Calculations In Infinite Matrix Groups Using Congruence Images

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Joint work with **Alla Detinko** and **Dane Flannery**,

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Setup

$$F \xrightarrow{\psi} \Gamma \xrightarrow{\varphi_m} A_m$$

Let Γ be discrete subgroup of Lie Group,

In particular, SL or SP over $\mathbb{Z}$ and its friends (rings of integers, localizations).

$G \leq \Gamma$ given by finite set of generating matrices.
Can we compute the index $[\Gamma : G]$?

Toolset: Computational Group Theory,

Isomorphism $\psi \colon F \to \Gamma$ with F finitely presented explicitly given. Reduction φ_m modulo integer m.

Algorithmic Tools

Finitely presented groups:

Coset Enumeration

Reidemeister-Schreier

Quotient Algorithms

Knuth-Bendix

Matrix groups over finite fields:

Composition tree

Homomorphisms

Finite group calculations

General Paradigms:

Groups given by small set of generators

Orbit algorithm, Schreier Generators.

Subgroup chain data structures for decomposition

Infinity representable by finite information only

Algorithmic Tools

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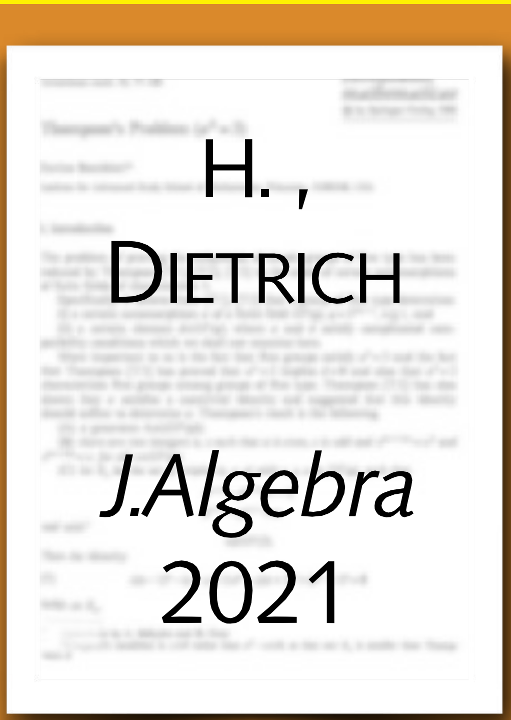
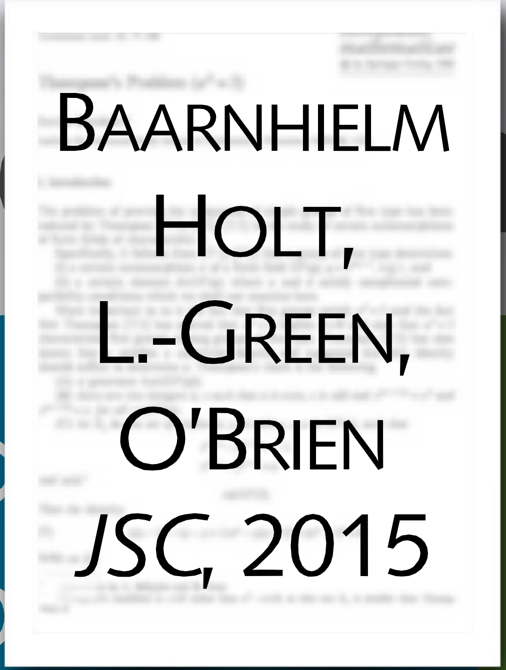
General Paradigm
Groups given by s
Orbit algorithm, S
Subgroup chain d
Infinity representa

Shameless Advertisement:
Nonsolvable (` Hybrid")
quotient algorithm.
(Implemented!)


H. ,
DIETRICH
J.Algebra
2021

Alg ic Tools



BAARNHIELM

HOLT,

L.-GREEN,

O'BRIEN

JSC, 2015

Finitely presente

Coset Enumeratio

Reidemeister-Schreier

Quotient Algorithms

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Matrix groups over finite ~~fields~~ Rings:

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Matrix groups over finite ~~fields~~ Rings:
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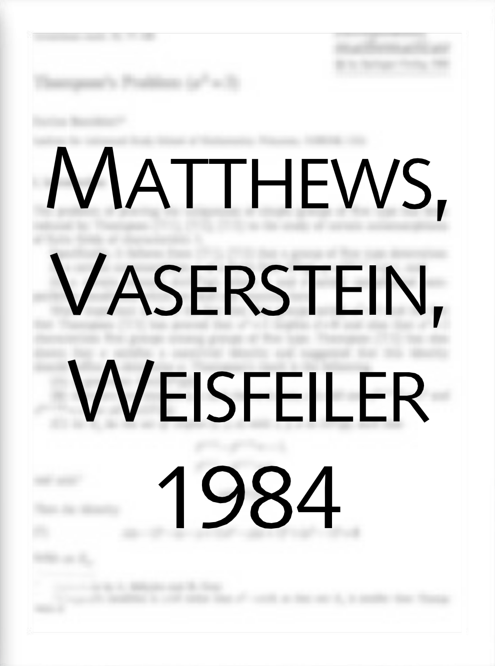
Finite group calculations

Local/Global:
Congruence images and their structure

rithmic Tools

Weigel
1995,96



Matthews,
Vaserstein,
Weisfeiler
1984

groups:

Reidemeister-Schreier
Quotient Algorithms
Knuth-Bendix

Matrix groups over finite ~~fields~~ Rings:

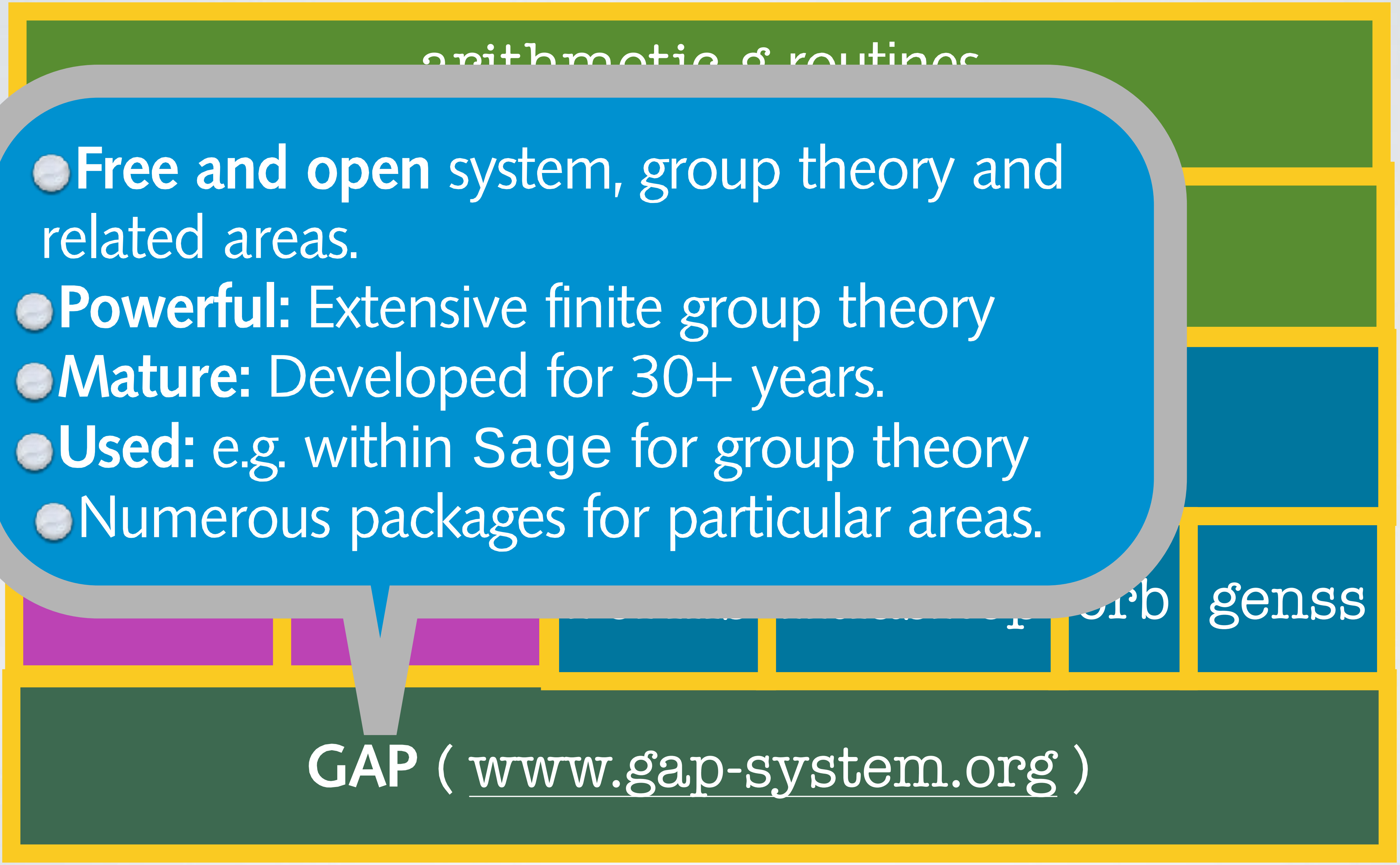
Composition tree
Homomorphisms
Finite group calculations

