Observers in dynamical systems and their application to geophysical models

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Prediction and Variability of Air-Sea Interactions: the South Asian Monsoon
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Outline

- Overview of some applications of observer-based ideas for PDEs
- A little bit of theory
- Another application

Two motivating questions

- How do we combine various multi-scale models (of same phenomenon)?

- How can observations be used with these models?

Symmetry-based observers: Shallow-water equations

Auroux & Bonnabel *IEEE Trans. Auto. Control* (2011)

- Given height field, deduce velocity field for Saint-Venant and SWE. SWE includes Coriolis force, viscous dissipation, friction and wind shear.
- Nudging term respects symmetries of underlying problem
- Reconstruction with perfect and noisy observations
- Easier to compute compared to 4D-Var; comparable error estimate.

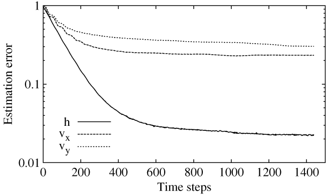


Fig. 3. Full nonlinear model: evolution of the estimation error in relative norm versus the number of time steps, in the case of noisy observations (20% noise), for $\beta_h = 5.10^{-6}\ \mathrm{s}^{-1}$ and $\alpha_h = \alpha_v = 10^3\ \mathrm{m}^{-2}$, for the three variables: height h, longitudinal velocity v_x and transversal velocity v_y.

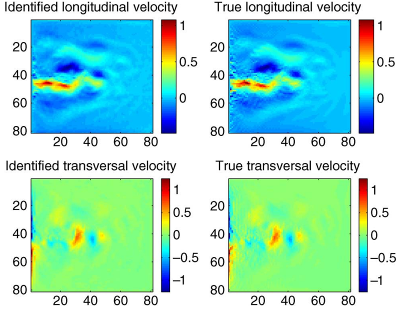


Fig. 6. Identification process for the velocity, in $\mathrm{m.s}^{-1}$: identified longitudinal (respectively, transversal) velocity at final time $(\hat{v}(T))$; true longitudinal (respectively, transversal) velocity at final time $(v(T))$.

Observers for compressible NavierStokes equation

Apte, Auroux & Ramaswamy *SIAM J Control & Optimization* (2018)

- Compressible NS: state recovery with either density observations or velocity observations
- Theorems for the linear case + numerical simulations for the nonlinear case
- Feedback term added to all equations
- Partial measurements: when the observations are made on a subset of the full computational domain

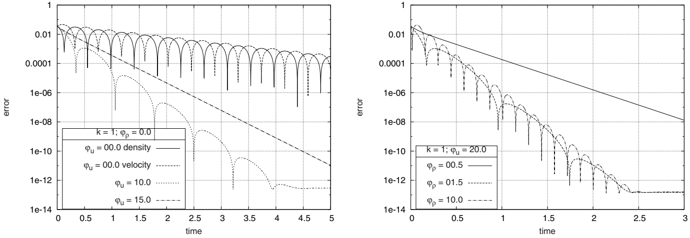


FIG. 1. *The L^2 norm of the difference between the observer $(\hat{\rho}, \hat{u})$ and the solution (ρ, u) versus time. Solid and dotted lines are the errors in ρ and u, respectively. The left panel is for fixed $\varphi_\rho = 0$ with varying φ_u while the right panel is for fixed $\varphi_u = 20$ with varying φ_ρ.*

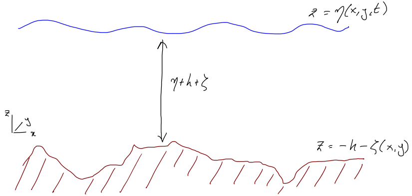
Ocean depth measurement

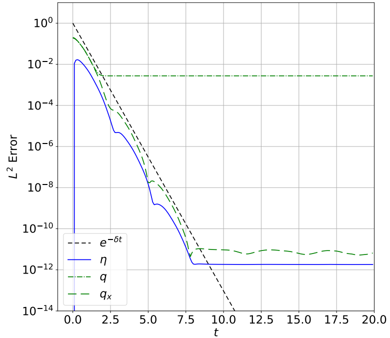
Vasan, Manisha & Auroux *Studies in Applied Math* (2021)

