On the derivation of the kinetic wave equation
Quadratic nonlinearity in the inhomogeneous setting

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Generic Behavior of Dispersive Solutions and Wave Turbulence

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Part 1 : The problem of justifying the kinetic wave regime for the weakly nonlinear Schrödinger equation

Weakly nonlinear Schrodinger equations with random data

$$(NLS) \quad i\partial_t u + \Delta_\omega u = \left| \begin{array}{l} \lambda^2 |u|^2 u, \\ \lambda M(u+\bar{u})^2, \\ \lambda M(Mu + M\bar{u})^2, \end{array} \right.$$

- $\lambda > 0$ ensures weakly nonlinear regime. Normalisation "$u_0 = O(1)$".
- $\widehat{M(u)}(\xi) = m(\xi)\hat{u}(\xi)$ Fourier multiplier.
- $\widehat{\Delta_\omega u}(\xi) = -\omega(\xi)\hat{u}(\xi)$ dispersion relation.
- $0 < \epsilon \ll 1$ correlation spatial scale. Spatial scale for enveloppe is 1.

Random initial data :

$$u(p, t=0, x) = u_0(p, x), \qquad p \in \text{Probability space}.$$

Homogeneous or inhomogeneous settings :

$$x \in \mathbb{T}^d \qquad \text{or} \qquad x \in \mathbb{R}^d.$$

Weak Wave Turbulence

Weak nonlinearity :

$$u_t = \frac{1}{T_{\text{lin}}}\text{linear effects} + \frac{1}{T_{\text{NL}}}\text{nonlinear effects}$$

Over a time interval $[t, t+T]$:

- Linear regime if $T \gg T_{\text{NL}}$.
- Strongly nonlinear regime if $T_{\text{NL}} \sim T_{\text{lin}} \lesssim T$.
  - *Stationary states, solitons, self-similar solutions...*
- **Weakly nonlinear regime if $T_{\text{lin}} \ll T_{\text{NL}} \lesssim T$.**
  - ***Nonlinear effects accumulate, filtered by linear effects***
  - ***Linear framework still applicable : superposition of eigenfunctions***

Randomness :

- Many degrees of freedom.
- **Random initial data** or random forcing.
- **Statistical description**.

Kinetic description in the homogeneous setting

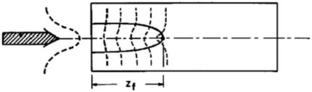
- Point energy spectrum : $\rho(k) = \mathbb{E}|\hat{u}(k)|^2$.
- Assumptions : RPA Random Phases and Amplitudes, weakly nonlinear regime.
- Kinetic wave equation : a *deterministic* effective equation for point energy spectrum, used in description of weakly nonlinear wave turbulence.

As $\epsilon \to 0$, and $\lambda \ll \epsilon^{-1}$ or ϵ^{-2},

$$\epsilon^{-d}\mathbb{E}\left|\widehat{u}(\lfloor \epsilon^{-1}k \rfloor)\,(T_{kin}t)\right|^2 \longrightarrow \rho(t,k), \qquad \text{for } T_{kin}(\epsilon,\lambda)$$

where ρ solves the kinetic wave equation, with $\mathscr{C}$ *nonlinear collision operator*

$$\begin{cases} \partial_t \rho(t,k) = \mathscr{C}[\rho](k), \\ \rho(0,k) = |A(k)|^2 = \lim_{\epsilon\to 0} \epsilon^{-d}\mathbb{E}\left|\widehat{u_0}(\lfloor \epsilon^{-1}k \rfloor)\right|^2, \end{cases} \qquad \text{(KWE)}$$



(NLS) toy model for turbulence (Zakharov- L'vov-Falkovich 92, Nazarenko '11, ...)

