Invariant Gibbs Measures for NLS and Hartree equations

Haitian Yue

ShanghaiTech University

Joint work with Yu Deng (USC) and Andrea Nahmod (UMass Amherst)

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Outline

1. Background
2. Main result
3. Random averaging operator method
4. Further discussions

The periodic NLS

Consider the defocusing nonlinear Schrödinger (NLS) equation on torus,

$$(i\partial_t + \Delta)\, u = |u|^{p-1}\, u, \quad (t, x) \in \mathbb{R} \times \mathbb{T}^d. \tag{NLS}$$

- (NLS) is an infinite dimensional Hamiltonian system with Hamiltonian

$$H[u](t) := \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2\, dx + \frac{1}{p+1} \int_{\mathbb{T}^d} |u|^{p+1}\, \mathrm{d}x = H[u](0).$$

  It also conserves the mass $m(u) := \int_{\mathbb{T}^d} |u|^2 dx$.

- The scaling critical threshold is $s_{cr} := \frac{d}{2} - \frac{2}{p-1}$. For $s > s_{cr}$ (*sub-critical* regime) local well-posedness is obtained (when $s_{cr} > 0$) in H^s. (Bourgain '93, Bourgain-Demeter '15)

  For $s < s_{cr}$ (*super-critical* regime) ill-posedness happens.

Random initial data

Here we focus on (NLS) with *random* initial data with canonical randomization. By this we mean

$$u(0) = f(\omega) = \sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{\langle k \rangle^{\alpha}} e^{ik \cdot x} \tag{ID}$$

and $\alpha - \frac{d}{2} := s$.

- In (ID) $\{g_k(\omega)\}$ are i.i.d. standard Gaussian random variables.
- Almost surely in ω, the initial data $f(\omega)$ belongs to H^{s-} (i.e. $\cap_{s'<s} H^{s'}$). Thus (ID) is in *super-critical* regime when taking $s < s_{cr}$.
- The random data (ID) with $\alpha = 1$ is linked to the Gibbs measure.

The Gibbs measure

The concept of the Gibbs measure lies nicely at the intersection the PDE and statistical physics directions alluded above. From the Hamiltonian structure of (NLS), it can be formally defined as

$$`\ \mathrm{d}\mu \sim e^{-\mathcal{H}[u]} \prod_{x \in \mathbb{T}^d} \mathrm{d}u(x)\ '$$

and $\mathcal{H}[u]$ is the renormalization of the Hamiltonian.

$$d\mu \sim \underbrace{\exp\left[-\frac{1}{p+1}\int_{\mathbb{T}^d} :|u|^{p+1}:\ \mathrm{d}x\right]}_{\text{weight}} \cdot \underbrace{\exp\left[-\frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2\,\mathrm{d}x\right]\prod_{x\in\mathbb{T}^d}\mathrm{d}u(x)}_{\text{Gaussian measure} =: d\rho}$$

- This definition is purely *formal*, but as such $\mathrm{d}\mu$ is *formally invariant* under the flow of (NLS), due to a formal "Liouville's theorem" and conservation of the renormalized Hamiltonian $\mathcal{H}[u]$.

Construction of Gibbs measure

Justifying the above definition of Gibbs measure (for some d and p) and studying their properties under various dynamics are major problems of constructive quantum field theory.

This intimately related to the so-called Φ_d^4 model. Seminal work by: Glimm-Jaffe, Lebowitz-Rose-Speer, Simon, Nelson, Wilson, Aizenman, Fröhlich, ...
Known results:

- Dimensions $d = 1, 2$: Φ_1^{p+1}, Φ_2^{p+1} for any p.
- Dimension $d = 3$: Φ_3^4 for $p = 3$ (it's not expected for any other p.)
- Dimension $d \geq 4$: Φ_d^{p+1} can not be done for any p (recently completed by Aizenman and Duminil-Copin).

In $d = 1, 2$, the Gibbs measure $\mathrm{d}\mu$ is **absolutely continuous** w.r.t. the Gaussian measure $\mathrm{d}\rho$, which is the law of distribution for the **random initial data** (ID) with $\alpha = 1$.
In $d = 3$, the Gibbs measure $\mathrm{d}\mu$ is singular w.r.t. Gaussian measure $\mathrm{d}\rho$ (but $\mathrm{d}\rho$ still linked to $\mathrm{d}\mu$ via some reference measure, see Barashkov-Gubinelli '20).

