Full description of Benjamin-Feir instability of Stokes wave in deep water

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joint work with M. Berti and P. Ventura

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ICERM, Brown, "Generic Behavior of Dispersive Solutions and Wave Turbulence"

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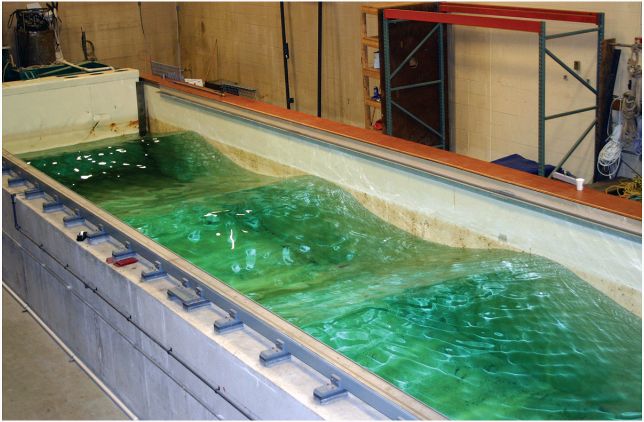
1967

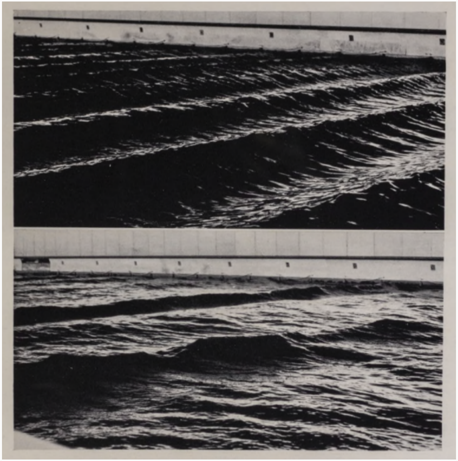
Stokes waves are traveling 1 dimensional, 2π-periodic solutions of water waves



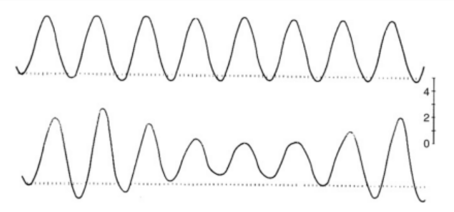
It was widely believed in the 60s that Stokes waves are **stable** solutions

Benjamin and **Feir**'s experiments to prove the *stability* of Stokes waves keep failing

Benjamin: *Instability of periodic wavetrains in nonlinear dispersive systems*, 1967



``Deep-water wave trains were generated at one end of a long tank and were observed travelling many wavelengths. It was found that such a wavetrain may develop conspicuous irregularities if it travels far enough, even when departures from periodicity can hardly be detected near the origin. And eventually, at great distances from the origin, the train may become completely disintegrated and its energy redistributed over a broad spectrum''



Benjamin-Feir (or moduational) instability

BF heuristic physical mechanism: "with long wave perturbations, Stokes wave becomes unstable"

- Many experimental and numerical results, possible mechanism for formation of rogue waves,

- Rigorous mathematical results for Water Waves:
  1. Bridges-Mielke '95 (finite depth), linear instability
  2. Nguyen-Strauss 2020 (infinite depth), linear instability
  3. Hur-Yang 2020 (finite depth), linear instability (different proof)
  4. Chen-Su 2020 (infinite depth), Nonlinear instability

- Many results for dispersive PDEs (NLS, gKdV, Whitham, ...) by Segur-Henderson-Carter-Hammack, Gallay-Haragus, Haragus-Kapitula, Bronski-Johnson, Johnson, Hur-Johnson, Bronski-Hur-Johnson, Hur-Pandey, Leisman-Bronski-Johnson-Marangell, Jin-Liao-Lin

"Take home theorem": Berti, M., Ventura 2021

Complete description of the **spectrum** near zero of the linearized water waves at **small** amplitude Stokes waves acting on **long wave** periodic perturbations

Mathematics of Benjamin-Feir instability

Water Waves: Euler equations for an incompressible, irrotational fluid in deep water $\mathcal{D}_\eta(t) = \{y < \eta(t,x)\}$ under gravity.

Equation of motions for $\vec{u} = \binom{u}{v}$ in $y < \eta(t,x)$

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla P - g e_y \\ \mathrm{div} \vec{u} = 0 \\ \mathrm{rot} \vec{u} = 0 \end{cases}$$

Boundary conditions:

$$\begin{cases} \eta_t = v - u\eta_x & \text{at } y = \eta(t,x) \\ P = P_0 & \text{at } y = \eta(t,x) \\ v = 0 & \text{at } y \to -\infty \end{cases}$$

g = gravity,

P = pressure of fluid, P_0 = atmospheric pressure,

Unknowns: free surface $y = \eta(t,x)$ velocity field $\vec{u}(t,x,y)$

- For irrotational fluids $\vec{u}$ is the gradient of a velocity potential

$$\vec{u}(t,x,y) = \nabla\Phi, \qquad \Phi(t,x,y) \text{ velocity potential}$$

- Define

$$\psi(t,x) = \Phi(t,x,\eta(t,x)) \quad \text{trace of potential at the border}$$

Φ solves elliptic problem with Dirichlet-Neumann bc

$$\begin{cases} -\Delta\Phi = 0 & \text{in } y < \eta(t,x) \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \partial_y\Phi \to 0 & y \to -\infty \end{cases}$$

- Zakharov's key observation:

$\vec{u}(t,x,y)$ is completely determined by $\eta(t,x)$ and $\psi(t,x)$ (data at the surface)

Reformulate the equations in terms of (η,ψ): e.g.

