# Breakdown of small amplitude breathers for the nonlinear Klein-Gordon equation

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# Breathers and the Klein-Gordon equation

- Klein-Gordon equation

$$u_{tt} = u_{xx} - u + f(u), \qquad f(u) = \mathcal{O}(u^2), \qquad x \in \mathbb{R}$$

- Breathers: Periodic in time spatially localized solutions.
- Breathers for the Sine-Gordon equation $u_{tt} = u_{xx} - \sin u$:

$$u(x,t) = 4 \arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right), \quad m, \omega > 0, \;\; m^2 + \omega^2 = 1.$$

- What about other nonlinearities $f$?
- Families of breathers should be unlikely to happen.

# Importance of the existence/non-existence of breathers

- Per se: Existence of periodic orbits in a dynamical system.
- For the linear equation there is decay in time. If breathers exist are an obstruction for decay in time for the nonlinear equation (even for small data).
- If breathers exist they should play an important role in the long term dynamics (Soliton resolution conjecture).
- This is the case for the Sine-Gordon equation in suitable weighted Sobolev spaces (Chen, Liu, Lu 2020).

# Spatial dynamics: Breathers as homoclinic orbits

$$u_{tt} = u_{xx} - u + f(u), \qquad f(u) = \mathcal{O}(u^2)$$

- Dynamical system with $x$ as time: phase space is space of $2\pi/\omega$-periodic functions in $t$ for some $\omega > 0$.
- Breathers $\equiv$ Homoclinic orbits to the steady state $u = 0$.
- $u = 0$ has finite dimensional stable and unstable eigenspaces: the stable/unstable invariant manifolds unlikely to intersect.
- But this is hard to prove in general...

# Non-existence of breathers for the Klein Gordon eq.

Global results:

- Kowalczyk, Martel and Muñoz (2016): Nonexistence of odd (in $x$) breathers for any odd $f$.
- The breathers of the Sine-Gordon equation are even in $x$.

Perturbative results:

- Birnir–McKean–Weinstein and Denzler (1990's): Perturbed Sine-Gordon equation

$$u_{tt} = u_{xx} - \sin u + \epsilon \Delta(u), \quad \epsilon \ll 1, \quad \Delta \text{ analytic}$$

- Persistence of the family of breathers implies $\Delta(u)$ is a trivial perturbation.

# In higher dimensions and lattices

- Soffer–Weinstein, 1999 (also Bambusi–Cuccagna, 2011):
  In $\mathbb{R}^3$, non-existence of breathers if one adds a potential (under some hypotheses).

- Scheider (2020): The cubic Klein Gordon equation has very weakly localized breathers (do not belong to the energy space).

- Breathers do exist for Hamiltonian systems on lattices (McKay, Aubry,...).

- In this talk: Only the one dimensional Klein-Gordon equation.

# Small amplitude breathers for the odd Klein Gordon eq.

- What about small amplitude breathers?
- Spatial dynamics perspective: Small homoclinic loops to $u = 0$.
- Simplest setting: Analytic Odd Klein-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd analytic}$$

# Kruskal and Segur

- Kruskal–Segur (1987): Formal arguments for the $\phi^4$ model to indicate the breakdown of breathers with

$$\text{frequency } \omega : 0 < 1 - \omega \ll 1 \quad \text{and} \quad \text{amplitude } \sim \sqrt{1 - \omega^2}.$$

- Questions:

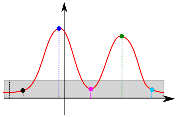
  1. How to make rigorous the formal arguments to prove the breakdown of breathers (and extend the proof to all possible $\omega$'s).
  2. Do small amplitude breathers with exponentially small (with respect to the amplitude) tails exist? $\leftarrow$ Generalized breathers

# Question 1: Breakdown of small amplitude breathers

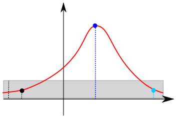
- Goal: For "typical" analytic odd $f$, small amplitude breathers do not exist.
- But we need to impose certain restrictions...
- Let $\sigma \in (0,1)$ and $\omega > 0$. A $\frac{2\pi}{\omega}$-periodic-in-$t$ function $u(x,t)$ is $\sigma$-multi-bump in $x$ if there exist $x_1 < x_2 < x_3 < x_4 < x_5$ such that

$$\|u(x_j,\cdot)\|_{H^1_t} \leq \sigma \|u(x_i,\cdot)\|_{H^1_t}, \quad \forall j \in \{1,3,5\},\ i \in \{2,4\}.$$

Otherwise, it is said to be $\sigma$-single-bump.



