A tale of two generalizations of Boltzmann equation

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Based on joint works
with Ampatzoglou and Miller

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Outline

The Boltzmann equation

The Boltzmann equation[1] :

$$\partial_t f + v \cdot \nabla_x f \;=\; Q_2(f,f) \tag{1.1}$$

- for the position $x \in \mathbb{R}^d$, the time $t \in \mathbb{R}^+$ and velocity $v \in \mathbb{R}^d$
- describes the evolution of the density $f(x,t,v)$ of gas particles
- where $Q_2(f,f)$ is a **quadratic integral operator that expresses the change of f due to instantaneous binary collisions of particles**. Its exact form depends on the type of interaction between particles.

[1] was introduced by Maxwell and Boltzmann in late 1860's - early 1870's as a model that describes a dynamics of a dilute gas.

Who is who...

For a pair of particles let:

- $(\mathbf{v}', \mathbf{v}_1')$ be their pre-collisional velocities, and $\mathbf{u}' = \mathbf{v}' - \mathbf{v}_1'$, $\sigma = \frac{\mathbf{u}'}{|\mathbf{u}'|}$.
- $(\mathbf{v}, \mathbf{v}_1)$ be their post-collisional velocities, and $\mathbf{u} = \mathbf{v} - \mathbf{v}_1$, $\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$.

For elastic interactions, momentum and energy are conserved, i.e.

$v + v_1 = v' + v_1'$,

$|v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2$,

thus,

$$\mathbf{v}' = \frac{\mathbf{v} + \mathbf{v}_1}{\mathbf{2}} + \frac{|\mathbf{v} - \mathbf{v}_1|\sigma}{2}$$

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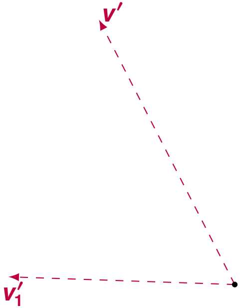

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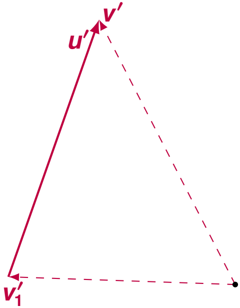

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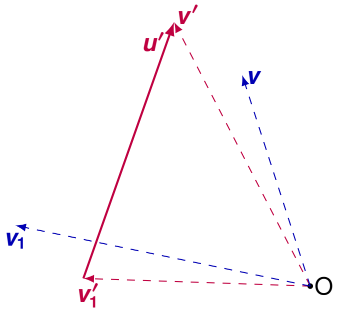

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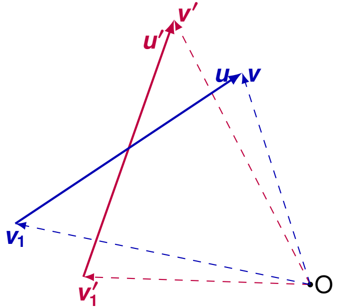

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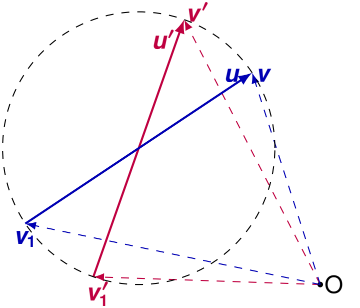

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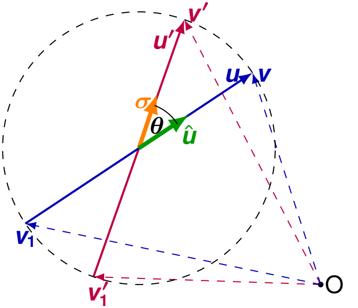


The collisional operator

The collisional operator $Q_2(f,f)$ is defined via:

$$(1.2) \qquad Q_2(f,f)(x,t,v) = \int_{\mathbb{R}^d} \int_{S^{d-1}} \left(f' f_1' - f f_1\right) b(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_1,$$

where

- $f' = f(x,t,v')$, $f_1' = f(x,t,v_1')$ etc.