$$\eta_t = v - u\eta_x \quad \rightsquigarrow \quad \eta_t = (\Phi_y - \eta_x\Phi_x)|_{y=\eta(t,x)} = G(\eta)\psi$$

Zakharov formulation of WW

$$\begin{cases} \eta_t = G(\eta)\psi \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} \end{cases}$$

Dirichlet–Neumann operator: $G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\Phi_y - \eta_x\Phi_x)|_{y=\eta(x)}$

WW is an infinite dimensional Hamiltonian system with $\eta(x)$ and $\psi(x)$ as canonical Darboux coordinates

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J \begin{pmatrix} \nabla_\eta H(\eta,\psi) \\ \nabla_\psi H(\eta,\psi) \end{pmatrix}, \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

Hamiltonian expressed in terms of (η, ψ)

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} \psi(x)\, G(\eta)\psi(x)\, dx + \frac{1}{2}\int_{\mathbb{T}} g\eta^2\, dx$$

$$\partial_t\eta = G(\eta)\psi = \nabla_\psi^{L^2} H(\eta,\psi), \qquad \partial_t\psi = -g\eta - \frac{\psi_x^2}{2} + \frac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{2(1+\eta_x^2)} = -\nabla_\eta^{L^2} H(\eta,\psi)$$

Reversibility

$$H \circ \rho = H, \quad \rho(\eta(x), \psi(x)) := (\eta(-x), -\psi(-x))$$

The equations are invariant under space translations

Space invariance

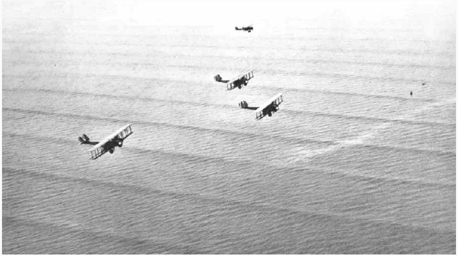
$$H \circ \tau_\theta = H\,, \quad (\tau_\theta u)(x) := u(x + \theta)$$

Stokes wave solutions

Periodic traveling waves solution of WW

$$\eta(t,x) = \breve{\eta}(x-ct),$$
$$\psi(t,x) = \breve{\psi}(x-ct)$$

2π-periodic *profiles* $\breve{\eta}(x), \breve{\psi}(x)$, *speed* $c \in \mathbb{R}$



In a reference frame in translational motion with constant speed c, the WW equations are

$$\begin{cases} \eta_t = c\eta_x + G(\eta)\psi \\ \psi_t = c\psi_x - g\eta - \dfrac{\psi_x^2}{2} + \dfrac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 \end{cases}$$

Stokes waves = equilibrium steady solutions

Theorem (Stokes, Levi-Civita, Struik, Nekrasov ....)

There exist $\epsilon_0 > 0$ *and analytic solutions* $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$, *parameterized by the amplitude* $|\epsilon| \leq \epsilon_0$ *with*

- $\eta_\epsilon(x)$, $\psi_\epsilon(x)$ 2π *periodic in* x
- $\eta_\epsilon(x)$ *even,* $\psi_\epsilon(x)$ *is odd*
- *expand as*

$$\eta_\epsilon(x) = \epsilon \cos(x) + \frac{\epsilon^2}{2}\cos(2x) + \mathcal{O}(\epsilon^3)$$

$$\psi_\epsilon(x) = \epsilon \sin(x) + \frac{\epsilon^2}{2}\sin(2x) + \mathcal{O}(\epsilon^3)\,,$$

$$c_\epsilon = 1 + \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)\,.$$

Extension:

• **Periodic 2D traveling waves:**

- *vorticity:* Dubreil-Jacotin '34, Goyon '58, Zeidler '73, Wahlen '09, Martin '13

- *large amplitude:* Krasovskii '71, Keady-Norbury '78, Toland '78, McLeod '97, Constantin-Strauss '04 , Constantin -Strauss -Varvaruca '18

• **2D quasi-periodic traveling waves:** Berti-Franzoi-M. '20, Berti-Franzoi-M. '21, Feola-Giuliani '20

Main question: are Stokes waves stable/unstable?

$$\begin{cases} \eta_t = c\eta_x + G(\eta)\psi \\ \psi_t = c\psi_x - g\eta - \dfrac{\psi_x^2}{2} + \dfrac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 \end{cases}$$

Linearized water waves equations in moving frame at Stokes waves

$$h_t = \mathcal{L}_\epsilon h$$

$\mathcal{L}_\epsilon$ = linear autonomous operator with 2π-periodic coefficients

Unstable "long wave" solutions

Look for solutions

$$h(t,x) = \mathrm{Re}\big(e^{\lambda t} e^{\mathrm{i}\mu x} v(x)\big)\,, \quad \mu \in \mathbb{R}\,,$$

where $v(x)$ is 2π-periodic, μ is **Floquet exponent**, and λ has positive real part.

Bloch-Floquet theory

Analyze the spectrum of

$$\mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\mu x} \circ \mathcal{L}_\epsilon \circ e^{\mathrm{i}\mu x}$$

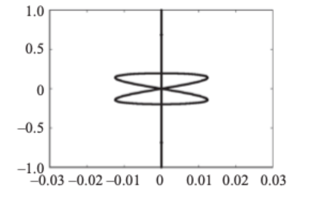
acting on 2π-periodic functions, for $0 \le \mu \le \frac{1}{2}$.