Multi-bump



Single-bump

# Main result

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd analytic}$$

**Theorem (G.-Gomide-Seara-Zeng)**

*There exists $\Theta_f \in \mathbb{C}$, depending analytically on $f$, such that if $\Theta_f \neq 0$: For any $\sigma \in (0,1)$, there exists $\rho^* > 0$ such that there does not exist any solution $u(x,t)$ which:*

1. *is $\frac{2\pi}{\omega}$-periodic in $t$ for some $\omega > 0$,*
2. *satisfies*

$$\|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \quad \text{as} \quad |x| \to +\infty,$$

3. *satisfies* $\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} < \rho^* \min\{1, \omega^{\frac{1}{2}}\}$,
4. *is $\sigma$-single-bump.*

# Some remarks

- $\Theta_f$ depends analytically on $f$ $\to$ For "typical" $f$, $\Theta_f \neq 0$.
- The result applies to any frequency $\omega$ and any decay rate as $|x| \to \infty$.
- So, for typical $f$, small amplitude breathers do not exist provided:
  - We restrict to single-bump breathers,
  - We admit the smallness to depend on $\omega$:

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} < \rho^* \min\{1, \omega^{\frac{1}{2}}\}.$$

- Typically, multi-bump breathers should not exist either.
- One should be able to rule out breathers such that

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} < \rho^*.$$

# Question 2: Generalized breathers

- Nan Lu (2014): There exist breathers with exponentially small tails for some periods.
- Fix the frequency $\omega = \sqrt{1-\epsilon^2}$ with $0 < \epsilon \ll 1$.
- There exist solutions $u$ such that are $2\pi/\sqrt{1-\epsilon^2}$ – periodic and

$$\frac{\epsilon}{2} \leq \sup \|u(x,\cdot)\|_{H^1_t} \leq 2\epsilon \text{ and } \limsup_{x\to\pm\infty} \|u(x,\cdot)\|_{H^1_t} \lesssim e^{-c/\epsilon}, \quad c > 0.$$

- Based on Shatah and Zeng (2003).
- Groves and Schneider (2000's): "modulated pulse" solutions with small (beyond all orders) tails for the nonlinear Klein Gordon equations (and quasilinear wave equations).

# Main result: Generalized breathers

Theorem (G.-Gomide-Seara-Zeng)

*Fix the frequency* $\omega = \sqrt{1-\epsilon^2}$ *with* $0 < \epsilon \ll 1$.

- *There exist* $2\pi/\omega$ *-periodic-in-t solutions u such that*

$$\frac{\epsilon}{2} \leq \sup \|u(x,\cdot)\|_{H^1_t} \leq 2\epsilon \qquad \textit{and}$$

$$\lim_{|x|\to\infty} \sup \left( \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} + \|\partial_x u(x,\cdot)\|_{L^2_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} \right) \leq M e^{-\frac{\sqrt{2}\pi}{\epsilon}}.$$

- *If* $\Theta_f \neq 0$, *they also satisfy*

$$\lim_{|x|\to\infty} \inf \left( \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} + \|\partial_x u(x,\cdot)\|_{L^2_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} \right) \geq M^{-1} e^{-\frac{\sqrt{2}\pi}{\epsilon}}.$$

# Some ideas about the proof

- The proofs of all the results rely on spatial dynamics techniques ($x$ as evolution variable).
- For the breakdown of breathers: We need to analyze the stable/unstable invariant manifolds associated to the steady state $u = 0$.
- For the generalized breathers: center-stable and center-unstable invariant manifolds.
- In this talk, we focus on the proof of the breakdown of breathers.
- We need to deal with exponentially small phenomena.

# Breather breakdown from spatial dynamics point of view

$$\partial_{tt} u - \partial_{xx} u + u - \frac{1}{3} u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5) \text{ odd analytic}$$

- Fix periodicity in $t = 2\pi/\omega$, $\omega > 0$.
- We want to rule out the existence of single-bump small homoclinic loops in $H^1_t\left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$.
- Eigenvalues of the linearized equation at $u = 0$:

$$\lambda_n^{\pm} = \pm\sqrt{1 - n^2\omega^2}, \quad n \in \mathbb{Z}.$$

- The number of hyperbolic eigenvalues is always finite and increases when $\omega \to 0$.

# Breather breakdown from spatial dynamics point of view

- Eigenvalues: $\lambda_n^\pm = \pm\sqrt{1 - n^2\omega^2}$.
- Bifurcations: At $\omega = \frac{1}{k}$, $k \in \mathbb{N}$, a new pair of (weakly) hyperbolic eigenvalues appears.
- Two settings:
  - Close to bifurcation: $0 < \frac{1}{k} - \omega \ll 1$, $k \in \mathbb{N}$.
  - Far from bifurcation: Otherwise (including $\omega = 1/k$, $k \in \mathbb{N}$).