The collisional kernel

In relevant physical applications, the collisional kernel $b(|u|, \hat{u} \cdot \sigma)$ is assumed to be of the form:

$$b(|u|, \hat{u} \cdot \sigma) = |u|^A \, \tilde{b}(\cos\theta), \tag{1.3}$$

where

1. $0 < A \leq 1$ corresponds to the variable hard potentials case
2. $A = 0$ corresponds to Maxwellian molecules
3. $-d < A < 0$ corresponds to the soft potentials case

First generalization of Boltzmann equation

Beyond binary interactions of particles - joint work with I. Ampatzoglou

Beyond binary interactions

- The Boltzmann equation takes into account only binary collisions of particles.
  - Ternary or higher order collisions are neglected due to lower probability of occurring compared to binary.
- However, when the gas is dense enough, higher order interactions are much more likely to happen.
  - An example of such a situation is a **colloid**, which is a homogeneous non-crystalline substance consisting of
    - ultramicroscopic particles of one substance dispersed through a second substance.
  - It was pointed out[2] by *Russ - Von-Günberg [2002]*, that multi interactions among particles significantly contribute to the grand potential of a colloidal gas and are modeled by a sum of higher order interaction terms.

[2] and further verified experimentally and numerically.

Our (long term) goal - joint work with I. Ampatzoglou

- Motivated by above, we aim to introduce and rigorously derive (from a system of classical particles) a kinetic model which goes beyond binary interactions, by incorporating a sum of higher order interaction terms in (1.1).
- Such an equation, which could serve as a toy model for a non-ideal gas, would be of the form:

$$(2.1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f = \displaystyle\sum_{k=2}^{m} Q_k(\underbrace{f, f, \cdots, f}_{k\text{-times}}), & (t,x,v) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0,x,v) = f_0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \text{ where} \end{cases}$$

  - $Q_k(\underbrace{f, f, \cdots, f}_{k\text{-times}})$ is the k-th order collisional operator, and
  - $m \in \mathbb{N}$ is the accuracy of the approximation.
- Equations similar to (2.1) were studied for Maxwell molecules in the works of Bobylev, Gamba and Cercignani using Fourier transform methods.

The case $m = 2$: Boltzmann equation

- The task of rigorously deriving an equation of the form (2.1) from a classical many particle system, even for the case $m = 2$, is a challenging problem[3] that has been settled only in certain situations.
- For hard-sphere interactions,
  - the analysis was pioneered by *Lanford [1975]*,
  - and recently completed by *Gallagher - Saint-Raymond - Texier [2012]*.
- For short-range potentials, the analysis has been done in e.g.
  - *King [1975]*,
  - *Gallagher - Saint-Raymond - Texier [2012]*,
  - *Pulvirenti - Saffirio - Simonella [2014]*.

[3] Hilbert described this task, in his famous sixth problem, as one of the main challenges for mathematicians of the twentieth century.

The case $m = 3$: binary-ternary Boltzmann equation

- Up to our knowledge, the case $m = 3$ has not been studied at all.

$$\partial_t f + v \cdot \nabla_x f = Q_2(f,f) + Q_3(f,f,f), \quad (t,x,v) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

- In addition to understanding
  - binary interactions, and
  - interactions among three particles,
  - the case $m = 3$ requires careful analysis of their mutual interactions.

- We will come back to the case $m = 3$ at the end of the talk.

Derivation of ternary Boltzmann - joint work with I. Ampatzoglou

- Now we focus on rigorously deriving a purely ternary equation.

Challenge #1

How to make sense of ternary interactions so that we can detect their contributions?

Ternary interaction

1. In a typical dilute hard-sphere gas, the probability of a simultaneous contact of three hard-spheres is very small compared to e.g. the situation when one of the three particles is in simultaneous contact with two other particles.
2. In some physical situations, e.g. when one considers colloids as in *Russ - Von-Günberg [2002]* **interactions among three particles are determined by the sum of the distances of the interacting particles**, rather than by different geometric configurations.