CSP (e.g. dim ≥ 3): Finite index ⇒ Dense ⇔
$\varphi_p(G) = \varphi_p(\Gamma)$ modulo almost all primes

Local/Global:

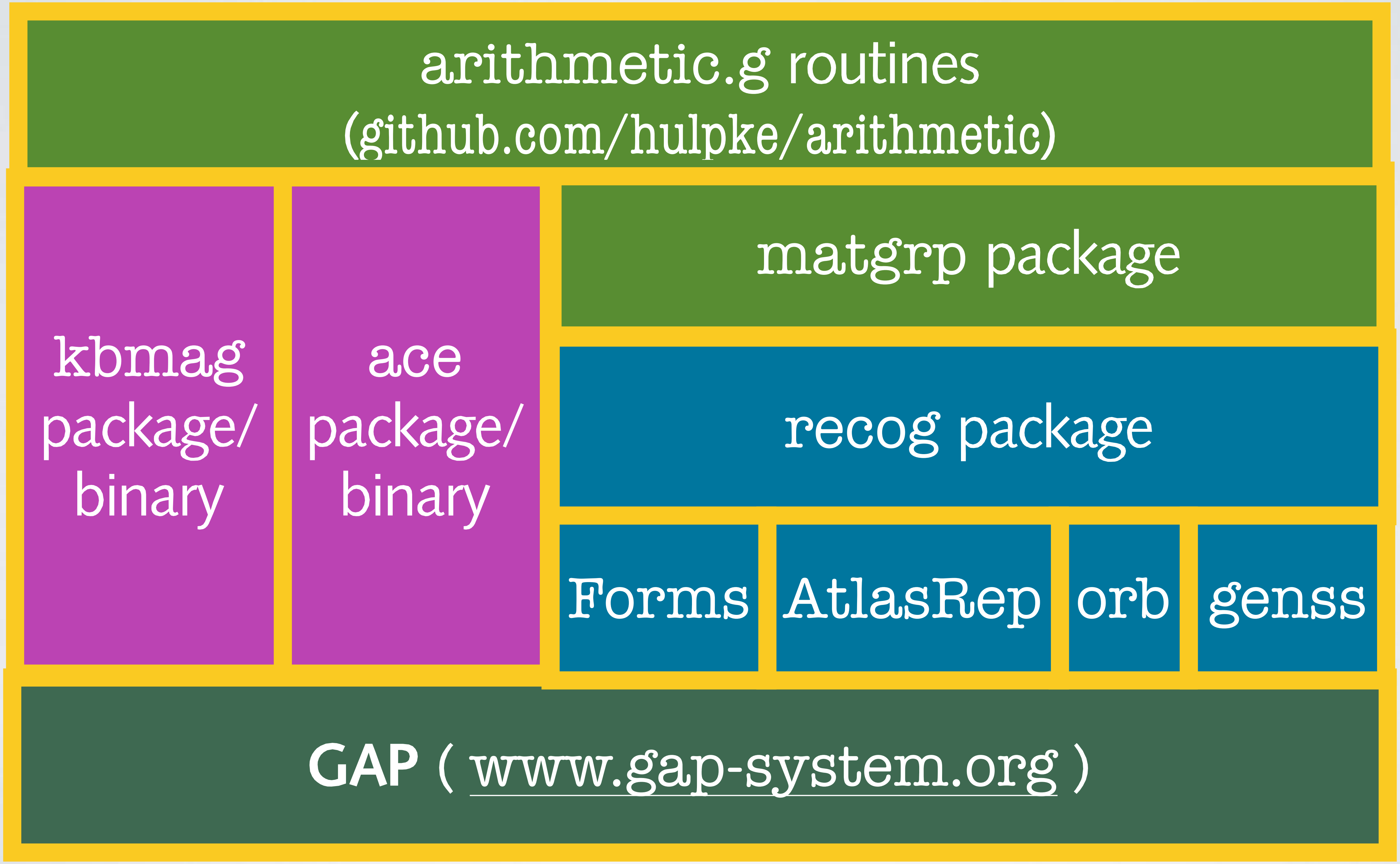
Congruence images and their structure

Software Stack Used



- **Free and open** system, group theory and related areas.
- **Powerful:** Extensive finite group theory
- **Mature:** Developed for 30+ years.
- **Used:** e.g. within Sage for group theory
- Numerous packages for particular areas.

Software Stack Used



arithmetic.g routines
(github.com/hulpke/arithmetic)

kbmag package/ binary

ace package/ binary

matgrp package

recog package

Forms

AtlasRep

orb

genss

GAP (www.gap-system.org)

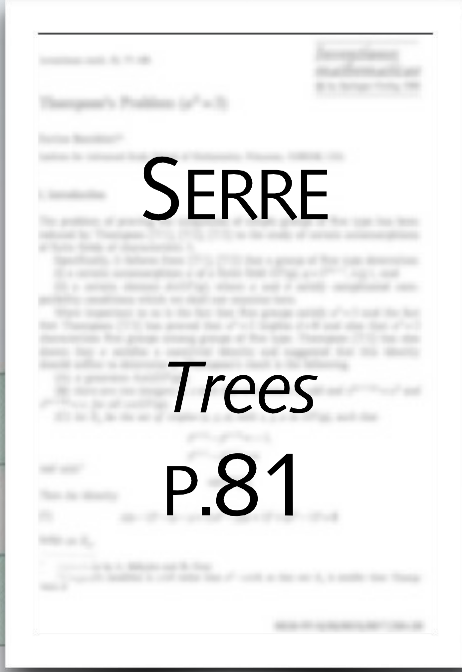
```
            GAP 4.12dev built on 2021-06-07
  GAP       https://www.gap-system.org
            Architecture: x86_64-apple-darwin20.5.0
gap> Read("arithmetic.g");
Loading 'Forms' 1.2.5
Loading  orb 4.8.3 (Methods to enumerate orbits)
Loading  genss 1.6.5 (Generic Schreier-Sims)
Loading  recog 1.3.2dev (Group recognition methods)
Matrix Group Interface routines
Arithmetic group routines, Version 1.11, Alexander Hulpke
gap> psi:=SLNZFP(3);    # also SPNZFP
[ t12, t13, t21, t23, t31, t32 ] ->
[ [ [ 1, 1, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],[…] ]
gap> F:=Source(psi);
<fp group on [ t12, t13, t21, t23, t31, t32 ]>
gap> Range(psi);
<matrix group with 6 generators>
gap> Image(psi,PseudoRandom(F:radius:=100));
[[1203,-1886,1904],[1381,-2181,2192],[-2498,3935,-3961]]
gap> PseudoRandom(Range(psi)); #forced short to 20 minutes
[ [ <integer 144...984 (161571960 digits)>,[…] ]
```

Constructing $SL_2(\mathbb{Z}[1/7])$

Following SERRE (choose $p = 7$).