λ has positive real part $\Rightarrow$ $h(t,x)$ grows exponentially in time

Previous results for water waves in deep water

- **Numerical:** Deconinck-Oliveras 2011:

Fix $\epsilon > 0$, then $\sigma(\mathcal{L}_{\mu,\epsilon})$ is *numerically* computed as μ changes: "figure 8"



- **Analytic:** Nguyen-Strauss 2020:

There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there exists $\mu_0 = \mu_0(\epsilon) > 0$ such that for all $0 < |\mu| < \mu_0$, $\mathcal{L}_{\mu,\epsilon}$ has 2 eigenvalues of the form

$$\lambda^{\pm}(\mu,\epsilon) = \begin{cases} \frac{1}{\sqrt{2}}\mathrm{i}\mu \pm \frac{1}{2\sqrt{2}}\mu\epsilon + O(\mu^2) + O(\mu\epsilon^2) & \text{if } \mu > 0 \\ \frac{1}{\sqrt{2}}\mathrm{i}\mu \mp \frac{1}{2\sqrt{2}}\mu\epsilon + O(\mu^2) + O(\mu\epsilon^2) & \text{if } \mu < 0 \end{cases}$$

Main result

Theorem (Berti - M. - Ventura, 2021)

There exist $\epsilon_0, \mu_0 > 0$ *such that,* $\forall (\mu, \epsilon) \in [0, \epsilon_0) \times [0, \mu_0)$*, the operator* $\mathcal{L}_{\mu,\epsilon}$ *has* 4 *eigenvalues close to 0 and*

- *2 eigenvalues* $\lambda_1^{\pm}(\mu, \epsilon)$ *have the form*

$$\begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2(1 + r_0(\epsilon,\mu)) - \mu^2(1 + r_0'(\epsilon,\mu))}\,, & 0 \le \mu < \underline{\mu}(\epsilon) \\ \frac{1}{2}\mathrm{i}\underline{\mu}(\epsilon) + \mathrm{i}r(\epsilon^3)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2(1 + r_0'(\epsilon,\mu)) - 8\epsilon^2(1 + r_0(\epsilon,\mu))}\,, & \mu > \underline{\mu}(\epsilon)\,, \end{cases}$$

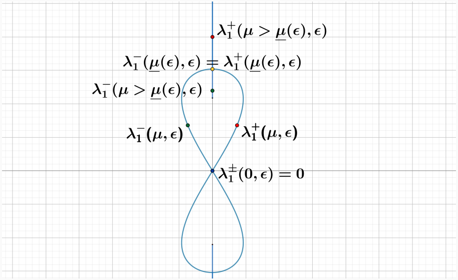
where $\underline{\mu}(\epsilon) = 2\sqrt{2}\epsilon(1 + r(\epsilon))$*. The function* $8\epsilon^2(1 + r_0(\epsilon,\mu)) - \mu^2(1 + r_0'(\epsilon,\mu)) > 0$*, respectively* < 0*, for* $0 < \mu < \underline{\mu}(\epsilon)$*, respectively* $\mu > \underline{\mu}(\epsilon)$*.*

- *2 eigenvalues are purely imaginary*

Notation: $|r(\epsilon^{m_1}\mu^{n_1}, \epsilon^{m_2}\mu^{n_2})| \le C\sum_{j=1}^{2} |\epsilon|^{m_j}|\mu|^{n_j}$ real analytic function

Curves of $\lambda^{\pm}(\mu,\epsilon) \in \mathbb{C}$ at fixed ϵ, as μ varies

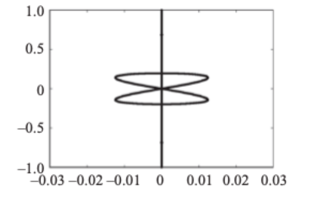
$$\lambda^{\pm}(\mu,\epsilon) \approx \begin{cases} \frac{1}{2}\mathrm{i}\mu \pm \frac{\mu}{8}\sqrt{8\epsilon^2 - \mu^2}\,, & 0 \le \mu < \underline{\mu}(\epsilon) \\ \frac{1}{2}\mathrm{i}\underline{\mu}(\epsilon)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2 - 8\epsilon^2}\,, & \mu > \underline{\mu}(\epsilon) \end{cases}$$



- For $0 < \mu < \underline{\mu}(\epsilon)$, $\lambda^{\pm}(\mu,\epsilon)$ have opposite non-zero real part
- As $\mu \to \underline{\mu}(\epsilon)$, the $\lambda^{\pm}(\mu,\epsilon)$ collide on $\mathrm{i}\mathbb{R}$ *far* from 0,
- For $\mu > \underline{\mu}(\epsilon)$ the $\lambda^{\pm}(\mu,\epsilon)$ are purely imaginary

Remarks

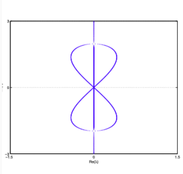
1. Our theorem describes ALL the eigenvalues close to 0, for (μ, ϵ) small
2. Complete accordance with numerical simulations by Deconinck-Oliveras, '11



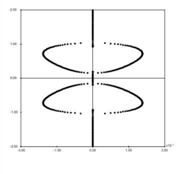
3. Nguyen-Strauss [CPAM, 22] describes the unstable eigenvalues $|\mu| \ll \epsilon$, namely the cross amid the "8". We extend these local branches to global ones

4. The eigenvalues $\lambda^{\pm}(\mu, \epsilon)$ are *not analytic* in (μ, ϵ) close to $(\underline{\mu}(\epsilon), \epsilon)$. In previous approaches the eigenvalues are a-priori supposed to be analytic in (μ, ϵ). The $\lambda^{\pm}(\mu, \epsilon)$ are eigenvalues of a 2×2 matrix *analytic* in (μ, ϵ).