- Given $\eta(\vec{x}, t)$ the free surface of the sea, find $\zeta(\vec{x})$ the bottom topography.
- Inviscid, irrotational, incompressible flow. Asymptotic reduction.
- ∇q =Surface tangential velocity

$$\partial_t \eta = \frac{\delta H}{\delta q}, \quad \partial_t q = -\frac{\delta H}{\delta \eta}$$

  for some appropriate class of Hamiltonians.
- Key sub-problem: for given $\zeta(\vec{x}, t), \eta(\vec{x}, t)$ find the associated ∇q using Observer framework





Observers

Luenberger Intro. to Observers (1971); Intro. to Dynamic Systems (1979)

$$y \in \mathbb{R}^n,\ f : \mathbb{R}^n \to \mathbb{R}^n$$

$$\frac{dy}{dt} = f(y)$$

$$\text{Observations } z = B(y)$$

Given z, find y

Observers

Linear model and Linear observations

$$y \in \mathbb{R}^n,\ A : \mathbb{R}^n \to \mathbb{R}^n$$

$$\frac{dy}{dt} = Ay$$

Observations $z = By$ where B is short and fat (p rows and n columns)

Is B good?

Luenberger suggests to look at the $pn \times n$ matrix

$$C = \begin{pmatrix} B \\ BA \\ BA^2 \\ \vdots \\ BA^{n-1} \end{pmatrix}$$

If this matrix is full-rank, then system is observable. We can solve $z^{(i)} = BA^{i-1}y$ for $i = 1, 2, \ldots$ where $z^{(i)}$ is the solution at the $i - th$ time step (essentially).

Observers

Linear model and Linear observations

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$$\frac{dy}{dt} = Ay$$

Observations $z = By$ where B is short and fat (p rows and n columns)

Goal Design a new system for $\tilde{y}$

$$\frac{d\tilde{y}}{dt} = A\tilde{y} - \Lambda(B\tilde{y} - z)$$

such that $\tilde{y} \to y$ as $t \to \infty$ for every initial condition $\tilde{y}(0)$

- $\tilde{y}$ is called the Observer (apologies for the confusing terminology!)
- Λ is the feedback

Observers

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The error $e = \tilde{y} - y$ satisfies

$$\frac{de}{dt} = (A - \Lambda B)e$$

Convergence guaranteed if eigenvalues of $A - \Lambda B$ are in left-half place

Observers

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$$y \in \mathbb{R}^n, \; f : \mathbb{R}^n \to \mathbb{R}^n$$

$$\frac{dy}{dt} = Ay + f(y)$$

Observations $z = By + g(y)$

Goal Design a new system for $\tilde{y}$

$$\frac{d\tilde{y}}{dt} = \tilde{f}(\tilde{y}, z)$$

such that $\tilde{y} \to y$ as $t \to \infty$ for every initial condition $\tilde{y}(0)$

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Observers

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Nudging

$$\tilde{f}(\tilde{y}, z) = A\tilde{y} + f(\tilde{y}) - \Lambda \left[B\tilde{y} + g(\tilde{y}) - z \right]$$

Same idea works for the nonlinear observer equation under suitable assumptions

Shallow-water equations

with rotation

$$\partial_t q = Lq + N(q), \quad q = [u\ v\ \eta]^T$$

$$\begin{aligned}
\partial_t u &= v - \frac{\mathrm{Fr}^2}{\mathrm{Ro}}\eta_x + \mathrm{Ro}\left(v\omega - \frac{1}{2}(u^2+v^2)_x\right), \\
\partial_t v &= -u - \frac{\mathrm{Fr}^2}{\mathrm{Ro}}\eta_y + \mathrm{Ro}\left(-u\omega - \frac{1}{2}(u^2+v^2)_y\right), \\
\partial_t \eta &= -\mathrm{Ro}\left(u_x + v_y + (\eta u)_x + (\eta v)_y\right)
\end{aligned}$$

where $\omega = v_x - u_y$.

- $\mathrm{Fr}^2 = gH/(fL)^2 \qquad \mathrm{Ro} = U/(fL)$
- Can the divergence alone or vorticity alone determine the full state?