Take $SL_2(\mathbb{Z}) = \langle a, b \mid a^4, a^2 = (ab)^3 \rangle$

```
gap> f:=FreeGroup("a","b");;
gap> g:=f/ParseRelators(f,"a4,a2=(ab)3");
<fp group on the generators [ a, b ]>
```



SERRE

Trees

P.81

Two matrix representations: $\alpha\colon a, b \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and

$\beta\colon a, b \mapsto \begin{pmatrix} 0 & 1/7 \\ -7 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1/7 \\ 0 & 1 \end{pmatrix}$.

```
gap> mats:=[[[0,1],[-1,0]],[[1,0],[1,1]]];;
gap> alpha:=GroupHomom...ByImages(g,Group(mats),GeneratorsOfGroup(g),mats);
[ a, b ] -> [[[0,1], [-1,0] ], [ [ 1, 0 ], [ 1, 1 ] ] ]
gap> new:=[[[0,1/7], [-7,0] ], [ [1,-1/7], [0,1] ] ];;
gap> beta:=GroupHomomorphismByImages(g,Group(new),GeneratorsOfGroup(g),new);
[ a, b ] -> [[[0,1/7],[-7,0]], [[1,-1/7],[0,1]]]
```

Nonprincipal congruence subgroup $\begin{pmatrix} * & * \\ 7* & * \end{pmatrix}$ modulo 7:

```
gap> one:=One(GF(7));;mare:=mats*one;;
gap> red:=GroupHomomorphismByImages(g,Group(mare),GeneratorsOfGroup(g),mare);
[ a, b ] -> [ [ [0*Z(7),Z(7)^0 ],[Z(7)^3,0*Z(7)]],[…]  ]
gap> sub:=Stabilizer(Range(red),[0,1]*one,OnLines);
Group([ [ Z(7)^2, Z(7)^3 ], [ 0*Z(7), Z(7)^4 ] ], […])
gap> sub:=PreImage(red,sub);
Group(<fp, no generators known>)
gap> gens:=GeneratorsOfGroup(sub);
[a^2,a*b/a,b^2/a^2/b^2,b^3/a/b^2,(a^-2)^(b^2),b^-3/a*b^2]
gap> gens:=gens{[1,2,4,6]};;
gap> suma:=List(gens,x->Image(beta,x));
[[[-1,0],[0,-1]],[[1,0],[7,1]],[[-3,-1],[7,2]],[…]]
```

Factor generators (done on the side — see later)

```
gap> wrd:=[a^2,b^7,(a^-1*b)^(b^2),(a^-1*b^-1)^(b^-2)];;
gap> List(wrd,x->ImagesRepresentative(alpha,x))=suma;
true # verified words
```

Redundancy amongst generators:

```
gap> suma:=List(gens,x->ImagesRepresentative(beta,x));
[ [ [ -1, 0 ], [ 0, -1 ] ], [ [ 1, 0 ], [ 7, 1 ] ],
  [ [ -1, 0 ], [ 0, -1 ] ], [ [ -3, -1 ], [ 7, 2 ] ],
  [ [ -1, 0 ], [ 0, -1 ] ], [ [ 3, -1 ], [ 7, -2 ] ] ]
```

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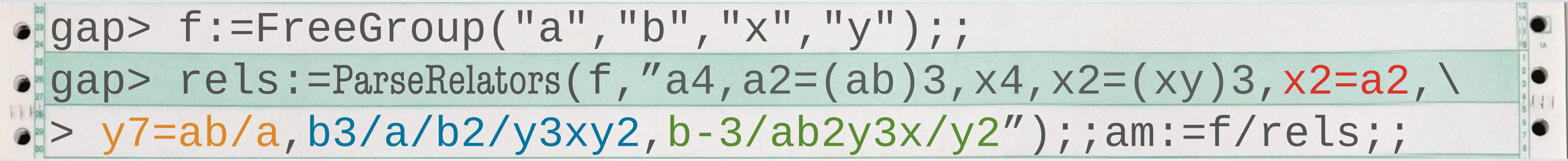
$$a^2, ab/a, b^3/a/b^2, b^{-3}a^{-1}b^2$$

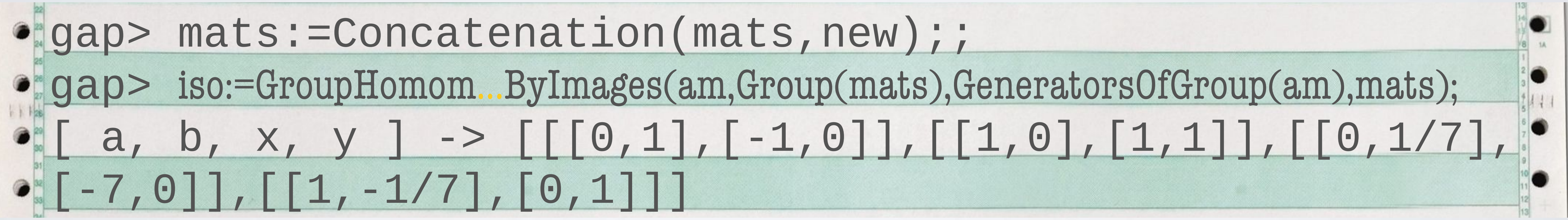
$$a^2, b^7, (a^{-1}b)^{b^2}, (a^{-1}b^{-1})^{b^{-2}}$$

Amalgamate the subgroups through generators



```
gap> f:=FreeGroup("a","b","x","y");;
gap> rels:=ParseRelators(f,”a4,a2=(ab)3,x4,x2=(xy)3,x2=a2,\
> y7=ab/a,b3/a/b2/y3xy2,b-3/ab2y3x/y2”);;am:=f/rels;;
```