Kinetic description in the inhomogeneous quadratic setting

- Wigner transform for local point energy spectrum [Spohn '06] :

$$W^\epsilon[u](x,v) = \epsilon^{-d}\mathbb{E}\left(\int_{\mathbb{R}^d} \overline{u(x+\frac{z}{2})}u(x-\frac{z}{2})e^{i\frac{v}{\epsilon}\cdot z}\,dz\right).$$

- Dispersion relation $\omega(\xi) = \omega_0 + \frac{|\xi|^2}{2}, \quad \omega_0 \geq 0.$

$$\text{As } \epsilon \to 0,\ \lambda \ll \epsilon^{-2}, \qquad W^\epsilon[u(t)](x,v) \to \rho(t,x,v),$$

where ρ solves the kinetic wave equation

$$\begin{cases} \partial_t \rho + \frac{1}{\epsilon} v \cdot \nabla_x \rho = \frac{8\pi}{T_{kin}} \mathscr{C}[\rho(x)] \\ \rho(t=0) = \rho_0, \end{cases} \qquad \text{where } T_{kin} = \frac{1}{\lambda^2 \epsilon^2} \qquad \text{(KWE)}$$

The collision operator $\mathscr{C}$ is given by ($M = 1$ here)

$$\mathscr{C}[\rho](t,x,v) = \int \Big[\delta(\Sigma_-)\delta(\Omega_-)\rho\rho_1\rho_2\left(\frac{1}{\rho} - \frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + 2\delta(\Sigma_+)\delta(\Omega_+)\rho\rho_1\rho_2\left(\frac{1}{\rho} + \frac{1}{\rho_1} - \frac{1}{\rho_2}\right.$$

$$\begin{cases} \Sigma_- = v - v_1 - v_2 \\ \Sigma_+ = v + v_1 - v_2 \end{cases} \quad \begin{cases} \Omega_- = \omega(v) - \omega(v_1) - \omega(v_2) \\ \Omega_+ = \omega(v) + \omega(v_1) - \omega(v_2), \end{cases} \quad \begin{cases} \rho = \rho(v) \\ \rho_i = \rho(v_i), \quad i = 1,2 \end{cases}$$

Initial data | Basic concepts (i)

Gaussian field at spatial scale ϵ, dW Wiener integral :

$$u(t=0,x) = u_0(x) = \int a(x,\xi) e^{i\frac{\xi}{\epsilon}\cdot x} dW(\xi),$$

$$W^{\epsilon}[u_0](x,v) \to |a|^2(x,v).$$

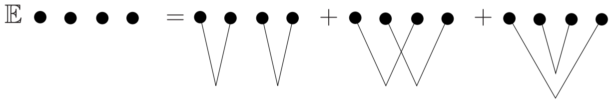
[Spectral representation Theorem (Herglotz 10's, Bochner 30's)]. Initial datum is of size one [Khinchine 20's, Paley-Zygmund 30's].

Weakly nonlinear regime if $T_{\text{lin}} \ll T_{\text{NL}} \lesssim T$.

- Nonlinear effects accumulate, filtered by linear effects.
- Linear framework still applicable : Fourier analysis.

Wick's rule for expectation of products of Gaussian variables [Isserlis 1918] :

$$\mathbb{E}(X_1X_2X_3X_4) = \mathbb{E}(X_1X_2)\mathbb{E}(X_3X_4) + \mathbb{E}(X_1X_3)\mathbb{E}(X_2X_4) + \mathbb{E}(X_1X_4)\mathbb{E}(X_2X_3).$$



Generalises to higher order products.

Initial data | Basic concepts (ii)

Iteration of Duhamel formula should be valid for large times

$$u = e^{it\Delta_\omega} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta_\omega} M(e^{is\Delta_\omega} u_0 + \overline{e^{is\Delta_\omega} u_0})^2 ds + \dots$$

$$= -i \frac{\lambda}{(2\pi)^{\frac{d}{2}}} e^{i\omega(\xi)t} \iint_{\xi_{0,1}+\xi_{0,2}=\xi} \int_0^t e^{-i\Omega s} ds \, \widehat{u_0}(\xi_{0,1}, \sigma_{0,1}) \widehat{u_0}(\xi_{0,2}, \sigma_{0,2}) d\xi_{0,1} d\xi_{0,2}$$

$$\Omega = \omega(\xi) - \sigma_{0,1}\omega(\xi_{0,1}) - \sigma_{0,2}\omega(\xi_{0,2})$$

$$\int_0^t e^{-i\Omega s} ds = \frac{1 - e^{-i\Omega t}}{i\Omega} \quad \text{of order} \quad \begin{cases} t & \text{if } |\Omega| \lesssim \frac{1}{t} \quad \text{(growth)}, \\ \frac{1}{|\Omega|} & \text{if } |\Omega| \gtrsim \frac{1}{t} \quad \text{(lower order)}. \end{cases}$$

Nonlinear effects accumulate near set $\{\Omega \approx 0\}$.