Shallow-water equations

with rotation

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$$\partial_t \eta = -\mathrm{Ro}\left(u_x + v_y + (\eta u)_x + (\eta v)_y\right)$$

where $\omega = v_x - u_y$.

- $\mathrm{Fr}^2 = gH/(fL)^2 \qquad \mathrm{Ro} = U/(fL)$
- Can the divergence alone or vorticity alone determine the full state?

Not solving a steady-state or elliptic PDE for the velocity field. This is not like 2D incompressible fluid flow

Shallow-water equations

with rotation

Focus on linear equation

$$\partial_t q = Lq, \quad q = [u\, v\, \eta]^T$$

$$L = \begin{pmatrix} 0 & 1 & -\mathrm{Fr}^2/\mathrm{Ro}\,\partial_x \\ -1 & 0 & -\mathrm{Fr}^2/\mathrm{Ro}\,\partial_y \\ -\mathrm{Ro}\,\partial_x & -\mathrm{Ro}\,\partial_y & 0 \end{pmatrix}$$

Divergence

$$\mathrm{z} \; = u_x + v_y = \begin{pmatrix} \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = Bq$$

Convenient to move into vorticity-divergence coordinates $y = [d\, \omega\, \eta]^T$

where now

$$\partial_t y = Ay := \begin{pmatrix} 0 & 1 & -\mathrm{Fr}^2/\mathrm{Ro}\,\Delta \\ -1 & 0 & 0 \\ -\mathrm{Ro} & 0 & 0 \end{pmatrix} \begin{pmatrix} d \\ \omega \\ \eta \end{pmatrix}$$

Shallow-water equations

with rotation

Focus on linear equation

$$\partial_t y = Ay := \begin{pmatrix} 0 & 1 & -\mathrm{Fr}^2/\mathrm{Ro}\,\Delta \\ -1 & 0 & 0 \\ -\mathrm{Ro} & 0 & 0 \end{pmatrix} \begin{pmatrix} d \\ \omega \\ \eta \end{pmatrix}$$

Divergence measurements

$$\mathrm{z} \; = u_x + v_y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d \\ \omega \\ \eta \end{pmatrix} = Bq$$

$$\text{Fourier transform of } \begin{pmatrix} B \\ B\tilde{L} \\ B\tilde{L}^2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha|k|^2 \\ -1-\alpha|k|^2 & 0 & 0 \end{pmatrix}, \quad \alpha > 0$$

The above matrix is rank deficient (for every wavevector k)

Shallow-water equations

with rotation

Focus on linear equation

$$\partial_t y = Ay := \begin{pmatrix} 0 & 1 & -\mathrm{Fr}^2/\mathrm{Ro}\,\Delta \\ -1 & 0 & 0 \\ -\mathrm{Ro} & 0 & 0 \end{pmatrix} \begin{pmatrix} d \\ \omega \\ \eta \end{pmatrix}$$

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Null space: velocity with div $= 0$ vorticity $= \alpha\Delta\eta$, arbitrary η

Geostrophic flow!

Shallow-water equations

with rotation

What about vorticity?

$$\partial_t q = Lq, \quad q = [u\, v\, \eta]^T$$

$$L = \begin{pmatrix} 0 & 1 & -\mathrm{Fr}^2/\mathrm{Ro}\,\partial_x \\ -1 & 0 & -\mathrm{Fr}^2/\mathrm{Ro}\,\partial_y \\ -\mathrm{Ro}\,\partial_x & -\mathrm{Ro}\,\partial_y & 0 \end{pmatrix}$$

Vorticity

$$\mathrm{z} \;= v_x - u_y = \begin{pmatrix} -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = Bq$$

Now $\begin{pmatrix} B \\ BL \\ BL^2 \end{pmatrix}$ is full-rank $\Rightarrow$ Vorticity is a good observable

$\longrightarrow$ Likewise $z = \eta$ also leads to a full-rank matrix

Shallow-water equations: vorticity observer

with rotation

$$\partial_t \tilde{q} = L\tilde{q} + \Lambda(B\tilde{q} - z), \quad z = \text{ vorticity measurements}$$

We can design Λ so that the error

$$\|e\|_2 \leq e^{-\delta t} \|e_0\|_2$$

for some δ where e_0 is the initial state-estimation error.