Combine the matrix representations:



```
gap> mats:=Concatenation(mats,new);;
gap> iso:=GroupHomom…ByImages(am,Group(mats),GeneratorsOfGroup(am),mats);
[ a, b, x, y ] -> [[[0,1],[-1,0]],[[1,0],[1,1]],[[0,1/7],
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Since we did not add an NC, this verified that the map is a homomorphism.

$$a^2, ab/a, b^3/a/b^2, b^{-3}a^{-1}b^2$$

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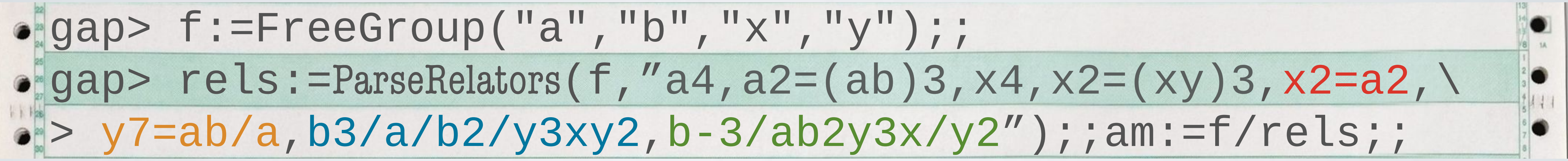

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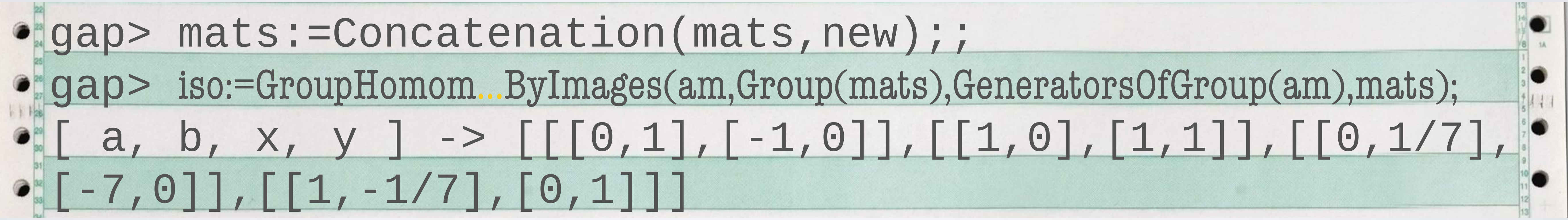

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String goes over more than one line

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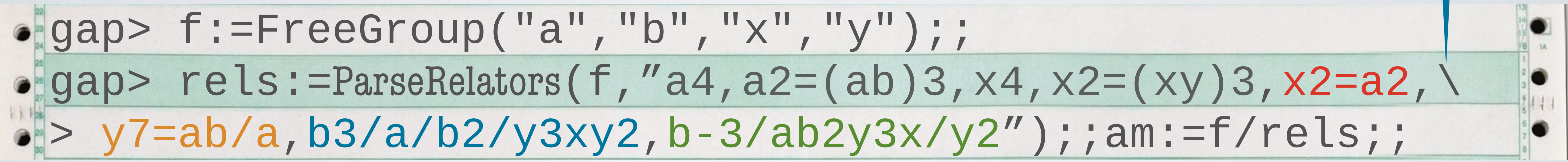

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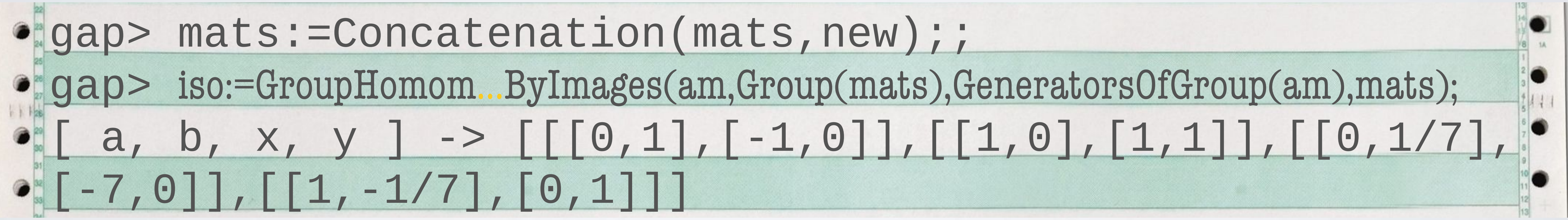

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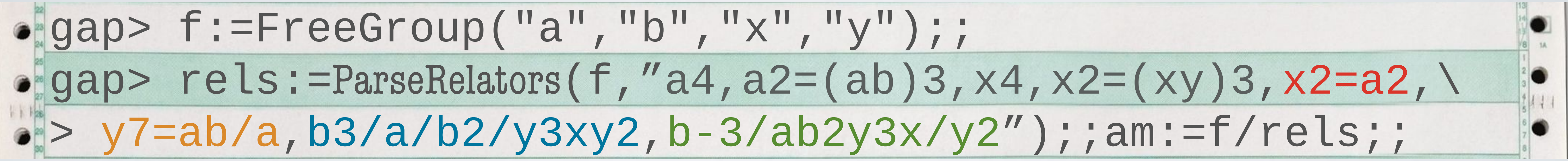

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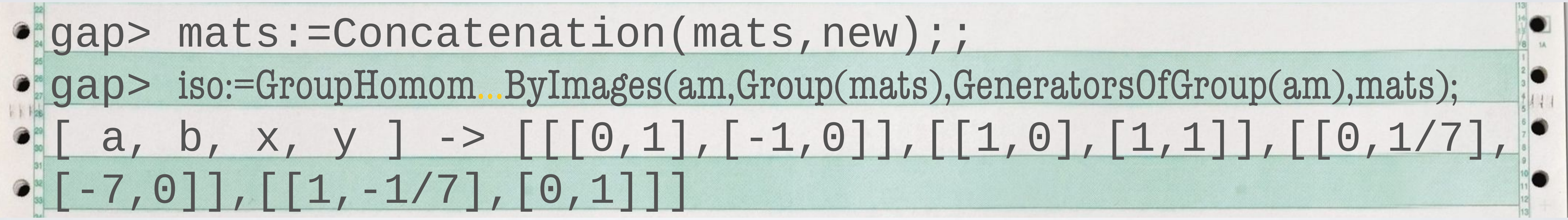

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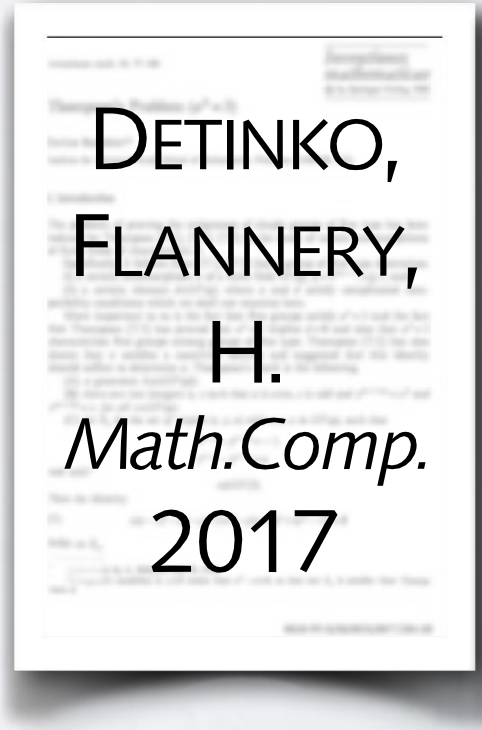
Index In Congruence Images

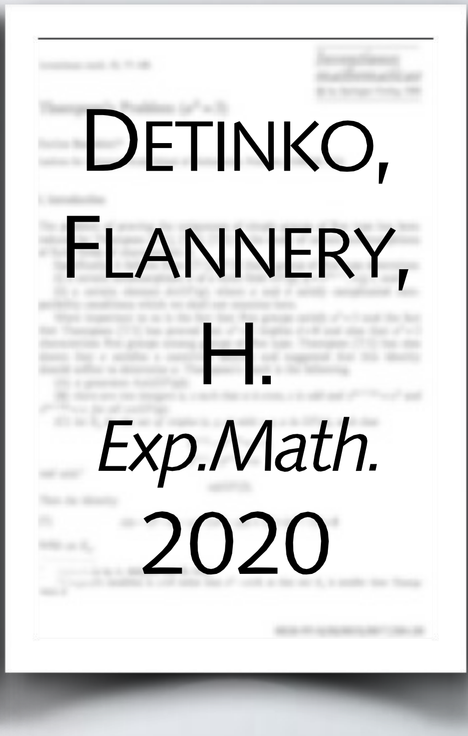
Congruence images are approximations. There is a largest principal congruence subgroup $\Gamma_l \leq \Gamma$ of level l, such that $\langle \Gamma_l, G \rangle \leq \langle \Gamma_k, G \rangle$ for any k. Find this level l, and index $[\Gamma{:}\langle \Gamma_l, G \rangle] = [\varphi_l(\Gamma){:}\varphi_l(G)]$.

Two part problem:

- Find primes dividing l
- Find l from primes.



DETINKO,
FLANNERY,
H.
Math.Comp.
2017



DETINKO,
FLANNERY,
H.
Exp.Math.
2020

Structure Of Congruence Image

Image $A_l = \varphi_l(\Gamma)$ is classical group (SL/Sp) over $Z_l = \mathbb{Z}/l\mathbb{Z}$.

A_l subdirect product of A_{p^e} for prime powers p^e in l.

A_{p^e} is iterated extension of adjoint module (Lie sense) by group A_p over prime field.

Generically, these extensions A_{p^e} are not supplemented, thus if $[A_{m\cdot p} : \varphi_{m\cdot p}(G)] = [A_m : \varphi_m(G)]$, no further factor of p in l. Thus find maximal prime powers.

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A_l subdire e powers

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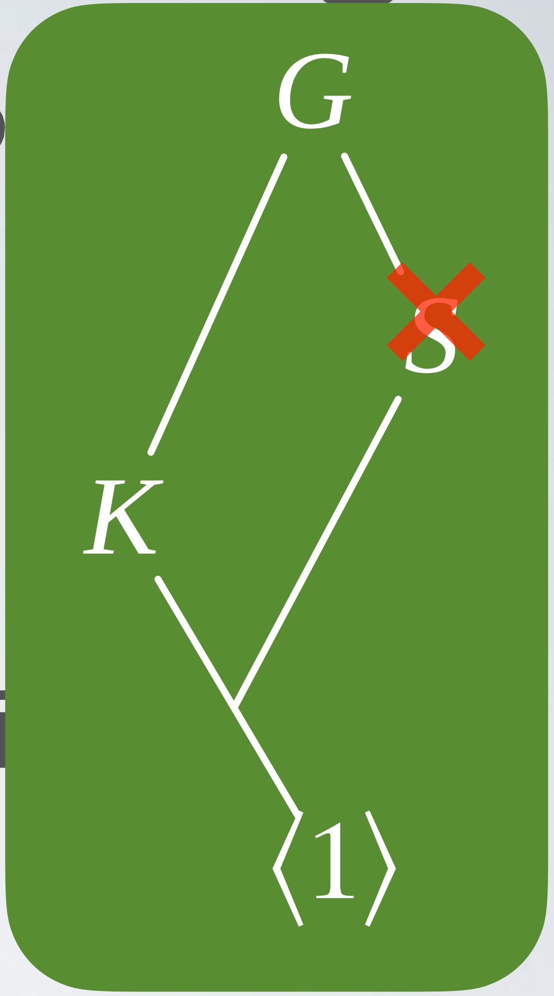
by group A_p

Exceptions for small primes (2,3) and small dimensions (≤ 3), resolved by starting with prime power p^2.



Generically, these extensions A_{p^e} are not supplemented, thus if $[A_{m \cdot p} : \varphi_{m \cdot p}(G)] = [A_m : \varphi_m(G)]$, no further factor of p in l. Thus find maximal prime powers.

Finding Primes

Same argument, primes dividing l are (disclaimer for $p = 2,3$) those p, such that $\varphi_p(G) \neq A_p$.

Criteria: a) Given transvection stays transvection

b) Adjoint repres. of G mod p abs. red.

Theorem: An (irreducible) representation $\alpha : G \to \mathrm{GL}_k(\mathbb{Z})$ is absolutely irreducible modulo every prime that does not divide the discriminant of the $\mathbb{Z}$-lattice spanned by $\alpha(G)$.

Method: Approximate this lattice until full rank.

Problem: Dimension k^2. (Too) costly for adj. rep.

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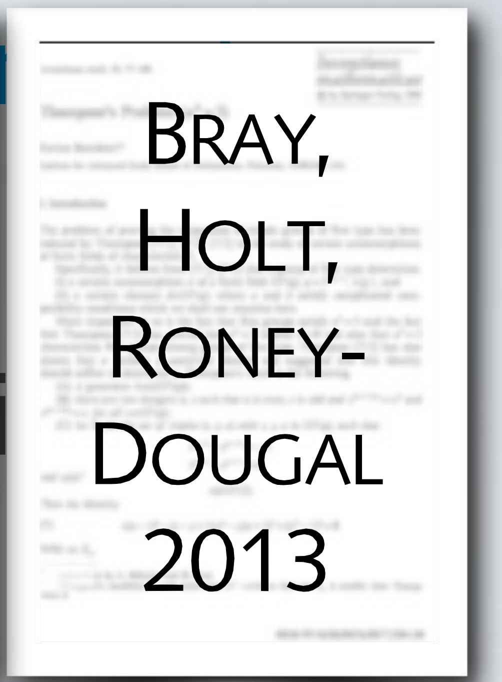

To find in practice:

- $\varphi_p(G)$ is in maximal subgroup of A_p.
- Aschbacher classes for maximal subgroups — stabilizers of geometric structures, order bounds.
- Translate stabilization to calculation on entries of element(s) in G. Test prime divisors of value.

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- Aschbacher cla es for maximal subgroups — stabilizers of geo etric structures, order bounds.
- Translate stabilization to calculation on entries of element(s) in G. Test prime divisors of value.

Any such prime is candidate only — test whether $\varphi_p(G) \neq A_p$

For Example

Convention: $a \in A_p$. $g \in G$ (pseudo-)random.

Order Bound: $|a| \leq b$ implies exponent $a^e = 1$. For $|g| = \infty$, consider lcm of entries of $g^e - 1$.

Tensor Product: If $a = a_1 \otimes a_2$, then coefficients of $\chi_a(x) = \prod_{i,j} (x - \alpha_i \beta_j)$ satisfy syzygy condition.

Reducible (not absolutely)**:** Run irreducibility test (MeatAxe) generically over $\mathbb{Z}$. Collect primes modulo which reduction fails.

+ further cases …

For Example

Hard (?) commutative algebra problem

Convention: $a \in A_p$. $g \in G$ (pseudo-)random.

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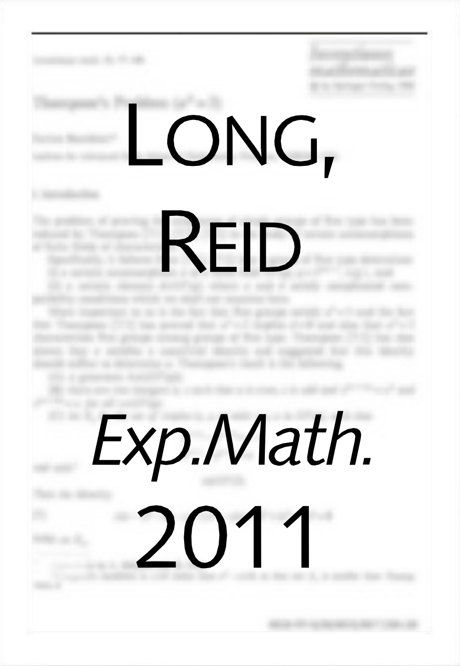
+ further cases …


GAP RD
STOP