Motivated by (1) and (2), we introduce the notion of a ternary interaction:

Definition

Let $\epsilon > 0$ and consider three particles i, j, k with positions and velocities $(x_i, v_i), (x_j, v_j), (x_k, v_k) \in \mathbb{R}^{2d}$. We say that the particles i, j, k are in $(i; j, k)$ ternary ϵ-interaction if the following geometric condition holds:

$$d^2(x_i; x_j, x_k) := |x_i - x_j|^2 + |x_i - x_k|^2 = 2\epsilon^2. \tag{2.2}$$

The parameter ϵ is called interaction zone.

Transformation of velocities at a ternary interaction

- Consider an $(i;j,k)$ ternary ϵ-interaction

(2.3) $$|x_i - x_j|^2 + |x_i - x_k|^2 = 2\epsilon^2.$$

- Let v_i^*, v_j^*, v_k^* denote the velocities of the interacting particles after the interaction. Assuming the particles are of equal mass $m = 1$, we consider the interaction to be elastic i.e. the three particle momentum-energy system is satisfied:

(2.4) $$v_i^* + v_j^* + v_k^* = v_i + v_j + v_k,$$

(2.5) $$|v_i^*|^2 + |v_j^*|^2 + |v_k^*|^2 = |v_i|^2 + |v_j|^2 + |v_k|^2.$$

- Introducing the re-scaled vectors[4]:

(2.6) $$\omega_1 := \frac{x_j(t) - x_i(t)}{\sqrt{2}\epsilon}, \quad \omega_2 := \frac{x_k(t) - x_i(t)}{\sqrt{2}\epsilon},$$

(2.3) is equivalent to $|\omega_1|^2 + |\omega_2|^2 = 1$.

[4] We call the vectors ω_1, ω_2 impact directions of the interaction.

More on transformations of velocities

- Q: How do we know that there is the unique solution to the system (2.4) - (2.5)?
- A: We impose an extra condition:
  - Since the i particle interacts with the pair of particles (j,k), we assume that the velocities v_j, v_k transform with respect to the impact directions unit vector

    $$\begin{pmatrix} v_j^* \\ v_k^* \end{pmatrix} = \begin{pmatrix} v_j \\ v_k \end{pmatrix} - c \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \tag{2.7}$$

    for some $c \in \mathbb{R}$.
  - Once we add condition (2.7) to the system (2.4) - (2.5), the new system has a unique solution that algebraically characterizes the conservation of momentum and energy for the type of ternary interaction defined in (2.2).
- From now on, when an $(i;j,k)$ interaction happens, we assume that velocities of the interacting particles instantaneously transform according to the collisional law

$$(v_i, v_j, v_k) \to (v_i^*, v_j^*, v_k^*). \tag{2.8}$$

The phase space

Definition

Let $d \in \mathbb{N}$, with $d \geq 2$, $N \in \mathbb{N}$ and $\epsilon > 0$. The phase space of the N-particle system of ϵ-interaction zone is defined as:

$$\mathcal{D}_{N,\epsilon} = \left\{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d^2(x_i; x_j, x_k) \geq 2\epsilon^2 \quad \forall (i,j,k) \in \mathcal{I}_N \right\}, \tag{2.9}$$

where

$$X_N = (x_1, ..., x_N) \in \mathbb{R}^{dN},$$
$$V_N = (v_1, ..., v_N) \in \mathbb{R}^{dN},$$

represent the positions and velocities of the N-particles and

$$\mathcal{I}_N = \left\{ (i,j,k) \in \{1, ..., N\}^3 : i < j < k \right\}.$$

Evolution of the system of N particles

Consider an initial configuration $Z_N \in \mathcal{D}_{N,\epsilon}$. The motion is described as follows:

(I) Particles perform rectilinear motion as long as there is no interaction i.e.

$$\dot{x}_i = v_i, \quad \dot{v}_i = 0, \quad \forall i \in \{1, ..., N\}.$$

(II) Assume now that

- an initial configuration $Z_N = (X_N, V_N)$ has evolved until time $t > 0$, reaching $Z_N(t) = (X_N(t), V_N(t))$, and
- there is an $(i; j, k)$ interaction at time t.