- Caveat 1: cannot make δ arbitrarily large. Depends on the physical parameter; better in low Ro.
- Caveat 2: cannot recover the zero mode of η (must assume it is zero).
- If (initial error + true solution) not too large, then

$$\partial_t \tilde{q} = L\tilde{q} + N(\tilde{q}) + \Lambda(B\tilde{q} - z)$$

gives $\tilde{q} \to q \Rightarrow$ full state recovery for the nonlinear problem too!

* technical statements avoided for brevity.

What does viscous dissipation do?

$$\partial_t q = Lq + Dq, \quad q = [u\ v\ \eta]^T$$

$$D = \begin{pmatrix} \mathrm{Ek}\Delta & 0 & 0 \\ 0 & \mathrm{Ek}\Delta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Ek : Ekman number}$$

represents impact of momentum diffusion. Turns out now

$$\begin{pmatrix} B \\ B(L+D) \\ B(L+D)^2 \end{pmatrix}$$

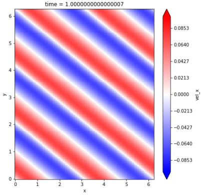
is ill-conditioned with one eigenvalue (for every wavenumber) going to zero as $|k| \to \infty$.

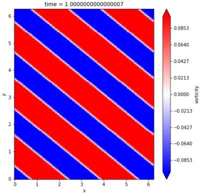
- Cannot guarantee uniform (over wavevectors) error decay-rate
- Error still decreases
- Physical interpretation: energy at smallest lengthscales vanishes fastest
- All due to zero at $D_{3,3}$

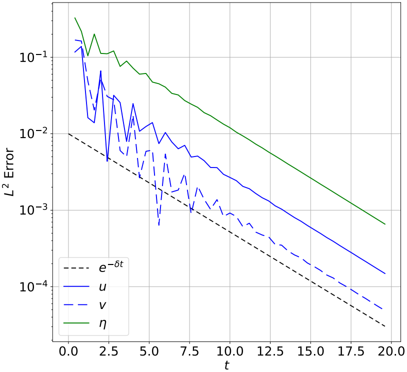
State estimation: vorticity measurements

$$\partial_t \tilde{q} = (L + D)\tilde{q} + N(\tilde{q}) + \Lambda(B\tilde{q} - z), \quad q = [u\ v\ \eta]^T$$

$$\mathrm{Ro} = 1,\ \mathrm{Fr}^2 = 1,\ \mathrm{Ek} = 10^{-4}$$







Decay-rate of error: $\delta \approx 0.3$

State estimation: vorticity measurements

$$\partial_t \tilde{q} = (L + D)\tilde{q} + N(\tilde{q}) + \Lambda(B\tilde{q} - z), \quad q = [u\; v\; \eta]^T$$
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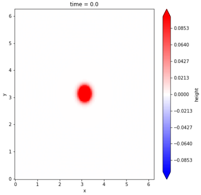

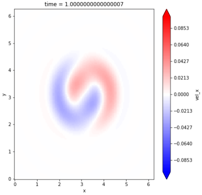

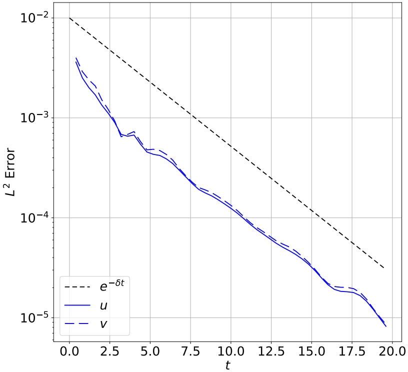

Decay-rate of error: $\delta \approx 0.3$

Summary

- There is a systematic way to confirm if (linear) observations are useful to reconstruct full state.
- ODE→PDE involves technicalities but basic ideas similar
- Linear→Nonlinear doable under some cases

- For SWE:
  - If we knew the geostrophic mode, then the divergence of the fluid velocity is sufficient to determine the full state
  - Vorticity and height field allow full state determination
- Conjecture: PV could determine full state of SWE...future work to confirm.
- What about proxies? Which observations could be used?