```
gap> g:=BetaT(27);
<matrix group with 3 generators>
gap> pr:=PrimesNonSurjective(g);
#I  Absolute irreducibility - found: [ 17, 23,
#I  Element Order - found: [ 3 ] new:[  ]
#I  Similarity - found: [ 3, 17, 23, 151 ] new:
#I  Monomial - found: [ 3 ] new:[  ]
#I  Solvable - found: [ 3 ] new:[  ]
[ 3, 17, 23, 151 ]
gap> MaxPCSPrimes(g,pr);
#I  Try 3^1, Try 3^2, […] Try 3^10
#I  Try 17^1, Try 17^2
#I  Try 23^1, Try 23^2
#I  Try 151^1, Try 151^2
#I  Try 1162104003 3
#I  Try 1162104003 17
#I  Try 1162104003 23
#I  Try 1162104003 151
#I  Try extra 2
#I  Level = 1162104003 =3^9*17*23*151
[ 1162104003, 9854089154390043625196124257904 ]
```



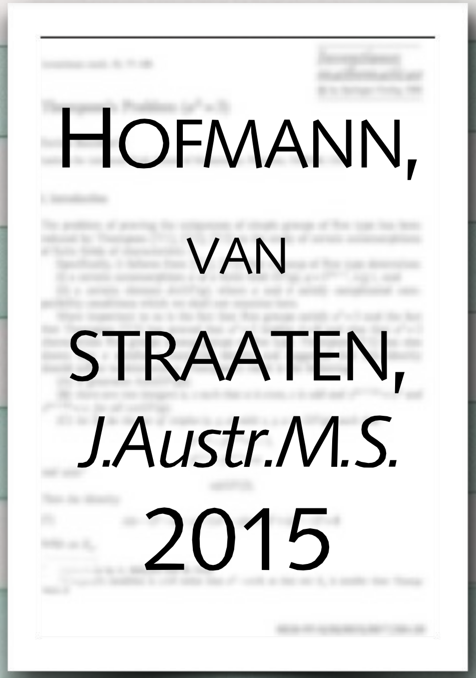
LONG, REID

Exp.Math. 2011

Examples chosen to illustrate, even if results known.