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Justification of the invariance of Gibbs measure

Bourgain ('94, '96) developed a systematic way of showing the invariance of Gibbs measure, provided one has local well-posedness or almost sure local well-posedness with respect to Gibbs measure data.

The remaining main difficulty of the justification is that the support of the Gibbs measure is **rough** ($d \geq 2$) and so controlling the **local in time dynamics** in its statistical ensemble is a hard problem.

- In dimension d, the support of $\mathrm{d}\mu$ is $H^{1-\frac{d}{2}-}$ (negative when $d \geq 2$); In particular, this regularity admits deterministic local well-posedness (recall that $s_{cr} = \frac{1}{2} - \frac{2}{p-1} > \frac{1}{2}-$) only when $d = 1$.
- The invariance of $\mathrm{d}\mu$ was justified for $d = 1$, all $p \geq 3$ odd (Bourgain '94) and for $d = 2$, $p = 3$ (Bourgain '96).
- In 2019, Deng-Nahmod-Y. justified the invariance of Gibbs measure for $d = 2$, $p \geq 5$ (odd) and proved a.s. global well-posedness.
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Outline

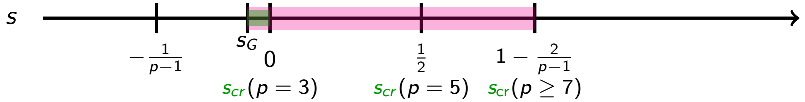
Main result

Theorem (Deng–Nahmod–Y. 2019)

Let $d = 2$ *and* $p \geq 3$ *odd. Then the renormalized (Wick ordered)* (NLS) *is almost surely globally well-posed on the support of the Gibbs measure* $\mathrm{d}\mu$ *(which is in* H^{0-}*).*
The global flow Φ_t *maps a full measure set* Σ *to itself, forms a one-parameter group (i.e.* $\Phi_{t+s} = \Phi_t \Phi_s$*), and keeps the Gibbs measure* $\mathrm{d}\mu$ *invariant under the flow:*

$$\mu(E) = \mu(\Phi_t(E))$$

for any Borel set $E \subset \Sigma$*.*



Outline

Review: Bourgain's re-centering idea

Bourgain ('96) considered the L^2 critical cubic NLS on $\mathbb{T}^2$

$$iu_t + \Delta u \ =: |u|^2 u := \mathcal{N}(u)$$

with the random initial data

$$f(\omega) = \sum_{k \in \mathbb{Z}^2} \frac{g_k(\omega)}{\langle k \rangle} e^{ik \cdot x}$$

where g_k are i.i.d centered complex Gaussian random variables.

- The data $f(\omega) \in H^{0-}(\mathbb{T}^2)$ almost surely.
- However, Bourgain constructed solutions u of the form:

$$u = u^\omega_{\mathrm{lin}} + z, \quad z \in X^{s,\frac{1}{2}+}, \text{ (for some small } s > 0\text{)},$$

where the center term $u^\omega_{\mathrm{lin}} := e^{it\Delta} f(\omega)$ has Gaussian structure, and the remainder term z can be proved in a smoother space $X^{s,\frac{1}{2}+}$.

Review: Bourgain's re-centering idea

Then solve the difference initial value problem via a Banach fixed point argument on a ball in **a smoother space**:

$$\begin{cases} iz_t + \Delta z = \mathcal{N}(\underbrace{e^{it\Delta} f(\omega)}_{\text{R:=rough-random}} + \underbrace{z}_{\text{D:=smoother-`deterministic'}}) \\ z(x,0) = 0, \qquad x \in \mathbb{T}^2 \end{cases}$$

Denote $D_i = P_{N_i} D$, $R_i = P_{N_i} R$ (where P_{N_i} are Littlewood-Paley projections) and assume $N_1 \geq N_2 \geq N_3$. Then we need to consider all the following cases:

1. $RRR \leftarrow$ multilinear large deviation estimates + integer lattice counting estimates
   - (R_1, R_2, R_3);
2. $RRD \leftarrow$ TT^* arguments $\leftrightarrow$ random matrix estimates
   - (D_1, R_2, R_3), (R_1, D_2, R_3), (R_1, R_2, D_3);
3. RDD
   - (D_1, D_2, R_3), (R_1, D_2, D_3), (D_1, R_2, D_3);
4. $DDD \leftarrow$ Deterministic H^s local theory (note that $s > s_{cr} = 0$).
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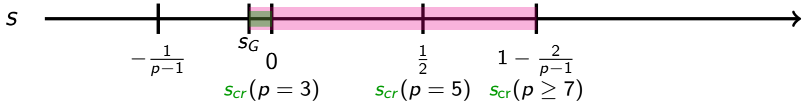
Difficulties for higher powers ($p \geq 5$)



- Let us consider the case $d = 2, p = 5$ and recall that $s_{cr} = \frac{1}{2}$.
- If we follow Bourgain's approach above, at least we need to control the following terms in H^s: (WLOG assume $N_1 \geq \cdots \geq N_5$)
  1. $\mathcal{IN}(D_1, D_2, \cdots, D_5) \to$ require $s > s_{cr} = \frac{1}{2}$ (deterministic local theory)
  2. $\mathcal{IN}(R_1, R_2, \cdots, R_5) \to \| \cdot \|_{H^s} \lesssim N_1^{(s-\frac{1}{2})+\varepsilon} N_2^{-\frac{1}{2}+\varepsilon}$
- When $N_1 \sim N \gg L \sim N_2 \geq \cdots \geq N_5$, $\mathcal{IN}(R_1, R_2, \cdots, R_5)$ is impossible to be controlled in H^s when $s > \frac{1}{2}$.
- In summary, the tricky terms are **the high-low-$\cdots$-low interactions** $\mathcal{IN}(R_N, u_L, \cdots, u_L)$, which is in $H^{\frac{1}{2}-\varepsilon}$ and hence cannot be put in the smooth remainder.

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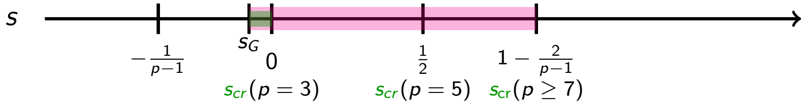

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Random averaging operator (RAO)

To handle these high-low-$\cdots$-low interactions, we need to introduce the **random averaging operator (RAO)**:

$$\mathcal{P}_{NL} : y \longrightarrow \mathcal{IN}(P_N(y), u_L \cdots, u_L).$$

Then the high-low-$\cdots$-low interaction term is

$$\mathcal{P}_{NL}(u_{\mathrm{lin}}) = \mathcal{IN}(P_N(e^{it\Delta} f(\omega)), u_L \cdots, u_L).$$

Recall Bourgain's ansatz

$$u = \underbrace{u^{\omega}_{\mathrm{lin}}}_{\in H^{0-}} + \underbrace{z}_{\in H^{\frac{1}{2}-}},$$

and then the **RAO ansatz**:

$$u = \underbrace{u^{\omega}_{\mathrm{lin}}}_{\in H^{0-}} + \underbrace{\textstyle\sum_N \sum_{L \ll N} \mathcal{P}_{NL} u^{\omega}_{\mathrm{lin}}}_{\in H^{\frac{1}{2}-}} + \underbrace{z}_{\in H^{1-}}.$$

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The key to run the above RAO ansatz under the dynamics is the structure of $\mathcal{P}_{NL}$:

- $\mathcal{P}_{NL}$ are the random averaging operators, whose coefficients are independent with $P_N(u^{\omega}_{\mathrm{lin}})$.
- Inducting on frequency we show that $\mathcal{P}_{NL}$ satisfy the bounds of Hilbert-Schmidt and Operator norms:

$$\|\mathcal{P}_{NL}\|_{\mathrm{OP}} \lesssim L^{-\delta_0}, \quad \|\mathcal{P}_{NL}\|_{\mathrm{HS}} \lesssim N^{1/2+\delta_1} L^{-1/2}. \qquad (\dagger)$$

How does the RAO ansatz run?

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Toy Model

Start with the truncated equation

$$(i\partial_t + \Delta)u_N =: |u_N|^{p-1}u_N :, \quad u_N(0) = f_N(\omega),$$

where f_N is the truncation of f, as in (ID), to frequencies $|k| \leq N$.

Goal: Prove that $\{u_N\}$ converges.