Then the velocities $v_i(t), v_j(t), v_k(t)$ instantaneously transform according to

$$(v_i(t), v_j(t), v_k(t)) \to (v_i^*(t), v_j^*(t), v_k^*(t)). \tag{2.10}$$

Challenge #2

- It is not obvious that (I)-(II) produce a well defined dynamics, since the system can possibly run into pathological configurations.
- In the case of binary interactions, *Alexander [1975]* established global well-defined dynamics, but that work does not apply to the notion of our ternary interaction.
- For ternary interactions:
  - We prove that a global in time, measure-preserving flow can be defined for almost all initial configurations.
  - In order to go from local to global in time flow we establish the following crucial fact - **if a triplet of particles was in the interaction then as the system evolves in time the subsequent interaction cannot involve the same triplet of particles**.

The Liouville equation

The global measure-preserving interaction flow yields the Liouville equation for the N-particle probability density f_N:

(2.11)
$$\begin{aligned}
&\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0, \quad (t, Z_N) \in (0,\infty) \times \mathring{\mathcal{D}}_{N,\epsilon},\\
&f_N(t, Z_N) = f_N(t, Z_N^*), \quad \text{whenever } d^2(x_i; x_j, x_k) = 2\epsilon^2 \text{ for } (i,j,k) \in \mathcal{I}_N.\\
&f_N(0, Z_N) = f_{N,0}(Z_N), \quad Z_N \in \mathring{\mathcal{D}}_{N,\epsilon},
\end{aligned}$$

where

$$\begin{aligned}
Z_N^* &= (X_N, V_N^*),\\
V_N^* &= (v_1, ..., v_{i-1}, v_i^*, v_{i+1}, ..., v_{j-1}, v_j^*, v_{j+1}, ..., v_{k-1}, v_k^*, v_{k+1}, ..., v_N).
\end{aligned}$$

Ternary Boltzmann equation

- Integrating by parts the Liouville equation, we derive a linear finite hierarchy of equations, referred to as BBGKY hierarchy.
- By considering the number of particles $N \to \infty$, and the interaction zone $\epsilon \to 0^+$ in the **new scaling**:

$$N\epsilon^{d-\frac{1}{2}} \simeq 1, \tag{2.12}$$

  we arrive at the infinite hierarchy of equations, called Boltzmann hierarchy.
- Boltzmann hierarchy admits a special class of factorized solutions with the factor solving the new **ternary** Boltzmann equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0, x, v) = f_0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{2.13}$$

Ternary kernel

The expression $Q_3(f,f,f)$ is the cubic collisional kernel, given by:

$$Q_3(f,f,f) = \int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}} \frac{b_+(\omega_1,\omega_2,v_1-v,v_2-v)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} \left(f^* f_1^* f_2^* - f f_1 f_2\right)\, d\omega_1\, d\omega_2\, dv_1\, dv_2,$$

where

$$b_+ = \max\{b,0\},$$
$$b = b(\omega_1,\omega_2,v_1-v,v_2-v) := \langle\omega_1, v_1-v\rangle + \langle\omega_2, v_2-v\rangle,$$

$$f^* = f(t,x,v^*), \quad f = f(x,t,v),$$
$$f_1^* = f_1(t,x,v_1^*), \quad f_1 = f(t,x,v_1),$$
$$f_2^* = f_2(t,x,v_2^*), \quad f_2 = f(t,x,v_2),$$

Challenge #3

- Proving convergence of the BBGKY hierarchy to the (infinite particle) Boltzmann hierarchy - the heart of which lies in

  1. Identifying good configurations[5] and studying their stability under adjunction of a collisional pair of particles, which turns out to be possible thanks to
  2. New geometric estimates
     - Spherical estimates of the intersection of $\mathbb{S}_1^{2d-1}$ with $K_\rho^d \times \mathbb{R}^d$, where K_ρ^d is a solid cylinder in $\mathbb{R}^d$ of radius ρ.
     - A transition map (which allows us to control post-collisional configurations using new ellipsoidal estimates)
     - Ellipsoidal estimates of the intersection of $(2d-1)$-dimensional ellipsoids with $K_\rho^d \times \mathbb{R}^d$.