```
gap> g:=BetaT(27);
<matrix group with 3 generators>
gap> pr:=PrimesNonSurjective(g);
#I  Absolute irreducibility - found: [ 17, 23, 151 ]
#I  Element Order - found: [ 3 ] new:[  ]
#I  Similarity - found: [ 3, 17, 23, 151 ] new:[  ]
#I  Monomial - found: [ 3 ] new:[  ]
#I  Solvable - found: [ 3 ] new:[  ]
[ 3, 17, 23, 151 ]
gap> MaxPCSPrimes(g,pr);
#I  Try 3^1, Try 3^2, […] Try 3^10
#I  Try 17^1, Try 17^2
#I  Try 23^1, Try 23^2
#I  Try 151^1, Try 151^2
#I  Try 1162104003 3
#I  Try 1162104003 17
#I  Try 1162104003 23
#I  Try 1162104003 151
#I  Try extra 2
#I  Level = 1162104003 =3^9*17*23*151
[ 1162104003, 9854089154390043625196124257904 ]
```

```
gap> g:=HofmannStraatenExample(3,4);  # Calabi-Yau Threefolds
<matrix group with 2 generators>
gap> RankMat(g.2-g.2^0);
1
gap> PrimesForDense(g,g.2,SP); # use transvection
[ 2, 3 ]
gap> MaxPCSPrimes(g,pr,SP);
#I  Try 2^1, 2^2, 2^3
#I  Try 3^1, 3^2, 3^3
#I  Try 36 2
#I  Try 36 3
#I  Level = 36 = 2^2*3^2
#I  Index = 3110400 =2^9*3^5*5^2
[ 36, 3110400 ]
```



HOFMANN,
VAN
STRAATEN,
J.Austr.M.S.
2015

```
ga                           mple(3,4);  # Calabi-Yau Threefolds
<m                           rators>
gap>                         ,
1
gap> Primes    Dense(g,g.2,SP); # use transvection
[ 2, 3 ]
gap> MaxPCS   imes(g,pr,SP);
#I  Try 2^1   2^2, 2^3
#I  Try 3^1   3^2, 3^3
#I  Try 36  2
#I  Try 36  3
#I  Level  = 36 = 2^2*3^2
#I  Index  = 3110400 =2^9*3^5*5^2
[ 36, 3110400 ]
```

This
ONLY is the index of
the smallest congruence
subgroup above, not
of G

Verifying The Index

$$F \xrightarrow{\psi} \Gamma \xrightarrow{\varphi_m} A_m$$

Unless we know the subgroup to be a congruence subgroup , the maximal index in a congruence image is only the index of the smallest congruence subgroup above, and will never show infinite index.

Thus consider problem in finitely presented pre-image under ψ :

- Pre-image of matrix is word in generators
- Hope fp-group methods finish

Finding Words

$F \xrightarrow{\psi} \Gamma \xrightarrow{\varphi_m} A_m$

To find pre-image under ψ, need to express matrix in Γ as product of elementary generators:

Try possible words of short length: Limited scope

Word in A_m: Deterministic, but might need huge m.

Hermite NF: Only for SL, Long Words: $\begin{pmatrix} 1 & 34 \\ 1000 & 21 \end{pmatrix}$, We care not about Normal Form but transform matrix.

Ultimately Use: Norm-based reduction (row+column) heuristics. Produces reasonable words for SL and Sp.

Finding Words

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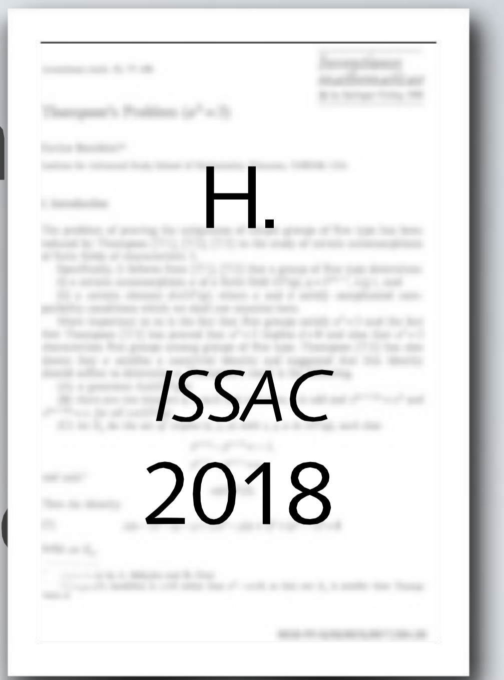
Γ as product of elementary generators:



H.

ISSAC

2018

Try possible words of short length: Limited sc

Word in A_m: Deterministic, but might need huge m.

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Ultimately Use: Norm-based reduction (row+column) heuristics. Produces reasonable words for SL and Sp.

```
gap> g:=HofmannStraatenExample(3,4);;  # as before — 36, 3110400
gap> psi:=SPNZFP(4);;F:=Source(psi);
<fp group on the generators [ Y1, Y2, U1, U2, Z1 ]>
gap> w:=List(GeneratorsOfGroup(g),
>     x->PreImagesRepresentative(psi,x));
[ U1*(U2^-1*U1*U2^-1)^2*Y1^-1*Y2^-1*U2^-1*Y2^-1*Z1*U2*Y2, Y2^-1 ]
gap> S:=Subgroup(F,w);;
gap> Index(F,S);
3110400    # have proven the finite index, 5 minutes

gap> g:=HofmannStraatenExample(1,4);;  # known infinite index
gap> pr:=PrimesForDense(g,g.2,SP);
[ 2 ]
gap> MaxPCSPrimes(g,pr,SP);
[ 4, 160 ]  # level 4, index 160
gap> w:=List(GeneratorsOfGroup(g),x->PreImagesRepresentative(psi,x));
[ U2^-3*U1*U2^-1*Y1^-1*Y2^-1*U2^-1*Y2^-1*Z1*U2*Y2, Y2^-1]
gap> Index(F,S);
Error, coset enumeration defined more than 4096000 cosets
```

```
gap> g:=HofmannStraatenExample(3,4);;  # as before — 36, 3110400
gap> psi:=SPNZFR(4);;F:=Source(psi);
<fp gr                    ators [ Y1, Y2, U1, U2, Z1 ]>
gap                       fGroup(g),
>                         tative(psi,x));
[ U                       -1*U2^-1*Y2^-1*Z1*U2*Y2, Y2^-1 ]
gap>
gap> Ind
3110400    # h   e proven the finite index, 5 minutes

gap> g:=Hofma  StraatenExample(1,4);;  # known infinite index
gap> pr:=Prim  ForDense(g,g.2,SP);
[ 2 ]
gap> MaxPCSPri es(g,pr,SP);
[ 4, 160 ]   #  evel 4, index 160
gap> w:=List(Genera orsOfGroup(g),x->PreImagesRepresentative(psi,x));
[ U2^-3*U1*U2^ 1*Y1^-1*Y2^-1*U2^-1*Y2^-1*Z1*U2*Y2, Y2^-1]
gap> Index(F,S);
Error, coset enumeration defined more than 4096000 cosets
```

This is **NOT** a proof that the index is infinite, just failure so far.