Imagine $u_N(t,x) = \chi(t) \cdot e^{it\Delta} v_N(x)$ where χ is a Schwartz function, and if we identify different Schwartz functions, then by Duhamel's formula we deduce that

$$v_N = f_N + \mathcal{M}(v_N, \cdots, v_N) \qquad \text{where,}$$

(recall $\Omega = |k|^2 - |k_1|^2 + \cdots - |k_p|^2$)

$$\mathcal{M}(v^1, \cdots, v^p)_k = \sum_{\substack{k_1 - \cdots + k_p = k \\ \Omega = 0}} (v^1)_{k_1} \overline{(v^2)_{k_2}} \cdots (v^p)_{k_p}.$$

Assume there is no pairings: $k_j \notin \{k_{j'}, k\}$ for any odd j and even j' ($\sim$ Wick ordering).

Imagine v_N is the "frequency $\leq N$" part of the full solution v.

- Let $y_N := v_N - v_{N/2}$, then y_N can be viewed as the "frequency $\sim N$" part of v. Then y_N satisfies the equation

$$y_N = F_N + \sum_{\max(N_1,\cdots,N_p)=N} \mathcal{M}(y_{N_1},\cdots,y_{N_p}), \tag{TOY}$$

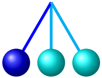
where F_N is the frequency N part of initial data,

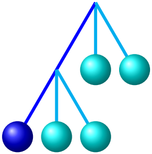
$$F_N = f_N - f_{N/2}, \qquad (F_N)_k \sim \langle k\rangle^{-\alpha} g_k(\omega) \text{ for } |k| \sim N.$$

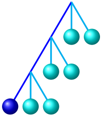
- Our goal is to prove suitable estimates for $\{y_N\}$ so that they are summable in N (hence $\{v_N\}$ will converge).

RAO Revisited $\longrightarrow$ random matrix ($p = 3$)

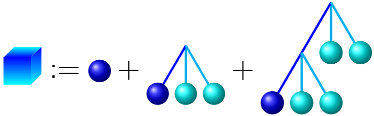
Let $\mathcal{I}$ = Duhamel operator. Denote  $:= F_N(\omega)$ and  $:= v_{N^\delta}$ (choose $N \gg L := N^\delta$ with some $1 > \delta > 0$). Naturally we can also define

 $:= \mathcal{M}(F_N, v_{N^\delta}, v_{N^\delta})$,

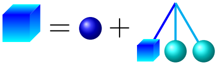
 $:= \mathcal{M}(\mathcal{M}F_N(\omega), v_{N^\delta}, v_{N^\delta}), v_{N^\delta}, v_{N^\delta})$,

and so on  $, \cdots .$

The sum of these tree forms on an infinite series of trees:

$$\Psi_{N,N^\delta} =$$  $$+ \cdots .$$

which is equivalent to the para-linearized equation:

$$\begin{cases} \Psi_{N,N^\delta} = \mathcal{M}(\Psi_{N,N^\delta}, v_{N^\delta}, v_{N^\delta}); \\ \Psi_{N,N^\delta}(0) = F_N(\omega). \end{cases} \quad \Longleftrightarrow$$ 

By solving this equation, we have that the k-th Fourier mode of Ψ_{N,N^δ} is in the following form:

$$\boxed{\mathcal{F}(\ \blacksquare\ )(k) = \sum_{k_1} h_{kk_1} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha}}$$

where h_{kk_1} is the $(1,1)$ random tensor (matrix); indep. of $g_{k_1}(\omega)$.

RAO Ansatz Revisited

If we just stop here, we could have the ansatz for RAO method,

RAO Ansatz

$$(y_N)_k = (F_N)_k + \sum_{k_1} h_{kk_1} \cdot (F_N)_{k_1} + (z_N)_k, \tag{RAO}$$

- Here h_{kk_1} is the $(1,1)$ random tensor (matrix), which is a Borel function of y_L for $L \leq N^\delta$.
- The $(1,1)$ random tensor (matrix) h_{kk_1} satisfies some Hilbert-Schmidt norm and operator norm bounds:

$$\|h_{kk_1}\|_{\ell^2_k \to \ell^2_{k_1}} \lesssim L^{-\delta_0}, \quad \|h_{kk_1}\|_{\ell^2_{kk_1}} \lesssim N^{1/2+\delta_1} L^{-1/2},$$

$$\left\| \left(1 + \frac{|k - k_1|}{L}\right)^\kappa h_{kk_1} \right\|_{\ell^2_{kk_1}} \lesssim N, \quad \text{for some large } \kappa.$$

Outline

Probabilistic scaling

Recall NLS with nonlinear power p on $\mathbb{T}^d$ with the random data:

$$u(0) = f(\omega) = \sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{\langle k \rangle^\alpha} e^{ik\cdot x} \tag{ID}$$

and $\alpha - \frac{d}{2} := s$.