[5] configurations that do not run into any kind of interactions under backwards time evolution.

A remark about symmetrized equation

- The phase space (2.9) will produce the kinetic equation (2.13), in which the tracked particle is always the central particle of the interactions occurring.
- By working in the phase space

$$\widetilde{D}_{N,\epsilon} = \{Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d_l^2(x_i, x_j, x_k) \geq 2\epsilon^2, \quad \forall (i,j,k,l) \in \widetilde{\mathcal{I}}_N\},$$

  where

$$\widetilde{\mathcal{I}}_N = \{(i,j,k,l) : (i,j,k) \in \mathcal{I}_N \text{ and } l : \{i,j,k\} \to \{i,j,k\} \text{ is a permutation}\},$$

$$d_l(x_i, x_j, x_k) = \sqrt{|x_{l_i} - x_{l_j}|^2 + |x_{l_i} - x_{l_k}|^2},$$

  and using similar arguments, one can derive a symmetrized version of (2.13), in which the tracked particle can be either central or adjacent.
- It has been shown by Ampatzoglou, that the symmetrized ternary equation satisfies similar statistical and entropy production properties as the classical Boltzmann equation.

Derivation of the binary-ternary Boltzmann equation - joint work with I. Ampatzoglou

Recently we have completed the next step in our program - derivation of equation (2.1) for $m = 3$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_2(f, f) + Q_3(f, f, f), \\ f(0) = f_0, \end{cases} \tag{2.14}$$

We call this equation the **binary–ternary Boltzmann equation**.

Challenge #1 - detecting both binary and ternary interactions

The first challenge we face in deriving (2.14) is to provide a mathematical framework allowing us to detect **both**:

- binary
$$d_2(x_i, x_j) := |x_i - x_j| = \epsilon,$$
- and ternary
$$d_3(x_i; x_j, x_k) := \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2} = \sqrt{2}\epsilon$$

interactions among particles.

A crucial, conceptual obstacle is the apparent incompatibility of the Boltzmann-Grad scaling dictated by binary interactions

$$N\epsilon^{d-1} = 1, \tag{2.15}$$

and the scaling of ternary interactions

$$N\epsilon^{d-1/2} = 1. \tag{2.16}$$

Addressing incompatibility of scalings

We overcome the scaling problem by assuming that

- particles are hard spheres of diameter ϵ_2,
- that can also interact as triplets via the interaction zone ϵ_3, in the common scaling

$$N\epsilon_2^{d-1} = N\epsilon_3^{d-1/2} = 1. \tag{2.17}$$

- Note that (2.17) implies that $\epsilon_2 << \epsilon_3$.

Challenge #2 - decoupling binary and ternary interactions

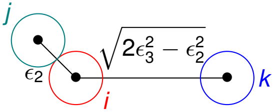
Our framework a-priori allows i.e. that particles i and j interact as hard spheres:

$$d_2(x_i, x_j) = \epsilon_2,$$

while at the same time there is another particle k such that the particle i interacts with the particles j and k:

$$d_3(x_i; x_j, x_k) = \sqrt{2}\epsilon_3,$$

as illustrated here:



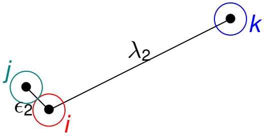
Pathological configurations, including the one above, are going to be shown to be negligible (far from trivial).

Decoupled interactions

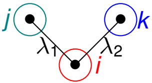
We show that as long as $0 < \epsilon_2 < \epsilon_3 < 1$, only the following two interaction scenarios are possible with non-trivial probability under time evolution:

- **Pure binary:** Two particles interact as hard-spheres while all other particles are not involved in any binary or ternary interactions at the same time.
- **Pure ternary:** Three particles interact via an interaction zone, while none of them is involved in a binary interaction with either of the other two particles of the interaction zone at the same time.