```
gap> g:=HofmannStraatenExample(4,4);;  # known finite index
gap> pr:=PrimesForDense(g,g.2,SP);
[2]
gap> MaxPCSPrimes(g,pr,SP);
[ 64, 47185920 ]
gap> w:=List(GeneratorsOfGroup(g),x->PreImagesRepresentative(psi,x));
[ U1^2*(U2^-1*U1*U2^-1)^2*Y1^-1*Y2^-1*U2^-1*Y2^-1*Z1*U2*Y2, Y2^-1 ]
gap> S:=Subgroup(F,w);;
gap> Index(F,S);
Error, coset enumeration defined more than 4096000 cosets
brk> return;
[…]    coset enumeration defined more than 16384000 cosets
gap> LoadPackage("ace");
Loading    ACE (C code by  George Havas, Colin Ramsay)
gap> TCENUM:=ACETCENUM;; #change default TC
```

Could use external binary for more powerful coset enumerator. Note that it requires to set options to use more than basic memory:

`:hard, Wo:=10^8`

```
gap> g:=HofmannStraatenE
gap> pr:=PrimesForDense(
[2]
gap> MaxPCSPrimes(g,pr,
[ 64, 47185920 ]
gap> w:=List(GeneratorsOfGroup(       >PreImagesRepresentative(psi,x));
[ U1^2*(U2^-1*U1*U2^-1)^2*Y1^       Y2^-1*U2^-1*Y2^-1*Z1*U2*Y2, Y2^-1 ]
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```

Use Intermediate Subgroup

Coset enumeration must store row for every coset, and intermediately might need more than index.

5 generators, index 47185920 needs at least 30 GB.

Instead find a subgroup $S \lneqq T \lneqq \Gamma$. Rewrite presentation to T and then (in new presentation) calculate $[T:S] = \dfrac{[\Gamma:S]}{[\Gamma:T]}$.

Price: Generator number of T grows with index, but subsequent Tietze transformations tend to reduce.

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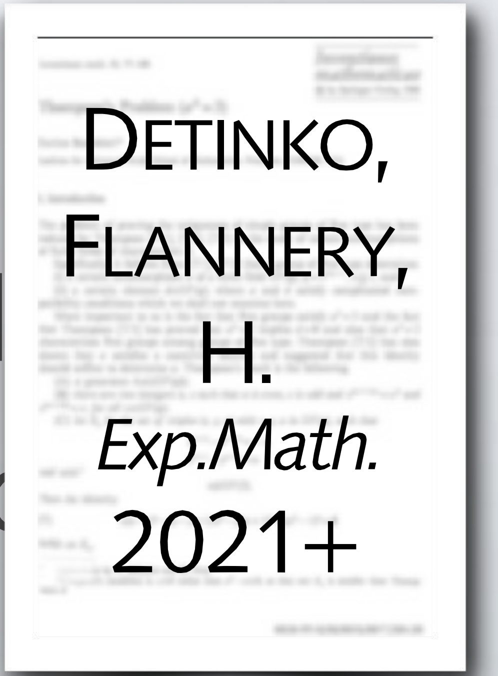

Detinko, Flannery, H.
Exp.Math.
2021+

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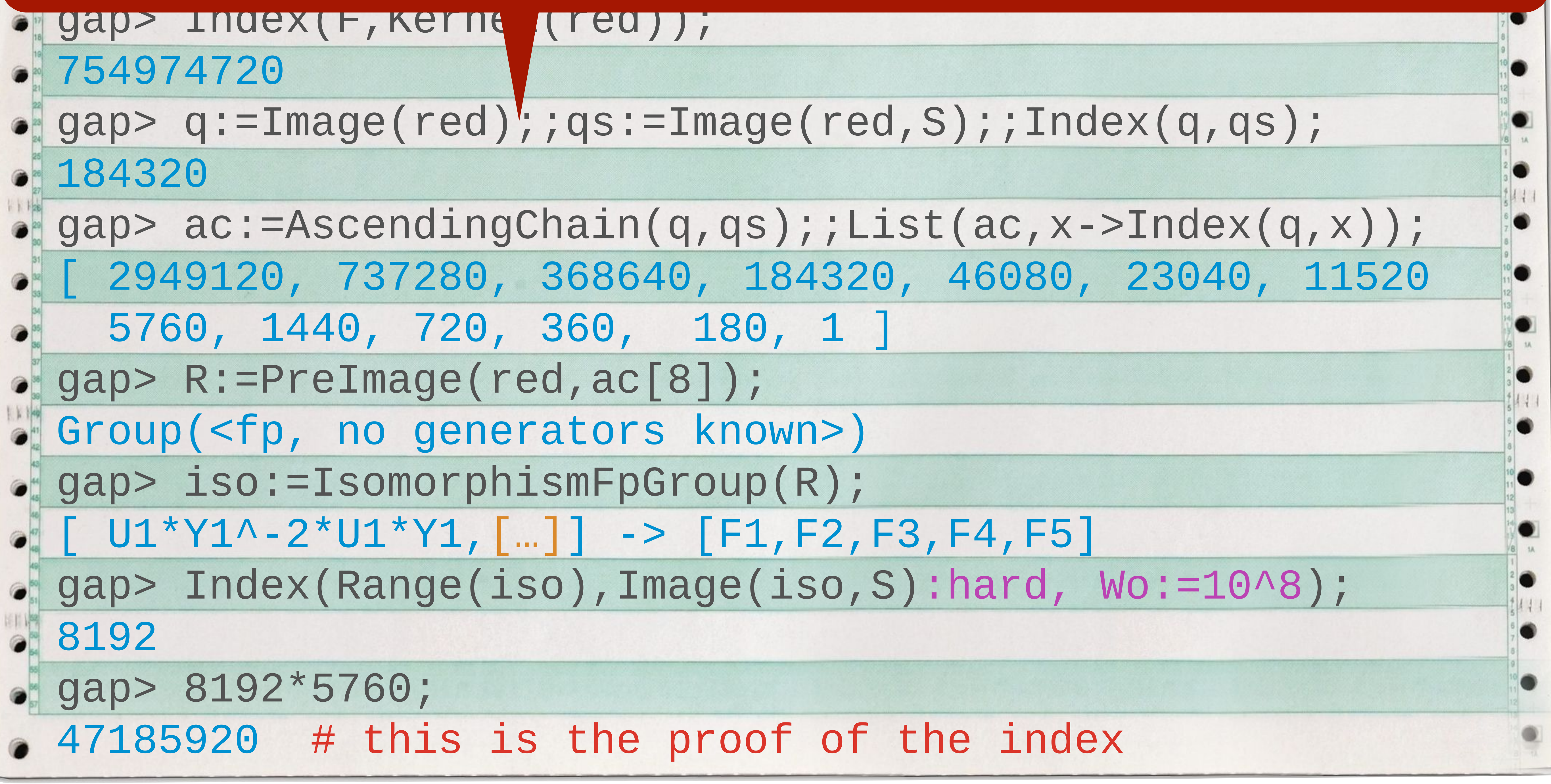
Price: Generator number of T grows with index, but subsequent Tietze transformations tend to reduce.

Change presentation in sequence of simple steps, trying to delete generators and reduce size.
Heuristic driven.

```
gap> ring:=Integers mod 8;;
gap> red:=MappingGeneratorsImages(psi)[2]*One(ring);
[ [ [ ZmodnZObj( 1, 4 ),[…]]
gap> red:=GroupHomomorphismByImagesNC(F,Group(red),
> MappingGeneratorsImages(psi)[1],red);;
gap> red:=red*IsomorphismPermGroup(Range(red));
gap> Index(F,Kernel(red));
754974720
gap> q:=Image(red);;qs:=Image(red,S);;Index(q,qs);
184320
gap> ac:=AscendingChain(q,qs);;List(ac,x->Index(q,x));
[ 2949120, 737280, 368640, 184320, 46080, 23040, 11520
  5760, 1440, 720, 360,  180, 1 ]
gap> R:=PreImage(red,ac[8]);
Group(<fp, no generators known>)
gap> iso:=IsomorphismFpGroup(R);
[ U1*Y1^-2*U1*Y1,[…]] -> [F1,F2,F3,F4,F5]
gap> Index(Range(iso),Image(iso,S):hard, Wo:=10^8);
8192
gap> 8192*5760;
47185920  # this is the proof of the index
```

Image is finite group, in which we can do all kinds of calculations. E.g. `IntermediateSubgroups(q,qs);` finds the 2301 subgroups (many b/c of the $p^{k-1} \bmod p^k$ layers) above $\langle \Gamma_8, S \rangle$. Use modulo level (64) to find all finite index subgroups above S.