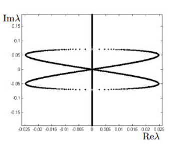
5. "Figure 8" is found *numerically* in many other models: we believe our method extends to these cases



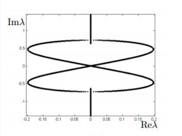
gKdV (Haragus-Kapitula)



NLS (Haragus-Kapitula)



Whitham (Deconinck Trichtchenko)



SG (Deconinck Trichtchenko)

Ideas of the proof

Difficulties:

- bifurcation problem from the defective eigenvalue 0
- the eigenvalues are not analytic “at the top of the 8”

Main ingredients:

1. Symplectic version of Kato’s similarity transformation theory
2. exploit Hamiltonian and reversibility structure
3. "KAM inspired" block diagonalization procedure

Preparation

1. Linearize the WW equations at the Stokes wave
2. Apply two changes of coordinates:
    - linear good unknown of Alinhac
    - Levi-Civita transformation

We get the system $h_t = \mathcal{L}_\epsilon h$

$\mathcal{L}_\epsilon$ Hamiltonian: $$\mathcal{L}_\epsilon = \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=:\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(1+p_\epsilon)(x)\partial_x \\ \partial_x \circ (1 + p_\epsilon(x)) & |D| \end{bmatrix}}_{=:\mathcal{B}_\epsilon}$$

$\mathcal{L}_\epsilon$ reversible: $$\mathcal{L}_\epsilon \circ \rho = -\rho \circ \mathcal{L}_\epsilon \, ,$$

where
$$p_\epsilon(x) = -2\epsilon \cos(x) + \epsilon^2 \Big(\frac{3}{2} - 2\cos(2x)\Big) + \mathcal{O}(\epsilon^3)$$
$$a_\epsilon(x) = -2\epsilon \cos(x) + \epsilon^2 \big(2 - 2\cos(2x)\big) + \mathcal{O}(\epsilon^3)$$

The linear operator $\mathcal{L}_\epsilon$ is autonomous, pseudodifferential, Hamiltonian and reversible, and has 2π-periodic coefficients

Bloch-Floquet expansion

Use Bloch-Floquet theory to study its spectrum

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_\epsilon) = \bigcup_{\mu \in [-\frac{1}{2}, \frac{1}{2})} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\epsilon}), \quad \mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\mu x}\, \mathcal{L}_\epsilon\, e^{\mathrm{i}\mu x}$$

where $\mathcal{L}_{\mu,\epsilon}$ acts on $L^2(\mathbb{T}, \mathbb{C}^2)$

- $A_\mu := e^{-\mathrm{i}\mu x}\mathrm{Op}(a(x,\xi))e^{\mathrm{i}\mu x} = \mathrm{Op}(a(x,\xi+\mu))$
- $\sigma(A_{-\mu}) = \overline{\sigma(A_\mu)} \implies \mu > 0$
- $\sigma(A_\mu)$ is 1-periodic $\implies \mu \in [-\frac{1}{2}, \frac{1}{2})$

$\Rightarrow$ We restrict to study $\sigma(\mathcal{L}_{\mu,\epsilon})$ for $\mu \in [0, \frac{1}{2})$

If λ is an eigenvalue of $\mathcal{L}_{\mu,\epsilon}$ with eigenvector $v(x)$, then

$$h(t,x) = e^{\lambda t} e^{\mathrm{i}\mu x} v(x) \quad \text{solves} \quad h_t = \mathcal{L}_\epsilon h$$

The Floquet operator associated to $\mathcal{L}_\epsilon$ is the complex *Hamiltonian* and *reversible* operator

$$\mathcal{L}_{\mu,\epsilon} = \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(1+p_\epsilon(x))(\partial_x + \mathrm{i}\mu) \\ (\partial_x + \mathrm{i}\mu)\circ(1+p_\epsilon(x)) & |D+\mu| \end{bmatrix}}_{=:\mathcal{B}_{\mu,\epsilon}}$$

domain $H^1(\mathbb{T}) := H^1(\mathbb{T}, \mathbb{C}^2)$ and range $L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C}^2)$

- **(Hamiltonian)** $\mathcal{B}_{\mu,\epsilon} = \mathcal{B}^*_{\mu,\epsilon}$
- **(Reversibility preserving)** $\mathcal{B}_{\mu,\epsilon}$ commutes with

$$\bar{\rho}\begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \bar{\eta}(-x) \\ -\bar{\psi}(-x) \end{bmatrix}$$

Goal: describe the spectrum of $\mathcal{L}_{\mu,\epsilon}$ on $L^2(\mathbb{T})$ when (μ, ϵ) small

- start from the unperturbed spectrum of $\mathcal{L}_{0,0}$
- switch on the parameters (μ, ϵ)

The unperturbed spectrum of $\mathcal{L}_{0,0}$

$$\mathcal{L}_{0,0} = \begin{bmatrix} \partial_x & |D| \\ -1 & \partial_x \end{bmatrix}$$

- $\sigma(\mathcal{L}_{0,0})$ consists of the purely imaginary eigenvalues $\lambda_k^{\pm}(0,0) := \mathrm{i}(k \mp \sqrt{|k|})\,, \quad k \in \mathbb{Z}\,.$
- 0 is isolated eigenvalue of $\mathcal{L}_{0,0}$ with algebraic multiplicity 4 $\lambda_0^+(0,0) = \lambda_0^-(0,0) = \lambda_1^+(0,0) = \lambda_{-1}^-(0,0) = 0$
- 0 has geometric multiplicity 3. A real basis of Kernel of $\mathcal{L}_{0,0}$ is

$$f_1^+ := \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}, \quad f_1^- := \begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix}, \qquad f_0^- := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

together with the generalized eigenvector

$$f_0^+ := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathcal{L}_{0,0} f_0^+ = -f_0^-$$

We want to bifurcate from the *defective* eigenvalue 0

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Kato's theory of similarity transformations

how to prolong analytically a basis of the unperturbed spectral space
to a basis of the perturbed one

Kato's theory of similarity transformations: projectors

Define the *projectors*

$$P_{\mu,\epsilon} := -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{\mu,\epsilon} - \lambda)^{-1} d\lambda$$

- well defined, bounded $L^2 \to H^1$, commuting with $\mathcal{L}_{\mu,\epsilon}$
- analytic in (μ, ϵ)
- $\mathcal{V}_{\mu,\epsilon} := \mathrm{Rg}(P_{\mu,\epsilon})$ is an invariant subspace

$$\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$$

  and one has the direct sum decomposition $H^1 = \mathcal{V}_{\mu,\epsilon} \oplus \mathrm{Ker}(P_{\mu,\epsilon})$
- $\sigma(\mathcal{L}_{\mu,\epsilon}) \cap \{z \in \mathbb{C} \text{ inside } \Gamma\} = \sigma(\mathcal{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}})$

Goal: Construct a basis of $\mathcal{V}_{\mu,\epsilon}$ and represent the action of $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ over this basis as a finite matrix

Q: How to do construct such a basis, in an *analytic* way?