Pure binary interaction: $\epsilon_2^2 + \lambda_2^2 > 2\epsilon_3^2, \quad \lambda_2 > \epsilon_2$.

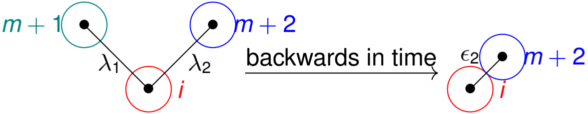


Pure ternary interaction: $\lambda_1^2 + \lambda_2^2 = 2\epsilon_3^2, \quad \lambda_1, \lambda_2 > \epsilon_2$.



Challenge #3 - the stability of a good configuration

- Assume that we have a good configuration[6] of m-particles and we add σ particles to the system, where $\sigma \in \{1, 2\}$, such that a binary or ternary interaction is formed among one of the existing particles and the σ new particles.
- Under backwards time evolution, the system could run into another binary or ternary interaction, e.g.



$$\lambda_1^2 + \lambda_2^2 = 2\epsilon_3^2, \quad \lambda_1, \lambda_2 > \epsilon_2$$

- This is the first time there was the need to address the possibility of a newly formed interacting configuration running into an interaction of a different type.
- We develop novel algebraic and geometric tools to show that outside of a small measure set, negligible in the limit, the newly formed configuration does not run into any additional interactions backwards in time.

[6] a configuration which does not run into any kind of interactions under backwards time evolution.

Current and future work

- Small data global well-posedness for the kinetic equation involving binary and ternary interactions (joint with Ampatzoglou, Gamba, and Tasković).
- Generation and propagation of moments for a binary-ternary Boltzmann equation (Ampatzoglou and Tasković).
- Derivation of a kinetic equation involving a linear combination of *m* interactions (joint work with Ampatzoglou)

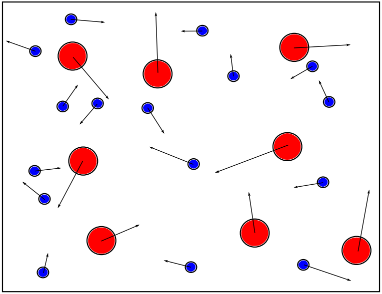
Second generalization of Boltzmann equation

Mixture of gases - joint with I. Ampatzoglou and J. Miller

Physical Motivation

Much effort has been put into studying the dynamics of a collection of interacting gases.

- Gas mixtures such as helium and xenon were studied as a possible coolant for nuclear reactors in spacecraft and thermoacoustic refrigerators.
- Sound propagation in binary mixtures and hypersonic shockwave analysis for aerospace applications have also been studied.



Boltzmann's Equation for Mixtures

Let us consider two gases, one of type A and the other of type B. If

- $g_0(x,v) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an initial density distribution on phase space of the type A gas, and
- $h_0(x,v) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the distribution of the type B gas,

the evolution of the two gases is modeled by the Boltzmann system for mixtures:

$$
\begin{cases} \partial_t g + v \cdot \nabla_x g = c_1 Q_{1,1}(g,g) + c_{1,2} Q_{1,2}(g,h) \\ \partial_t h + v \cdot \nabla_x h = c_2 Q_{2,2}(h,h) + c_{2,1} Q_{2,1}(h,g), \\ g(0,x,v) = g_0(x,v), \qquad h(0,x,v) = h_0(x,v). \end{cases} \tag{3.1}
$$

Results about (3.1) include:

- First formulations [Chapman, Cowling - 1950's], [Hamel - 1960's]
- LWP of system [Ha, Noh - 2010]
- Stability around Maxwellians [Briant, Daus - 2016]
- Propogation of moments [de la Canal, Gamba, Pavić Čolić - 2020].

Rigorous Derivation?

Despite the mathematical progress on the subject of the system (3.1), no work has been done on rigorously deriving the system from a system of particles.