# Far from bifurcation

- Far from bifurcation: All hyperbolic eigenvalues are “strong”.
- All orbits in the stable/unstable invariant manifolds of $u = 0$ escape “far away” from $u = 0$.
- If homoclinic loops exist, they must be large.
- Equivalently: If breathers exist, they must have large amplitude.
- Small homoclinic loops may only appear when $\omega$ is close to bifurcation.

# Close from bifurcation

- Kruskal-Segur setting: Close to the first bifurcation i.e. periodicity in $t = 2\pi/\omega$ with
$$0 < 1 - \omega \ll 1.$$

- Key setting: Close to the first bifurcation in the odd in $t$ setting:
$$u(x,t) = \sum_{n\geq 1} u_n(x)\sin(n\omega t).$$

- For the first bifurcation: Take
$$\omega = \sqrt{1-\epsilon^2} \qquad \text{with} \qquad 0 < \epsilon \ll 1.$$

# First bifurcation in the odd in $t$ setting

$$\partial_{tt} u - \partial_{xx} u + u - \frac{1}{3} u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd analytic}$$

- Eigenvalues: $\lambda_1^\pm = \pm\epsilon$ and $\lambda_n^\pm = \pm i\sqrt{n^2(1-\epsilon^2)-1}$, $n \geq 2$.
- Spatial dynamics ($x$ as evolution variable): one dimensional (weak) stable and unstable invariant manifolds.
- Weakness $\lambda_1^\pm \to$ The invariant manifolds have "size" $\mathcal{O}(\epsilon)$.
- Scaling: $u = \epsilon v$, $y = \epsilon x$ and $\tau = \omega t$:

$$\partial_y^2 v - \frac{\omega^2}{\epsilon^2} \partial_\tau^2 v - \frac{1}{\epsilon^2} v + \frac{1}{3} v^3 + \frac{1}{\epsilon^3} f(\epsilon v) = 0,$$

# Equation for the Fourier coefficients (odd setting)

- Equation for the Fourier coefficients $v(y,\tau) = \sum_{n\geq 1} v_n(y)\sin(n\tau)$:

$$\begin{cases} \ddot{v}_1 = v_1 - \Pi_1\left[\dfrac{v^3}{3} + \mathcal{O}(\epsilon^2)\right], \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\epsilon^2} v_n - \Pi_n\left[\dfrac{v^3}{3} + \mathcal{O}(\epsilon^2)\right], \;\; n \geq 2, \end{cases}$$

  with $\cdot = d/dy$ and $\lambda_n = \sqrt{n^2(1-\epsilon^2)-1}$.

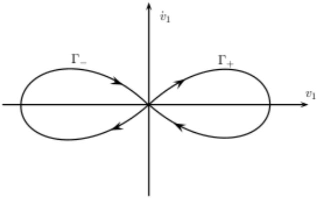
- Equivalently, for $v_n$:

$$\epsilon^2 \ddot{v}_n = -\lambda_n^2 v_n - \Pi_n\left[\epsilon^2 \frac{v^3}{3} + \mathcal{O}(\epsilon^4)\right], \;\; n \geq 2.$$

- Singular limit $\epsilon \to 0$: the plane $\mathcal{M} = \{v_n = \dot{v}_n = 0, \;\; n \geq 2\}$ is the slow manifold with dynamics $\ddot{v}_1 = v_1 - \dfrac{v_1^3}{4}$ (Duffing equation).

# Homoclinic breakdown

- The Duffing equation $\ddot{v}_1 = v_1 - \frac{v_1^3}{4}$ has two homoclinic loops.




$$v_1 = \pm \frac{2\sqrt{2}}{\cosh(y)}$$

- Do these homoclinic loops persist for the full problem?

# Homoclinic breakdown

- Singular perturbation problem:

  Fast rotation versus weak hyperbolicity $\longrightarrow$ Exponentially small phenomena.

- How to measure the distance between the perturbed invariant manifolds?

- Classical perturbative methods (Melnikov Theory) cannot be applied.