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Define the transformation operators

$$U_{\mu,\epsilon} := \left(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\right)^{-1/2}\left[P_{\mu,\epsilon}P_{0,0} + (\mathrm{Id} - P_{\mu,\epsilon})(\mathrm{Id} - P_{0,0})\right]$$

- well defined, bounded $H^1 \to H^1$, invertible, analytic in (μ, ϵ)
- conjugate the spectral projectors:

$$U_{\mu,\epsilon}P_{0,0}U_{\mu,\epsilon}^{-1} = P_{\mu,\epsilon}\,, \qquad U_{\mu,\epsilon}^{-1}P_{\mu,\epsilon}U_{\mu,\epsilon} = P_{0,0}$$

- the subspaces $\mathcal{V}_{\mu,\epsilon} = \mathrm{Rg}(P_{\mu,\epsilon})$ are isomorphic one to each other:

$$\mathcal{V}_{\mu,\epsilon} = U_{\mu,\epsilon}\mathcal{V}_{0,0}$$

Transform the unperturbed basis $\{f_1^+, f_1^-, f_0^+, f_0^-\}$ of $\mathcal{V}_{0,0}$ via $U_{\mu,\epsilon}$:

Kato basis of $\mathcal{V}_{\mu,\epsilon}$, $\dim \mathcal{V}_{\mu,\epsilon} = \dim \mathcal{V}_{0,0} = 4$, for any (μ, ϵ)

$$U_{\mu,\epsilon}f_1^+ \;,\; U_{\mu,\epsilon}f_1^- \;,\; U_{\mu,\epsilon}f_0^+ \;,\; U_{\mu,\epsilon}f_0^- \,.$$

In addition, since $\mathcal{L}_{\mu,\epsilon}$ is Hamiltonian and reversible ($\bar{\rho}\mathcal{L}_{\mu,\epsilon} = -\mathcal{L}_{\mu,\epsilon}\bar{\rho}$):

- The projectors $P_{\mu,\epsilon}$ are skew-Hamiltonian, namely

$$\mathcal{J} P_{\mu,\epsilon} = P^*_{\mu,\epsilon}\mathcal{J}$$

  and reversibility preserving, i.e.

$$\bar{\rho} P_{\mu,\epsilon} = P_{\mu,\epsilon}\bar{\rho}$$

- The transformation operators $U_{\mu,\epsilon}$ are symplectic, namely

$$U^*_{\mu,\epsilon}\mathcal{J} U_{\mu,\epsilon} = \mathcal{J}$$

  and reversibility preserving.

$\Rightarrow \{U_{\mu,\epsilon} f_1^{\pm},\ U_{\mu,\epsilon} f_0^{\pm}\}$ is a symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$

A basis $\mathtt{F} := \{\mathtt{f}_1^+, \mathtt{f}_1^-, \mathtt{f}_0^+, \mathtt{f}_0^-\}$ of $\mathcal{V}_{\mu,\epsilon}$ is

- *symplectic* if $\quad (\mathcal{J}\mathtt{f}_k^-, \mathtt{f}_k^+) = 1\,, \quad (\mathcal{J}\mathtt{f}_k^{\pm}, \mathtt{f}_k^{\pm}) = 0\,, \quad (\mathcal{J}\mathtt{f}_k^{\sigma}, \mathtt{f}_{k'}^{\sigma'}) = 0\,, \ k \neq k'$
- *reversible* if $\quad \bar{\rho}\mathtt{f}_1^+ = \mathtt{f}_1^+, \quad \bar{\rho}\mathtt{f}_1^- = -\mathtt{f}_1^-, \quad \bar{\rho}\mathtt{f}_0^+ = \mathtt{f}_0^+, \quad \bar{\rho}\mathtt{f}_0^- = -\mathtt{f}_0^-$

Next goal: Represent $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ on the basis $f_k^\sigma(\mu,\epsilon) = U_{\mu,\epsilon} f_k^\sigma$, $\sigma = \pm$, $k = 0, 1$

Lemma

The 4×4 matrix that represents the Hamiltonian and reversible operator $\mathcal{L}_{\mu,\epsilon} = \mathcal{J}\mathcal{B}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ with respect to the symplectic and reversible basis $\{f_k^\sigma(\mu,\epsilon)\}_{\sigma,k}$ is

$$\mathtt{J}_4 \mathtt{B}_{\mu,\epsilon}\,, \qquad \mathtt{J}_4 := \left(\begin{array}{c|c} \mathtt{J}_2 & 0 \\ \hline 0 & \mathtt{J}_2 \end{array}\right), \qquad \mathtt{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\mathtt{B}_{\mu,\epsilon} = \mathtt{B}_{\mu,\epsilon}^*$ is the matrix ($\zeta = (\mu,\epsilon)$)