A system of finitely many hard spheres

We derive the Boltzmann system (3.1) from a mixture of finitely many hard spheres. In order to do this, we consider

- N_1 hard spheres of mass M_1 and diameter ϵ_1, mixed with
- N_2 hard spheres of mass M_2 and diameter ϵ_2.
- For the i^{th} particle of mass M_1, we denote its center by x_i and its velocity by v_i.
- Similarly, for the i^{th} particle of mass M_2, we denote its center by y_i and its velocity by w_i.

For notational simplicity, we will write the vector of all positions and velocities by

$$Z_{(N_1,N_2)} = (x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2}, v_1, \dots, v_{N_1}, w_1, \dots, w_{N_2}).$$

The natural phase space for this collection of particles is

$$\mathcal{D}^{(N_1,N_2)}_{(\epsilon_1,\epsilon_2)} := \left\{ Z_{(N_1,N_2)} : \begin{array}{l} \forall i \neq j, \ |x_i - x_j| \geq \epsilon_1, \ |y_i - y_j| \geq \epsilon_2, \\ \forall i, j, \ |x_i - y_j| \geq \dfrac{\epsilon_1 + \epsilon_2}{2} \end{array} \right\}.$$

Evolution of the system of $N_1 + N_2$ particles

(I) If non-collisional, we assume that the particles perform rectilinear motion, i.e.

$$(3.2) \qquad \begin{cases} \dot{x}_i = v_i, & \dot{y}_j = w_j \\ \dot{v}_i = 0, & \dot{w}_j = 0. \\ (x_i(0), v_i(0)) = (x_{i,0}, v_{i,0}), & (y_i(0), w_i(0)) = (y_{i,0}, w_{i,0}). \end{cases}$$

(II) For a collisional pair of particles, we assume the collision is completely elastic, i.e. energy and momentum are conserved under collisions. Consequently, we have four collisional laws, including:

- If (x_i, v_i) and (y_j, w_j) are such that $|x_i - y_j| = (\epsilon_1 + \epsilon_2)/2$, then the pre-collisional velocities (v_i, w_j) give rise to the post-collisional velocities (v_i^*, w_j^*) by

$$(3.3) \qquad v_i^* = v_i - \frac{2M_2}{M_1 + M_2}\left((v_i - w_j) \cdot \frac{(x_i - y_j)}{\|x_i - y_j\|}\right)\frac{(x_i - y_j)}{\|x_i - y_j\|}$$

$$(3.4) \qquad w_j^* = w_j + \frac{2M_1}{M_1 + M_2}\left((v_i - w_j) \cdot \frac{(x_i - y_j)}{\|x_i - y_j\|}\right)\frac{(x_i - x_j)}{\|x_i - y_j\|}.$$

Liouville equation

We prove that for almost every initial configuration in phase space, the flow (I)-(II) is well defined and measure preserving. Consequently, an initial density $f_{(N_1,N_2),0}$ on the phase space $\mathcal{D}_{(\epsilon_1,\epsilon_2)}^{(N_1,N_2)}$ evolves according to the Liouville equation:

$$(3.5) \qquad \begin{cases} \partial_t f_{(N_1,N_2)} + \sum_{k=1}^{N_1} v_k \cdot \nabla_{x_k} f_{(N_1,N_2)} + \sum_{k=1}^{N_2} w_k \cdot \nabla_{y_k} f_{(N_1,N_2)} = 0 \text{ on } \mathring{\mathcal{D}}_{(\epsilon_1,\epsilon_2)}^{(N_1,N_2)} \\ f_{(N_1,N_2)}(Z^*_{(N_1,N_2)}) = f_{(N_1,N_2)}(Z_{(N_1,N_2)}) \text{ on } \partial\mathcal{D}_{(\epsilon_1,\epsilon_2)}^{(N_1,N_2)}, \\ f_{(N_1,N_2)}(0) = f_{(N_1,N_2),0} \end{cases}$$

where $Z^*_{(N_1,N_2)}$ is the post-collisional configuration related to the pre-collisional configuration $Z_{(N_1,N_2)}$ by the collisional laws.