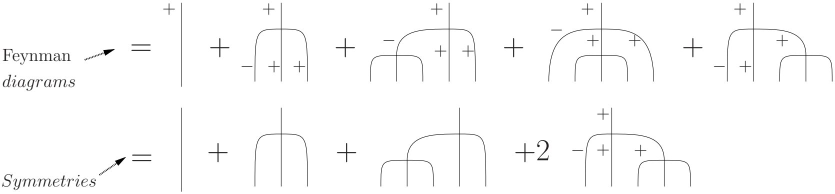
Part 2 : Statistics of second order quasisolution
Appearance of kinetic wave equation

Statistics of Quasisolution

Standard derivation of kinetic wave equation for the cubic (NLS) :

- Physics [Peierls 29, Hasselman 62, Zakharov- L'vov-Falkovich 92 and Nazarenko '11 textbooks]

$$u = u^0 + u^1 + u^2$$



Statistics :

$$\mathbb{E}\,|\hat{u}|^2(k) = \mathbb{E}\overline{(\widehat{u^0}(k)+\widehat{u^1}(k)+\widehat{u^2}(k))}(\widehat{u^0}(k)+\widehat{u^1}(k)+\widehat{u^2}(k))$$

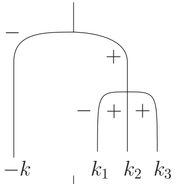
$$= \mathbb{E}|\widehat{u_0}(k)|^2 + 2Re\,\mathbb{E}\,(\overline{\widehat{u^0}(k)}\widehat{u^1}(k)) + \mathbb{E}\,|\widehat{u^1}(k)|^2 + 2Re\,\mathbb{E}\,(\overline{\widehat{u^0}(k)}\widehat{u^2}(k))$$

$$+h.o.t.$$

Statistics of Quasisolution

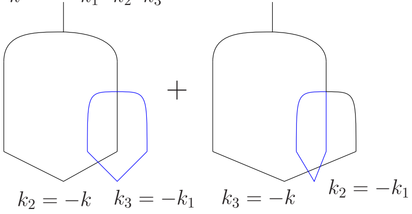
Diagrammatical computation :

$$\mathbb{E}\,\overline{\widehat{u^0}(k)}\,\widehat{u^1}(k) \quad = $$



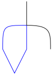
Expectation of i-i-d Gaussians forces $-k, -k_1, k_2, k_3$ to be pairwise equal.

$$=$$



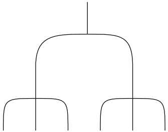
$$= \quad 0$$

Wick renormalisation removes terms involving



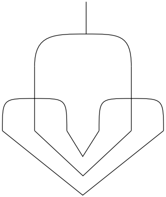
Statistics of Quasisolution

$$\mathbb{E}\,\overline{\widehat{u^1}(k)}\widehat{u^1}(k) \quad =$$

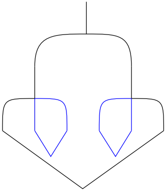


Pairing and symmetries

$$= 2$$

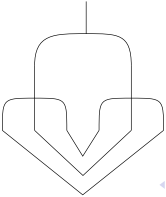


$$+4$$



Wick renormalisation

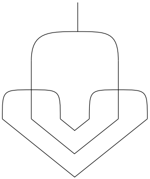
$$= 2$$



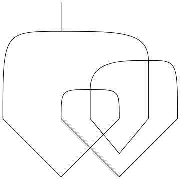
Statistics of Quasisolution

Finally we arrive at :

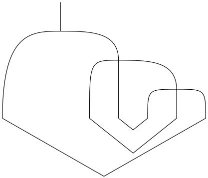
$$\mathbb{E}\,|\hat{u}(k)|^2 = \mathbb{E}\,|\widehat{u_0}(k)|^2 + 2$$



$$+4Re$$



$$+8Re$$



Statistics of Quasisolution

Explicit computation, then continuum approximation and stationary phase :

$$\mathbb{E}\,|\hat{u}(k)|^2$$

$$= \epsilon^d A^2(\epsilon k) + \epsilon^d \frac{2}{(2\pi)^{2d}} \lambda^4 \epsilon^{2d} \sum_{k=k_1+k_2+k_3} \left| \int_0^t e^{-i\Omega(k_1,k_2,k_3)s} ds \right|^2$$