$$\begin{pmatrix} (\mathcal{B}_\zeta f_1^+(\zeta), f_1^+(\zeta)) & (\mathcal{B}_\zeta f_1^-(\zeta), f_1^+(\zeta)) & (\mathcal{B}_\zeta f_0^+(\zeta), f_1^+(\zeta)) & (\mathcal{B}_\zeta f_0^-(\zeta), f_1^+(\zeta)) \\ (\mathcal{B}_\zeta f_1^+(\zeta), f_1^-(\zeta)) & (\mathcal{B}_\zeta f_1^-(\zeta), f_1^-(\zeta)) & (\mathcal{B}_\zeta f_0^+(\zeta), f_1^-(\zeta)) & (\mathcal{B}_\zeta f_0^-(\zeta), f_1^-(\zeta)) \\ (\mathcal{B}_\zeta f_1^+(\zeta), f_0^+(\zeta)) & (\mathcal{B}_\zeta f_1^-(\zeta), f_0^+(\zeta)) & (\mathcal{B}_\zeta f_0^+(\zeta), f_0^+(\zeta)) & (\mathcal{B}_\zeta f_0^-(\zeta), f_0^+(\zeta)) \\ (\mathcal{B}_\zeta f_1^+(\zeta), f_0^-(\zeta)) & (\mathcal{B}_\zeta f_1^-(\zeta), f_0^-(\zeta)) & (\mathcal{B}_\zeta f_0^+(\zeta), f_0^-(\zeta)) & (\mathcal{B}_\zeta f_0^-(\zeta), f_0^-(\zeta)) \end{pmatrix}$$

The entries of the matrix $\mathtt{B}_{\mu,\epsilon}$ are alternatively real or purely imaginary

- Use the expansion of $f_k^\sigma(\mu,\epsilon)$

$$\Rightarrow U_{\mu,\epsilon} f_k^\sigma = \Big(\mathrm{Id} + \epsilon \partial_\epsilon P|_{0,0} + \mu \partial_\mu P|_{0,0} + \mu\epsilon\big(\partial_{\epsilon,\mu} P|_{0,0} - \frac{1}{2} P_{0,0}\, \partial_{\epsilon,\mu} P|_{0,0}\big)\Big) f_k^\sigma + \mathcal{O}(\mu^2, \epsilon^2)$$

- at $\mu = 0$ use also the information on the generalized kernel of $\mathcal{L}_{0,\epsilon}$ *for any* $\epsilon > 0$ by Nguyen-Strauss: (this is *not* a Taylor expansion)

0 is eigenvalue of $\mathcal{L}_{0,\epsilon}$ with algebraic multiplicity 4 and geometric multiplicity 2

Lemma: Matrix expansion

In a symplectically modified basis of $\mathcal{V}_{\mu,\epsilon}$ obtained from $\{U_{\mu,\epsilon} f_k^\sigma\}_{k=0,1,\sigma=\pm}$, the operator $\mathcal{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}}$ is represented by the Hamiltonian and reversible matrix

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \begin{pmatrix} E & F \\ F^* & G \end{pmatrix} \equiv \begin{pmatrix} \mathtt{J}_2 E & \mathtt{J}_2 F \\ \mathtt{J}_2 F^* & \mathtt{J}_2 G \end{pmatrix},$$

where $E = E^*, F, G = G^*$ are the 2×2 matrices

$$E := \begin{pmatrix} \epsilon^2(1 + r_1'(\epsilon, \mu\epsilon^2)) - \frac{\mu^2}{8}(1 + r_1''(\epsilon, \mu)) & -\mathrm{i}\big(\frac{1}{2}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big) \\ \mathrm{i}\big(\frac{1}{2}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big) & -\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix}$$

$$G := \begin{pmatrix} 1 + r_8(\epsilon^3, \mu^2\epsilon, \mu\epsilon^2, \mu^3) & -\mathrm{i}\mu - \mathrm{i}r_9(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ \mathrm{i}\mu + \mathrm{i}r_9(\mu\epsilon^2, \mu^2\epsilon, \mu^3) & \mu + r_{10}(\mu^2\epsilon, \mu^3) \end{pmatrix}$$

$$F = \begin{pmatrix} r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3) & \mathrm{i}r_4(\mu\epsilon, \mu^3) \\ \mathrm{i}r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \end{pmatrix}$$

Rk1: because of the Hamiltonian and reversible structure

$$E = \begin{pmatrix} \alpha & -\mathrm{i}\beta \\ \mathrm{i}\beta & \gamma \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R} \quad \Rightarrow \quad \mathtt{J}_2 E = \begin{pmatrix} \mathrm{i}\beta & \gamma \\ -\alpha & \mathrm{i}\beta \end{pmatrix}$$

Rk2: Modified basis of $\mathcal{V}_{\mu,\epsilon}$ given by ($n(\mu,\epsilon) := \frac{(f_1^-(\mu,\epsilon), f_0^-(\mu,\epsilon))}{\|f_0^-(\mu,\epsilon)\|^2}$ is a real number)

$$g_1^+(\mu,\epsilon) := f_1^+(\mu,\epsilon), \qquad g_1^-(\mu,\epsilon) := f_1^-(\mu,\epsilon) - n(\mu,\epsilon) f_0^-(\mu,\epsilon),$$
$$g_0^+(\mu,\epsilon) := f_0^+(\mu,\epsilon) + n(\mu,\epsilon) f_1^+(\mu,\epsilon), \qquad g_0^-(\mu,\epsilon) := f_0^-(\mu,\epsilon),$$

- The top-left 2×2 block of $\mathtt{L}_{\mu,\epsilon}$ shows the BF phenomenon

$$\mathtt{J}_2 E = \mathrm{i}\big(\frac{1}{2}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big)\mathrm{Id} + \begin{pmatrix} 0 & -\frac{\mu^2}{8}(1 + r_5(\epsilon,\mu)) \\ -\epsilon^2(1 + r_1'(\epsilon, \mu\epsilon^2)) + \frac{\mu^2}{8}(1 + r_1''(\epsilon,\mu)) & 0 \end{pmatrix}.$$