- One fundamental question: what is the optimal value of s for almost-sure LWP to hold?
- Our results fully answer this question, at least in the setting of (NLS). In particular we find the value

$$s_{pr} := -\frac{1}{p-1} \leq s_{cr}$$

the critical index in *probabilistic scaling*, as the threshold for (NLS) with random data.

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Probabilistic scaling

If NLS is a.s. locally well-posed with $u(0) = f(\omega)$, then the second iteration

$$u^{(1)}(t) = \mathcal{IN}(e^{it\Delta} f(\omega), \cdots, e^{it\Delta} f(\omega))$$

should be bounded in H^s for fixed time t.

Fix $|t| \sim 1$, and for $|k| \sim N$, it is easy to check that

$$\|u^{(1)}\|_{H^s} \lesssim N^{-(p-1)(s-s_{pr})},$$

where $s_{pr} := -\frac{1}{p-1}$.

Random tensor theory

To study the full probabilistic subcritical (all $s > s_{pr}$), we need to introduce the higher order random tensors (more than just random matrix) and the corresponding ansatz:

$$(y_N)_k = (F_N)_k + \sum_q \sum_{k_1,k_2,\cdots,k_q} h_{kk_1\cdots k_q} \cdot (F_N)_{k_1} + (z_N)_k. \quad \text{(Random tensor ansatz)}$$

Theorem (Deng–Nahmod–Y. 2020)

On $\mathbb{T}^d$, any $d, p \geq 3$, odd, we obtain **optimal prob. LWP** *for α–random data on H^s, $s > s_{pr}$ and $\alpha = s + \frac{d}{2}$ where $s_{pr} := -\frac{1}{p-1}$.*

- In $d = 3, p = 3$, the invariance of the Gibbs measure is very hard to justified. (probababilistic critical!)
- The justification of WKE (wave kinetic equation) at the kinetic timescale T_{kin} is related to random data problem of NLS with probababilistic critical data (but not with the full nonlinearity). (as in Deng-Hani '21)
- The **random tensor theory** can be viewed as the dispersive counterpart of the existing parabolic theories (in particular, Hairer's theory of regularity structures).

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Periodic Hartree equation

Consider the Hartree equation on torus,

$$(i\partial_t + \Delta)u = (|u|^2 * V)u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \qquad \text{(Hartree)}$$

where V is the periodic version of "$|x|^{-(3-\beta)}$" and acts like the Bessel potential $\langle \nabla \rangle^{-\beta}$ of order β on $\mathbb{T}^3$.

- (Hartree) is an infinite dimensional Hamiltonian system with Hamiltonian

$$\mathcal{H}(u)(t) = \int_{\mathbb{T}^3} |\nabla u|^2 + \frac{1}{2}|u|^2(V * |u|^2)\,\mathrm{d}x.$$

- The scaling critical threshold is

$$s_{cr} := \frac{1-\beta}{2}.$$

- V is like a Fourier multiplier $\langle k \rangle^{-\beta}$ in the Fourier fields.

Main result. Hartree equation

Theorem (Deng–Nahmod–Y. 2021)

On $\mathbb{T}^3$, $\beta > 1 - \varepsilon_0$ (ε_0 is a fixed small number), defocusing, we prove invariance of Gibbs measure and existence of global strong solutions in its statistical ensemble ($1 = \alpha$–random data).

- Note that for (Hartree),

$$s_{pr} = -\min\left(\frac{1+\beta}{2}, 1\right),$$
$$s_{cr} = \frac{1-\beta}{2}$$

  and the support of Gibbs measure: $s_G = -\frac{1}{2}^{-}$.