$$(A(\epsilon k_1)A(\epsilon k_2)A(\epsilon k_3) - A(\epsilon k)A(\epsilon k_2)A(\epsilon k_3) + A(\epsilon k)A(\epsilon k_1)A(\epsilon k_3) - A(\epsilon k)A(\epsilon k_1)A(\epsilon k_2$$

$$= \epsilon^d A^2(\epsilon k) + \epsilon^d c_0 \boxed{\frac{t}{\lambda^{-4}\epsilon^{-2}}} \int_{\epsilon k=k_1+k_2+k_3} dk_1 dk_2 dk_3 \delta(\Omega(k_1,k_2,k_3))$$

$$A(\epsilon k)A(k_1)A(k_2)A(k_3) \left(\frac{1}{A(k)} - \frac{1}{A(k_1)} + \frac{1}{A(k_2)} - \frac{1}{A(k_3)} \right)$$

$$+h.o.t$$

$$\boxed{\frac{t}{\lambda^{-4}\epsilon^{-2}}} = \frac{t}{T_{kin}}, \qquad T_{lin} \ll T_{NL} \ll T_{kin}.$$

Appearance of Kinetic Wave Equation from second order quasi-solution

continuum approximation requires equidistribution of resonant contributions :

$$\epsilon^{2d}\sum_{k=k_1+k_2+k_3}\left|\int_0^t e^{-i\Omega(k_1,k_2,k_3)s}ds\right|^2 f(\epsilon k_1,\epsilon k_2,\epsilon k_3)$$
$$=\int_{\epsilon k=k_1+k_2+k_3}\left|\int_0^t e^{-i\Omega\left(\frac{k_1}{\epsilon},\frac{k_2}{\epsilon},\frac{k_3}{\epsilon}\right)s}ds\right|^2 f(k_1,k_2,k_3)+lot$$

$$\Omega(k_1,k_2,k_3)=|k|_H^2+|k_1|_H^2-|k_2|_H^2-|k_3|_H^2.$$

- Resonant terms $|\Omega|\lesssim\frac{1}{t}$.
- Continuum approximation holds true for $t\lesssim\epsilon^{0+}$ for standard Laplacian.
- Continuum approximation fails for $t\gg 1$ for standard Laplacian.

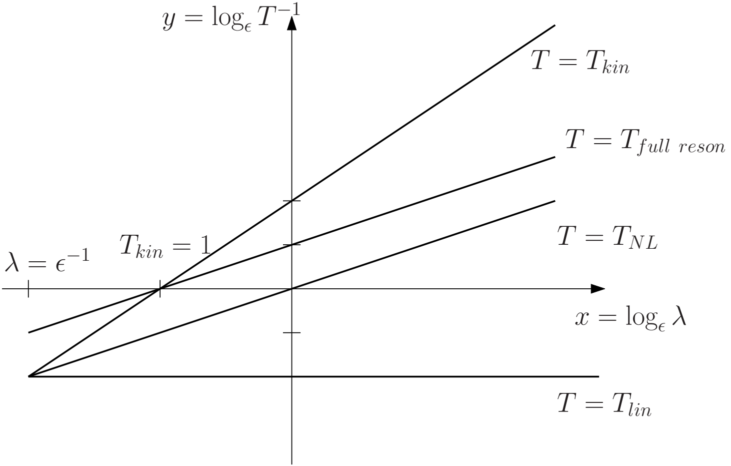
Buckmaster-Germain-Hani-Shatah '19

Continuum approximation holds true for generic diagonal ω up to time $t\lesssim\epsilon^{2-d+}$.

[Sarnak 96, Bourgain '16]

Part 3 : Results

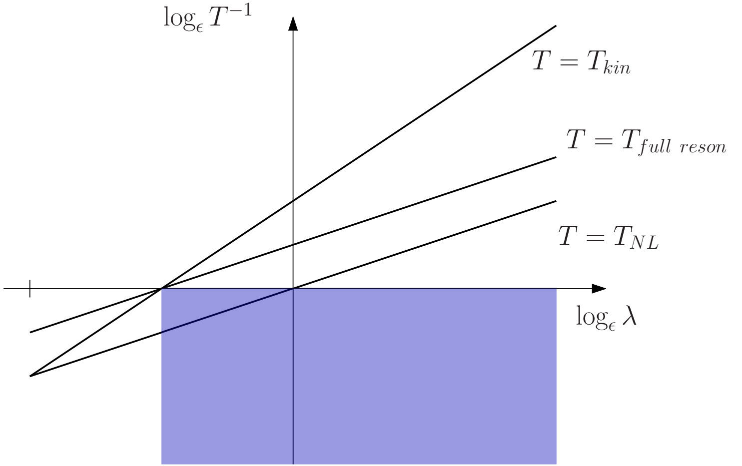
Lifespan of solutions (cubic homogeneous case)