Its eigenvalues are

$$\begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2(1 + r_0(\epsilon,\mu)) - \mu^2(1 + r_0'(\epsilon,\mu))}\,, & 0 \le \mu < \tilde{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\tilde{\mu}(\epsilon) + \mathrm{i}r_2(\epsilon^3)\,, & \mu = \tilde{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2(1 + r_0'(\epsilon,\mu)) - 8\epsilon^2(1 + r_0(\epsilon,\mu))}\,, & \mu > \tilde{\mu}(\epsilon)\,. \end{cases}$$

- Instead $\mathtt{J}_2 G$ has purely imaginary eigenvalues of size $\mathcal{O}(\sqrt{\mu})$

Idea: **Look for a perturbative block-decoupling**, cfr. KAM theory

Look for a symplectic and reversibility preserving transformation Φ s.t.

$$\Phi^{-1}\begin{pmatrix} \mathtt{J}_2 E & \mathtt{J}_2 F \\ \mathtt{J}_2 F^* & \mathtt{J}_2 G \end{pmatrix}\Phi = \begin{pmatrix} \mathtt{J}_2 E_{new} & 0 \\ 0 & \mathtt{J}_2 G_{new} \end{pmatrix}$$

and $\mathtt{J}_2 E_{new}$ with the same structure as $\mathtt{J}_2 E$

Block-decoupling

$$\mathrm{L}_{\mu,\epsilon} = \begin{pmatrix} \mathrm{J}_2 E & \mathrm{J}_2 F \\ \mathrm{J}_2 F^* & \mathrm{J}_2 G \end{pmatrix}$$

KAM Heuristic

Decoupling possible if

$$\frac{\|\mathrm{J}_2 F\|}{\mathrm{dist}(\sigma(\mathrm{J}_2 E), \sigma(\mathrm{J}_2 G))} \ll 1$$

- It is a SINGULAR perturbation problem

$$\sigma(\mathrm{J}_2 E) = \mathcal{O}(\mu), \quad \sigma(\mathrm{J}_2 G) = \mathcal{O}(\sqrt{\mu})$$

- Problem:

$$\mathrm{J}_2 F = \begin{pmatrix} \mathrm{i} r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \\ -r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3) & -\mathrm{i} r_4(\mu\epsilon, \mu^3) \end{pmatrix}$$

when $\mu \ll \epsilon \Rightarrow \|\mathrm{J}_2 F\| = \mathcal{O}(\epsilon^3)$

We would need to impose $\frac{\epsilon^3}{\sqrt{\mu}} \ll 1$: **WRONG!** want to keep μ, ϵ independent

Next Goal: find a transformation that eliminates $-r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3)$

First step of block-decoupling

- When $\mu = 0$ we have

$$\mathrm{L}_{0,\epsilon} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -\epsilon^2 + r_1'(\epsilon^3) & 0 & r_3(\epsilon^3) & 0 \\ \hline 0 & 0 & 0 & 0 \\ r_3(\epsilon^3) & 0 & -1 + r_8(\epsilon^3) & 0 \end{array}\right)$$

We find a *symplectic* transformation putting it into its Jordan normal form: exploit the information on the generalized kernel of $\mathcal{L}_{0,\epsilon}$!

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -\epsilon^2 + r_1'(\epsilon^3) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 + r_8(\epsilon^3) & 0 \end{array}\right)$$

- for $\mu \neq 0$ we continue this transformation to a symplectic and reversibility preserving transformation Φ_1 such that

Lemma

The 4×4 Hamiltonian and reversible matrix $\mathrm{L}^{(1)}_{\mu,\epsilon} := \Phi_1^{-1} \mathrm{L}_{\mu,\epsilon} \Phi_1$ has structure

$$\mathrm{L}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} \mathrm{J}_2 E^{(1)} & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & \mathrm{J}_2 G^{(1)} \end{pmatrix}$$

with $E^{(1)} \sim E$, $G^{(1)} \sim G$ and

$$\mathrm{J}_2 F^{(1)} = \begin{pmatrix} \mathrm{i} r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \\ 0 & -\mathrm{i} r_4(\mu\epsilon, \mu^3) \end{pmatrix} = \mathcal{O}(\mu\epsilon, \mu^3)$$

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Second step of block-decoupling

We have

$$\mathrm{L}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} \mathrm{J}_2 E^{(1)} & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & \mathrm{J}_2 G^{(1)} \end{pmatrix}$$

- The condition

$$\frac{\|\mathrm{J}_2 F^{(1)}\|}{\mathrm{dist}(\sigma(\mathrm{J}_2 E^{(1)}), \sigma(\mathrm{J}_2 G^{(1)}))} = \frac{\mathcal{O}(\mu\epsilon, \mu^3)}{\mathcal{O}(\sqrt{\mu})} \ll 1$$

is fulfilled uniformly in (μ, ϵ)!

