idea to 3-wave kinetic equations
Test 1 Here we choose initial condition

\[ g_0(k) = 1.26157 e^{-50(k-1.5)^2} \] (1)

with \( \Delta t = 0.05 \) for \( t \in [0, T] \), \( T = 10000 \) seconds, over a uniform grid, with \( \Delta k = 0.5 \), \( \gamma = 2 \), \( R = 50, 100, 200 \).
Numerical Tests

Figure: Initial Profile
Numerical Tests

**Figure:** Log of the decay rate
**Numerical Tests**

**Test 2** We consider the initial data given by

\[
g_0(k) = \begin{cases} 
1 & k \in [2n\pi, (2n + 1)\pi] \\
0 & k \in ((2n + 1)\pi, 2(n + 1)\pi) 
\end{cases}
\]

for \( n = 0, 1, 3, 5, \ldots \) \hspace{2cm} (2)

and perform test for \( t \in [0, T] \) for \( T = 100 \) and \( \Delta t = 0.0004 \), \( R = 50 \). The frequency step is \( \Delta k = 0.1 \) on the interval \([0, R]\).
Numerical Tests

Figure: Initial Profile
Numerical Tests

Figure: Log of the decay rate
Numerical Tests

Test 3 We consider the initial data given by

$$g_0(k) = \frac{k - 2n\pi}{2\pi} \quad k \in [2n\pi, 2(n + 1)\pi),$$

for $n \in \mathbb{N}_0$. We set $\Delta k = 0.1$, $T = 100$ and $\Delta t = 0.0004$, $R = 50$. 
Numerical Tests

**Figure:** Initial Profile
Numerical Tests

Figure: Log of the decay rate
APPLICATIONS IN BOSE-EINSTEIN CONDENSATES: WHAT’S NEW?
A Bose-Einstein condensate is a state of matter in which extremely cold atoms clump together and act as if they were a single atom. This state was first predicted, generally, in 1924-25 by Satyendra Nath Bose and Albert Einstein. On June 5, 1995, the first gaseous condensate was produced by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST-JILA lab, in a gas of rubidium atoms cooled to 170 nanokelvins (nK). Shortly thereafter, Wolfgang Ketterle at MIT realized a BEC in a gas of sodium atoms. For their achievements Cornell, Wieman, and Ketterle received the 2001 Nobel Prize in Physics.
Bose-Einstein condensate

Criterion for Bose-Einstein condensation. At high temperatures, a weakly interacting gas can be treated as a system of “billiard balls.” In a simplified quantum description, the atoms can be regarded as wavepackets with an extension $\lambda_{dB}$. At the BEC transition temperature, $\lambda_{dB}$ becomes comparable to the distance between atoms, and a Bose condensate forms. As the temperature approaches zero, the thermal cloud disappears leaving a pure Bose condensate.
Finite Temperature BEC

- \(f(t, r, p)\) density of the non-condensate: Kinetic equation, similar to 3 and 4-wave kinetic equations
- \(\Phi(t, r)\) wave function of the condensate: Gross-Pitaevski equation
$C_{12}$ (3-wave) and $C_{22}$ (4-wave)

Diagram for $C_{12}$:
- $p_2$ to $p_1$
- $p_3$ to $p_1$

Diagram for $C_{22}$:
- $p_1$ to $p_3$
- $p_2$ to $p_3$
- $p_4$ to $p_3$
$C_{31}$: The missing collision operator (Reichl-Gust’12, MBT-Pomeau’20’21)
**$C_{31}$: The missing collision operator (Reichl-Gust’12, MBT-Pomeau’20’21)**

- $C_{31}$: besides the $2 \leftrightarrow 2$ interaction, there should be another $3 \leftrightarrow 1$ one.
- The new kinetic operator should be
  
  $$C_{22}[f] + C_{31}[f] + C_{12}[f].$$

- However, the formal derivation of the new collision operator $C_{31}$ is very complicated, since the process generates around 40000 individual terms and one will need to do a combinatorics problem for all of them. Checking $C_{31}$ is a challenging problem. The paper (Reichl-Gust’12) cannot be published.

- MBT-Pomeau (Physical Review E 2020, EPJP 2021): The computations of $C_{31}$ reduce from 40000 to only around 30 terms, providing a full confirmation of $C_{31}$.
THANK YOU SO MUCH!