Question : at each fixed x, what is the largest T for the lifespan of solutions (how far can one go vertically) ?

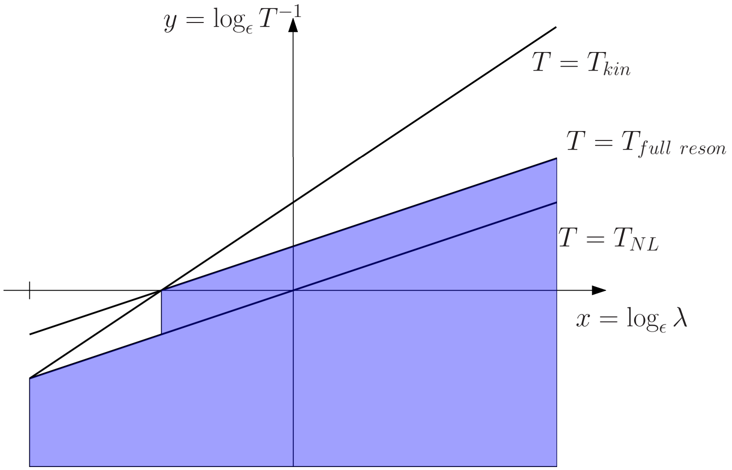
Lifespan of solutions (cubic homogeneous case)

Related works [de Suzzoni-Tzvetkov '14, Faou '18, Buckmaster-Germain-Hani-Shatah '19, Dymov-Kuksin '19, Rosenzweig-Staffilani 21, Staffilani-Tran '21]
$\Delta_\omega = \Delta$ Collot-Germain '19



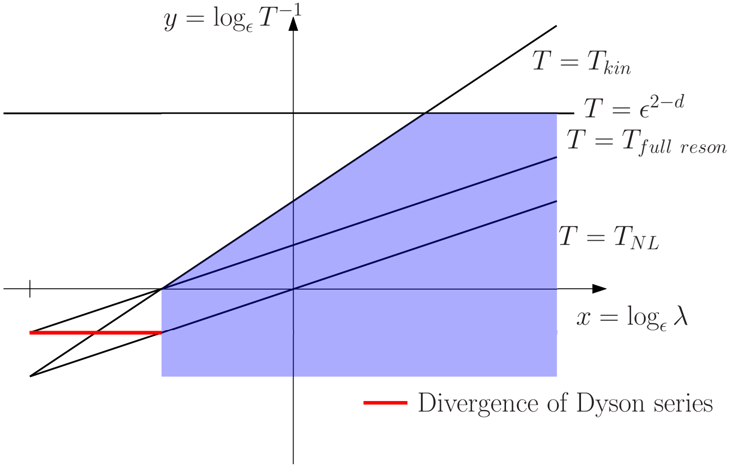
Lifespan of solutions (cubic homogeneous case)

$\Delta_\omega = \Delta$ Deng-Hani '19



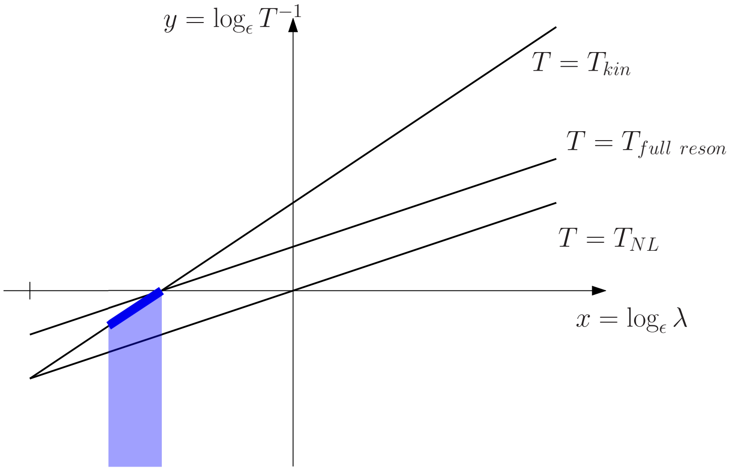
Lifespan of solutions (cubic homogeneous case)