# Formal series expansions

- Look for parameterizations of $W^{\mathrm{st}}(0)$ and $W^{\mathrm{uns}}(0)$ (one dimensional)

$$v_n^{\mathrm{uns}}(y,\epsilon),\ \ v_n^{\mathrm{st}}(y,\epsilon),\ \ n \geq 1.$$

- Look for formal solutions as a power series of $\epsilon$:

$$v_n^*(y,\epsilon) = v_{n,0}(y) + \epsilon v_{n,1}^*(y) + \epsilon^2 v_{n,2}^*(y) + \ldots \quad \text{for } * = \mathrm{uns}, \mathrm{st}$$

- One can check:

$$v_{n,m}^{\mathrm{uns}}(y) = v_{n,m}^{\mathrm{st}}(y) \quad \forall n, m \in \mathbb{N}$$

- Thus: their difference is beyond all orders:

$$v_n^{\mathrm{uns}}(y,\epsilon) - v_n^{\mathrm{st}}(y,\epsilon) = \mathcal{O}(\epsilon^m) \quad \forall m \in \mathbb{N}.$$

- Typically: the power series in $\epsilon$ are divergent and the difference between manifolds is flat with respect to $\epsilon$.

# Main result

- Take a section transversal to the solutions

$$\Sigma = \{(v, \partial_y v);\ \mathcal{H}(v, \partial_y v) = 0 \quad \text{and} \quad \Pi_1 [\partial_y v] = 0\}$$

**Theorem (G.-Gomide-Seara-Zeng)**

*Call $P^{uns,st}$ the first intersection points of $W^{uns,st}$ with $\Sigma$. Then, there exists a constant $\Theta_f$ such that, for $\epsilon \ll 1$, the distance $d(\epsilon) = P^{uns} - P^{st}$ satisfies*

$$\Pi_3[d(\epsilon)] = \frac{2}{\epsilon} e^{-\frac{\pi\sqrt{2}}{\epsilon}} \left( \Theta_f + \mathcal{O}\left(\frac{1}{\log \epsilon}\right)\right)$$

*and*

$$\Pi_n[d(\epsilon)] = \frac{2}{\epsilon} e^{-\frac{\pi\sqrt{2}}{\epsilon}} \mathcal{O}\left(\frac{1}{\log \epsilon}\right) \qquad n > 3.$$

# Implication on breathers (if $\Theta_f \neq 0$)

- The constant $\Theta_f$ is the one appearing in the main theorems.
- If $\Theta_f \neq 0$, then the invariant manifolds $W^{\mathrm{uns}}(0)$ and $W^{\mathrm{st}}(0)$ do not intersect the first time they reach $\Sigma$.
- It rules out the existence of single-bump homoclinic loops.
- Even if $\Theta_f \neq 0$, $W^{\mathrm{uns}}(0)$, $W^{\mathrm{st}}(0)$ may still coincide after more rounds.
- It implies non-existence of single–bump breathers with period $2\pi/\sqrt{1-\epsilon^2}$ but not of multi–bump breathers.

# Some ideas about the proof

- We follow the ideas by Lazutkin for the homoclinic breakdown for the Standard Map (also Kruskal and Segur).
- Mostly been applied to:
  - Invariant manifolds of nearly integrable Hamiltonian systems (Arnold diffusion)
  - Local bifurcations for Hamiltonian/Reversible/Volume preserving systems
- Homoclinic for the singular limit: $v_h(y,\tau) = \dfrac{2\sqrt{2}}{\cosh(y)} \sin\tau$.
- Look for solutions $v^{\mathrm{uns}}$ and $v^{\mathrm{st}}$ of Klein-Gordon eq. for $0 < \epsilon \ll 1$.
- $v^{\mathrm{uns}}$, $v^{\mathrm{st}}$ are $\epsilon$-close to $v_h$ and exponentially close to each other.

# Analytic continuation to complex domains

- $v_h(y,\tau) = \dfrac{2\sqrt{2}}{\cosh(y)} \sin\tau$ has singularities at $y = \pm i\pi/2$.
- Extend $v^{\text{uns}}, v^{\text{st}}$ to complex $y$ up to $y \pm i\pi/2 \sim \epsilon$.
- $v_h$ blows up at $y = \pm i\pi/2 \to v^{\text{uns}}, v^{\text{st}}$ is large at $y \pm i\pi/2 \sim \epsilon$.
- Analyze $v^{\text{uns}} - v^{\text{st}}$ for $y \pm i\pi/2 \sim \epsilon$.
- This analysis provides the constant $\Theta_f$ in the distance formula.
- $\Theta_f$ is a Stokes constant (Borel Resummation, Resurgence Theory).
- $\Theta_f$ depends on the full jet of the nonlinearity $f$.

# Last step

- This analysis is the starting point to deal with all bifurcations.
- In the $k$ bifurcation in the general (non-odd) setting, the invariant manifolds have dimension $2k + 1$.
- Two positive real eigenvalues are weak and the other strong.

Thank you for your attention