Partial symmetry

In order to understand the statistical behavior of the $N_1, N_2, \epsilon_1, \epsilon_2$ particle system, we require each of the particles of the same mass to behave identically. Namely:

- $f_{(N_1,N_2)}$ is invariant under permutations of the (x_i, v_i) variables and
- $f_{(N_1,N_2)}$ is invariant under permutations of the (y_j, w_j) variables.
- Note that we do not require that $f_{(N_1,N_2)}$ is invariant under interchanging (x_i, v_i) and (y_j, w_j) variables for any i, j.

This lack of symmetry forces $f_{(N_1,N_2)}$ to take into account the behavior of each type of particles, but places the $N_1, N_2, \epsilon_1, \epsilon_2$ particle system outside of the standard framework developed for hard sphere systems of a single type.

The concept of a mixed marginal

In order to handle this partial symmetry symmetry, we introduce a mixed marginal:

Definition

For each $s \in \{1, \dots, N_1 - 1\}$ and $\ell \in \{1, \dots N_2 - 1\}$ we define the *mixed marginal* of $f_{(N_1,N_2)}$ to be

$$f^{(s,\ell)}_{(N_1,N_2)} := \int \mathbb{1}_{\mathcal{D}^{(N_1,N_2)}_{(\epsilon_1,\epsilon_2)}} f_{(N_1,N_2)} dx_{s+1} \dots dx_{N_1} dv_{s+1} \dots dv_{N_1} dy_{\ell+1} \dots dy_{N_2} dw_{\ell+1} \dots dw_{N_2}.$$

This concept of a mixed marginal is key to our analysis and allows us to distinguish the behavior of both types of particles.

Convergence result

Theorem (Ampatzoglou, Miller, P. - 2021)

Let $N_1, N_2, \epsilon_1, \epsilon_2$ obey the mixed Boltzmann-Grad scalings. Then on certain Gaussian weighted L^∞ spaces, the following equations are locally well posed in the mild formulation:

1. *BBGKY Hierarchy,*
2. *Boltzmann Hierarchy,*
3. *Boltzmann System for Mixtures.*

Additionally, solutions of (1) converge in observables to solutions of (2) for a sequence of approximate initial data:

$$f^{(s,\ell)}_{N_1,N_2} \rightsquigarrow f^{(s,\ell)}, \qquad f^{(s,\ell)}_{N_1,N_2,0} := \mathbb{1}_{\mathcal{D}^{s,\ell}_{\epsilon_1,\epsilon_2}} f^{(s,\ell)}_0 \tag{3.6}$$

Propagation of Chaos as a corollary of the convergence result

Corollary (Ampatzoglou, Miller, P. - 2021)

Let (g, h) *solve the Boltzmann system* (3.1) *with initial data* $(g_0, h_0) \in \left(L^\infty_\omega(\mathbb{R}^{2d})\right)^2$*. Then solutions of the BBGKY hierarchy with initial data*

$$f^{(s,\ell)}_{N_1,N_2,0} := \mathbb{1}_{\mathcal{D}^{s,\ell}_{\epsilon_1,\epsilon_2}} g_0^{\otimes s} \otimes h_0^{\otimes \ell} \tag{3.7}$$

converge in the sense of observables

$$f^{(s,\ell)}_{N_1,N_2} \rightsquigarrow g^{\otimes s} \otimes h^{\otimes \ell}. \tag{3.8}$$

Relationship between constants

The constants $c_{1,2}, c_{2,1}$ are given by

$$c_{1,2} = c_2 \left(\frac{1+b}{2}\right)^{d-1}, \qquad c_{2,1} = c_1 \left(\frac{1+b^{-1}}{2}\right)^{d-1}, \tag{3.9}$$

where $\epsilon_1 \equiv b\epsilon_2$.

- Note that (3.9) shows that as b grows large, the constant $c_{1,2}$ grows large while the constant $c_{2,1}$ becomes small.
- This agrees with physical intuition that if one gas is comprised of larger particles than the other, it has a larger effect on the system as a whole.

Thank you!