Δ_ω generic non-diagonal (Lebesgue sense) Collot-Germain '20



Lifespan of solutions (cubic homogeneous case)

Δ_ω generic diagonal Deng-Hani '21 **first result reaching kinetic time.**



Result in the inhomogeneous setting, quadratic nonlinearity

No previous rigorous results. Earlier result [Spohn '06].

Theorem [Ampatzoglou-C.-Germain '21]

Let $d \geq 2$, $a \in \mathscr{C}_0^\infty(\mathbb{R}^{2d})$, $K > 0$, and $\omega(\xi) = -|\xi|^2 - \omega_0$ with

- either $\omega_0 = \epsilon^{-2}$ or $\omega_0 = 0$ and $m(0) = 0$.

Then there exist $\epsilon^* > 0$ and $K' > 0$ such that for any $0 < \epsilon < \epsilon^*$, for $T = \min\{\epsilon, \epsilon^K T_{kin}\}$, there exists a set E of probability $\mathbb{P}(E) > 1 - \epsilon^{K'}$ such that for any initial data in E, there exists a unique solution u in $[0, T]$.

Moreover, the solution u is approximated by the solution ρ of the corresponding kinetic wave equation in the following sense :

For any $t \in [0, T]$ and $\xi \in \mathbb{R}^d$, there holds :

$$\int_{\mathbb{R}^d} |\widehat{\rho}(t,\xi,v) - \widehat{W_E^\epsilon}[u](t,\xi,v)|\, dv \lesssim \epsilon^{K'} \left(\frac{T}{T_{kin}} + \frac{T}{\epsilon} \right),$$

where

$$W^\epsilon[u](x,v) = \frac{1}{(2\pi)^d} \mathbb{E}\left[\mathbf{1}_E \int_{\mathbb{R}^d} \overline{u\left(x + \frac{\epsilon y}{2}\right)} u\left(x - \frac{\epsilon y}{2}\right) e^{iv\cdot y}\, dy \right],$$

E is the exceptional set of existence obtained above and ρ solves (KWE) with initial data $|a|^2$.

Part 4 : Ideas of the proof

Dyson series expansion

Approximate solution :

$$u^0 = e^{it\Delta_\omega} u_0,$$

$$\begin{cases} i\partial_t u^n + \Delta_\omega u^n = \lambda \sum_{j+k=n-1} M(Mu^j + M\overline{u^j})(Mu^k + M\overline{u^k}) \\ u^n(t=0) = 0, \end{cases}, \quad n \geq 1$$

[Spohn '77, Erdos-Yau '00, Erdos-Salmhofer-Yau 07'-08', Lukkarinen-Spohn '09, Lukkarinen-Marcozzi '16.]. Decompose :

$$u = u^{app} + u^{err}, \qquad \text{where} \qquad u^{app} = \chi\left(\frac{t}{2T}\right) \sum_{n=0}^{N} u^n,$$

and stabilise u^{err} [Buckmaster-Germain-Hani-Shatah '19, C.-Germain '19, Deng-Hani '19, Deng-Hani '21] :

$$\begin{cases} i\partial_t u^{err} - \Delta_\omega u^{err} = \lambda \left[\mathfrak{L}_N(u^{err}) + \mathfrak{B}(u^{err}) + E_N\right], \\ u^{err}(0) = 0. \end{cases}$$

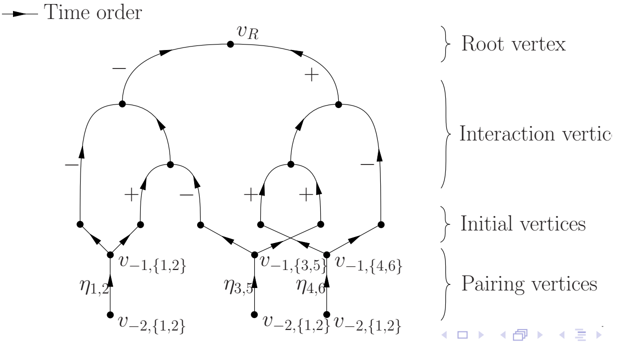
Bounds on the Dyson series | The example of $\mathbb{E}\|u^n\|_{L^2}^2$