- Bourgain ('97) proved invariance of Gibbs measure and existence of global strong solutions of (Hartree) for $\beta > 2$. (for $d = 2$, $\beta > 0$)
- Recently Bringmann ('20) proved the invariance of Gibbs measure and existence of global strong solutions of Hartree in the context of wave equation for $\beta > 0$ and Oh-Okamoto-Tolomeo ('20) proved similar result of Stochastic Hartree in the context of wave equation for $\beta > 1/2$.

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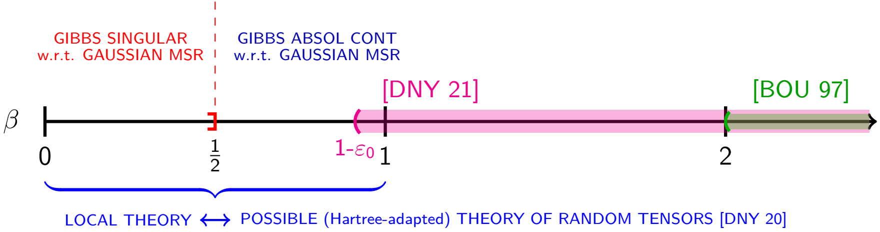

	s_{pr}	s_G	PROBABILISTICALLY	s_{det}
3D CUBIC NLS	$-\frac{1}{2}$	$-\frac{1}{2}-$	CRITICAL	$\frac{1}{2}$
3D HARTREE NLS	$-\min(\frac{1+\beta}{2}, 1)$	$-\frac{1}{2}-$	SUBCRITICAL ($\beta > 0$)	$\frac{1-\beta}{2}$

Hartree adapted RAO

Let us first write down the full ansatz of the solution to (Hartree):

$$u = \underbrace{u_{\mathrm{lin}}}_{\in H^{-\frac{1}{2}-}} + \underbrace{\sum_N \sum_{L \ll N} \mathcal{P}_{NL} u_{\mathrm{lin}}}_{\in H^{0-}} + \underbrace{\sum_N \rho_N}_{\in H^{\frac{1}{2}-\varepsilon_1-\varepsilon_2}} + \underbrace{z}_{\in H^{\frac{1}{2}-\varepsilon_1}}, \qquad \text{(RAO2)}$$

where

- $\mathcal{P}_{NL} u_{\mathrm{lin}}$ is the similar RAO term as in the ansatz of (NLS).
- ρ_N is a modified RAO term which arises from a 'critical' component (explained later) in (Hartree). It follows

$$(i\partial_t + \Delta)\rho_N = \Big(\Pi_{\lesssim N^\varepsilon}(|u_N|^2 - |u_{\frac{N}{2}}|^2)\Big) * V \cdot \rho_N.$$

- Note that $1 - \beta \ll \varepsilon_2 \ll \varepsilon_1$, which makes sure that both ρ_N and the remainder w are in the deterministic subcritical space ($\in H^{\frac{1-\beta}{2}}$).

The special term ρ_N: a 'critical' component

Suppose the frequency of $|u_N|^2 - |u_{\frac{N}{2}}|^2$ is s very small (say ~ 1), then the potential V does not lead to any gain of derivatives, and we will see that this particular term in fact exhibits some (probabilistically) 'critical' feature.
To see this, for simplicity, let us consider a simplified term:

$$\mathcal{I}\mathcal{N}_{cr}(P_N(u_{\text{lin}}), u_{N/2}, z) := \mathcal{L}(z),$$

where $\mathcal{N}_{cr}$ to be this 'critical' portion of nonlinearity:

$$\mathcal{N}_{cr}(v_1, v_2, v_3) := (\Pi_1(v_1\overline{v_2}) * V) \cdot v_3.$$

- $\mathcal{N}_{cr}$ can be understood as a cubic nonlinearity in the sense the potential V can be omitted in $\Pi_1(v_1\overline{v_2}) * V$. Hence $\mathcal{N}_{cr}$ acts as a part of 3D cubic NLS, which is probabilistic critical under the Gibbs measure.
- If $z \in H^s$, then $\mathcal{L}(z)$ can be shown in $H^{s-\varepsilon_2}$, while $\mathcal{P}_{NL}(z)$ is still in H^s. Hence
  1. $\mathcal{L}$ cannot be treated as a regular RAO as $\mathcal{P}_{NL}$;
  2. ρ_N, which satisfies $\rho_N = \mathcal{L}(\rho_N)$, cannot be treated as a part of the smooth remainder z.

Thank you!