- We look for an Hamiltonian and reversibility preserving matrix S such that

$$\mathrm{L}^{(2)}_{\mu,\epsilon} = e^S \, \mathrm{L}^{(1)}_{\mu,\epsilon} \, e^{-S} = \begin{pmatrix} \mathrm{J}_2 E^{(2)} & \mathrm{J}_2 F^{(2)} \\ \mathrm{J}_2 (F^{(2)})^* & \mathrm{J}_2 G^{(2)} \end{pmatrix}$$

and $E^{(2)} \sim E$, $\quad \|\mathrm{J}_2 F^{(2)}\| \ll \|\mathrm{J}_2 F^{(1)}\|$

- Lie expansion: $\quad \mathrm{L}^{(2)}_{\mu,\epsilon} = \mathrm{L}^{(1)}_{\mu,\epsilon} + [S, \mathrm{L}^{(1)}_{\mu,\epsilon}] + h.o.t.$

$$= \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} + \left[S, \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} \right] + \begin{pmatrix} 0 & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & 0 \end{pmatrix} + h.o.t.,$$

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$$= \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} + \left[S, \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} \right] + \begin{pmatrix} 0 & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & 0 \end{pmatrix} + h.o.t.,$$

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Homological equation

Choose S to solve the **homological equation**

$$\left[S, \begin{pmatrix} J_2 E^{(1)} & 0 \\ 0 & J_2 G^{(1)} \end{pmatrix}\right] + \begin{pmatrix} 0 & J_2 F^{(1)} \\ J_2 (F^{(1)})^* & 0 \end{pmatrix} = 0$$

Take $S = J_4 \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}$, then it is equivalent to the Sylvester equation

$$J_2 E^{(1)} X - X J_2 G^{(1)} = -J_2 F^{(1)}, \qquad \text{where } X := J_2 \Sigma$$

Sylvester equation

$$AX - XB = C$$

has a solution provided e.g. $\sigma(A) \subset \{z \colon |z| < \rho\}$ and $\sigma(B) \subset \{z \colon |z| > \rho\}$

Here $\sigma(J_2 E^{(1)}) = \mathcal{O}(\mu)$, $\quad \sigma(J_2 G^{(1)}) = \mathcal{O}(\sqrt{\mu})$, so OK!

Rk: differently from KAM theory, to solve the homological equation we DO NOT diagonalize $J_2 E^{(1)}$ and $J_2 G^{(1)}$, which is not even possible when $\mu \sim 2\sqrt{2}\epsilon$ since Jordan block appears. We compute X explicitly and prove it is analytic in (μ, ϵ)

Homological equation

Choose S to solve the **homological equation**

$$\left[S, \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} \right] + \begin{pmatrix} 0 & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & 0 \end{pmatrix} = 0$$

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Lemma

The 4×4 *Hamiltonian and reversible matrix* $\mathrm{L}^{(2)}_{\mu,\epsilon} := e^{S}\mathrm{L}^{(1)}_{\mu,\epsilon}e^{-S}$ *has structure*

$$\mathrm{L}^{(2)}_{\mu,\epsilon} = \begin{pmatrix} \mathrm{J}_2 E^{(2)} & \mathrm{J}_2 F^{(2)} \\ \mathrm{J}_2 (F^{(2)})^* & \mathrm{J}_2 G^{(2)} \end{pmatrix}$$

with $E^{(2)} \sim E$, $G^{(2)} \sim G$ *and*

$$\mathrm{J}_2 F^{(2)} = \begin{pmatrix} \mathrm{i} r_6(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) & r_7(\mu^3\epsilon^3, \mu^4\epsilon^2, \mu^6\epsilon, \mu^8) \\ -r_3(\mu^2\epsilon^3, \mu^3\epsilon^2, \mu^5\epsilon, \mu^7) & -\mathrm{i} r_4(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) \end{pmatrix} = \mathcal{O}(\mu^2\epsilon^2)$$

Now the size of $\mathrm{J}_2 F^{(2)}$ is sufficiently small to completely remove the off diagonal terms via a standard implicit function theorem

Lemma

There exists a 4×4 reversibility-preserving Hamiltonian matrix S_2 such that $\mathtt{L}^{(3)}_{\mu,\epsilon} := e^{S_2}\mathtt{L}^{(2)}_{\mu,\epsilon}e^{-S_2}$ is Hamiltonian, reversible and it has structure

$$\mathtt{L}^{(3)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2 E^{(3)} & 0 \\ 0 & \mathtt{J}_2 G^{(3)} \end{pmatrix}$$

with

$$\mathtt{J}_2 E^{(3)} := \begin{pmatrix} \mathrm{i}\big(\tfrac{1}{2}\mu + r(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big) & -\frac{\mu^2}{8}(1 + r_5(\epsilon,\mu)) \\ \frac{\mu^2}{8}(1 + r_1(\epsilon,\mu)) - \epsilon^2(1 + r_1'(\epsilon, \mu\epsilon^2)) & \mathrm{i}\big(\tfrac{1}{2}\mu + r(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big) \end{pmatrix}$$

$$\mathtt{J}_2 G^{(3)} := \begin{pmatrix} \mathrm{i}\mu\big(1 + r_9(\epsilon^2, \mu\epsilon, \mu^2)\big) & \mu + r_{10}(\mu^2\epsilon, \mu^3) \\ -1 - r_8(\epsilon^2, \mu^2\epsilon, \mu^3) & \mathrm{i}\mu\big(1 + r_9(\epsilon^2, \mu\epsilon, \mu^2)\big) \end{pmatrix}$$

The eigenvalues of $\mathtt{J}_2 E^{(3)}$ are

$$\lambda^{\pm}(\mu, e) = \begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2\big(1 + r_0(\epsilon,\mu)\big) - \mu^2\big(1 + r_0'(\epsilon,\mu)\big)}\,, & 0 \le \mu < \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\underline{\mu}(\epsilon) + \mathrm{i}r(\epsilon^3)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2\big(1 + r_0'(\epsilon,\mu)\big) - 8\epsilon^2\big(1 + r_0(\epsilon,\mu)\big)}\,, & \mu > \underline{\mu}(\epsilon)\,, \end{cases}$$

The eigenvalues of $\mathtt{J}_2 G^{(3)}$ are purely imaginary

$$\lambda_0^{\pm}(\mu,\epsilon) = \pm\mathrm{i}\sqrt{\mu}\big(1 + r'(\epsilon^2, \mu\epsilon, \mu^2)\big) + \mathrm{i}\mu\big(1 + r_9(\epsilon^2, \mu\epsilon, \mu^2)\big)\,.$$

Thanks for your attention!

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