Diagrammatic computation :

$$\mathbb{E}\|u^n(t)\|_{L^2}^2 = \sum_{G\in\mathscr{G}_n^p} \mathscr{F}_t(G).$$

where for each paired diagram $G\in\mathscr{G}_n^p$, there holds the formula :

$$\mathscr{F}_t(G) = (2\pi)^{\frac{d}{2}}\lambda^{2n}\epsilon^{d(n+1)}\int_{\underline{\eta}\in\mathbb{R}_0^{d(n+1)}}\int_{\underline{\xi}^f\in\mathbb{R}^{d(n+1)}}\int_{\underline{s}\in\mathbb{R}_+^{2n}}\Delta_t(\underline{s})d\underline{\xi}^f\,d\underline{\eta}\,d\underline{s}$$
$$M_G(\underline{\xi})\prod_{\{i,j\}\in P}\widehat{W_0^\epsilon}(\eta_{i,j},\frac{\epsilon}{2}(\sigma_{0,i}\xi_{0,i}+\sigma_{0,j}\xi_{0,j}))\prod_{v\in\mathscr{V}_i} e^{-is_v\sum_{\tilde v\in p^+(v)}\Omega_{\tilde v}}$$



- Solve Kirchhoff's laws in the graph : adapt minimal spanning tree from [Lukkarinen-Spohn '11]

$$(\underline{\xi}) \mapsto (\underline{\xi}^f, \underline{\eta}) \qquad \text{(isomorphism)}.$$

- Represent oscillatory phases as contour integrals [Erdos-Salmhofer-Yau, Buckmaster-Germain-Hani-Shatah '19]

$$\mathscr{F}_t(G) = \frac{(-1)^{\sigma_G} c_G}{(2\pi)^{n_m - \frac{d}{2}}} \lambda^{2n} \epsilon^{d(n+1)} \int_{\underline{\eta} \in \mathbb{R}_0^{d(n+1)}} \int_{\underline{\xi}^f \in \mathbb{R}^{d(n+1)}} \int_{\underline{\alpha} \in \mathbb{R}^{n_m}} d\underline{\xi}^f \, d\underline{\eta} \, d\underline{\alpha}$$

$$M(\underline{\xi}) \prod_{\{i,j\} \in P} \widehat{W_0^\epsilon}(\eta_{i,j}, \frac{\epsilon}{2}(\sigma_{0,i}\xi_{0,i} + \sigma_{0,j}\xi_{0,j}))$$

$$\prod_{\mathscr{p} \in \mathscr{P}_m} \frac{i}{\alpha_{\mathscr{p}} + \frac{ic_{\mathscr{p}}}{t}} \prod_{k=1}^{2n} \frac{i}{\alpha_{\mathscr{p}(v_k)} - \sum_{\tilde{\mathscr{p}} \triangleleft v_k} \alpha_{\tilde{\mathscr{p}}} - \sum_{\tilde{v} \in \mathscr{p}^+(v_k)} \Omega_{\tilde{v}} + \frac{ic_k}{t}}$$

- Classification of oscillatory phases at vertices : degree 0 and degree 1 vertices, linear and quadratic vertices.

$$\Omega_v(\xi^f) = \pm \tilde{\xi}.\xi^f + \tilde{\Omega}_v \qquad \text{for a linear vertex}$$

$$\Omega_v(\xi^f) = \pm \xi^f.(\xi^f + \tilde{\xi}) + \tilde{\Omega}_v \qquad \text{for a quadratic vertex.}$$

Bounds on the Dyson series | Novelties

Main novelties in inhomogeneous quadratic setting :

- New "algorithm" to estimate diagrammatic quantities.
- New slow variables $\eta_{i,j}$ appear.

- To avoid problem with low frequencies :
  - Introduce a new formulation of the time constraints function $\Delta_t(\underline{s})$.
  - Introduce "bad" clusters that needs specific bounds (case by case study).
  - Decomposition Lemma : Graphs are disjoint unions of "good" vertices and "bad" clusters. "Bad" clusters are mutually disjoints.

Thank you for your attention ! !