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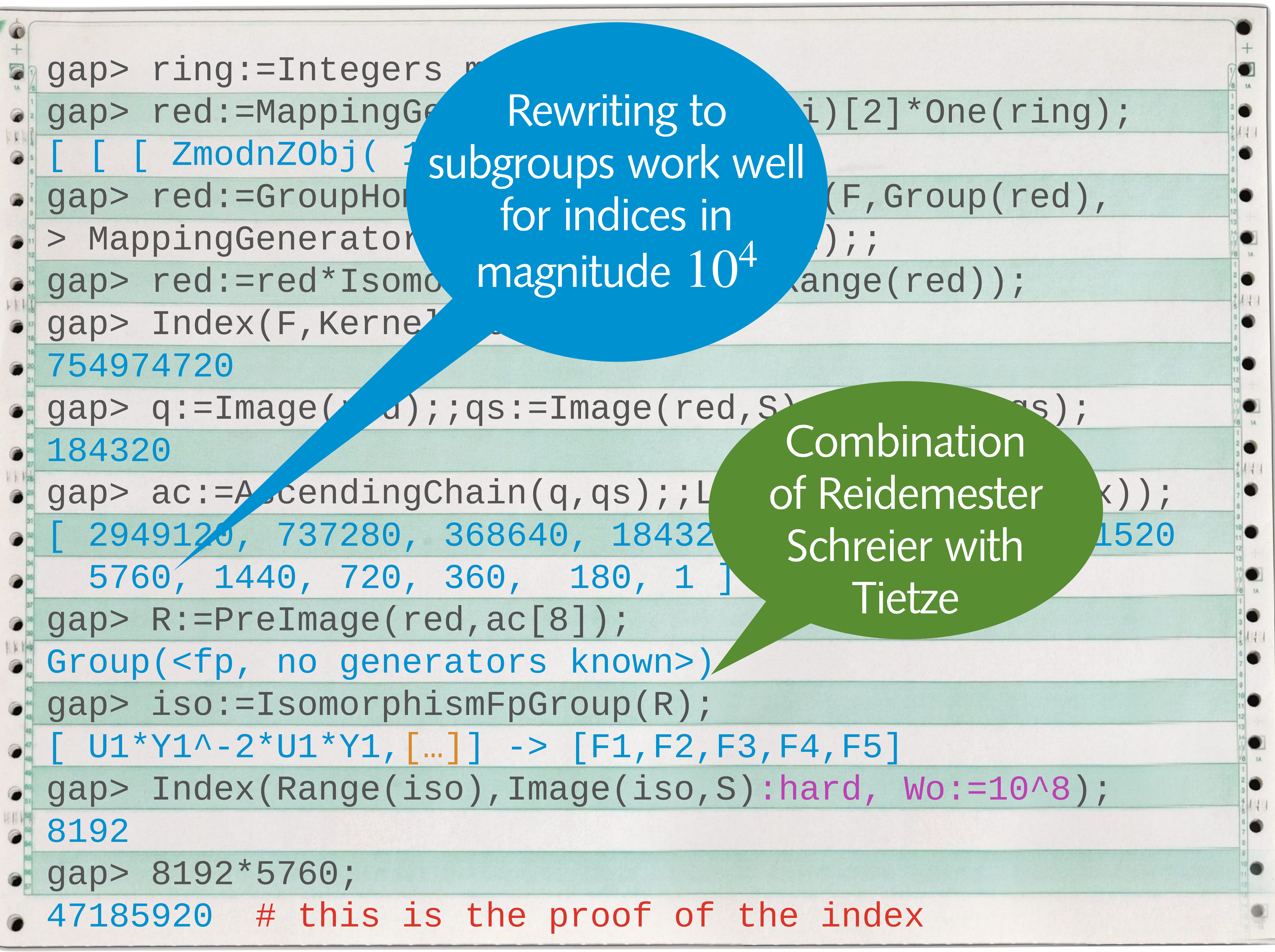
Combination of Reidemester Schreier with Tietze



```
gap> ring:=Integers m
gap> red:=MappingGe                            i)[2]*One(ring);
[ [ [ ZmodnZObj( 1
gap> red:=GroupHo                              (F,Group(red),
> MappingGenerator                             ));;
gap> red:=red*Isomo                            ange(red));
gap> Index(F,Kernel
754974720
gap> q:=Image(    d);;qs:=Image(red,S)              gs);
184320
gap> ac:=AscendingChain(q,qs);;L                    x));
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```

Rewriting to subgroups work well for indices in magnitude 10^4

Combination of Reidemester Schreier with Tietze

Proving Infinite Index

Some attempts of proving infinite index:

- Find an automatic structure that proves infinite index. (Requires suitable embedding.)
  Normal forms for subgroup elements might show that there are infinitely many cosets.
- Show that one can find an infinite index in a suitable quotient image of normal subgroup. (Applicable, if CSP does not hold.)

Proving

The
impossibility results for
algorithms for FPGroups
mean it can never be more
than attempts.

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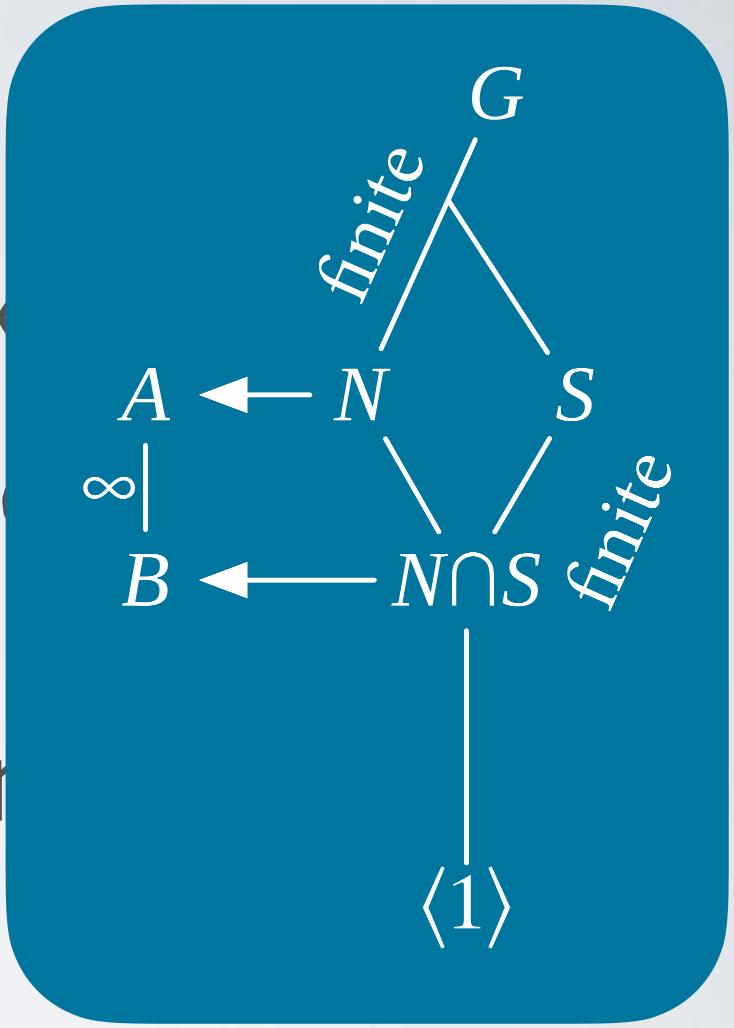

- Show that one can find an infinite index in a suitable quotient image of normal subgroup. (Applicable, if CSP does not hold.)

$$\left\langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\rangle \leq \mathrm{SL}_2(\mathbb{Z})$$

```
gap> psi:=SLNZFP(2);;F:=Source(psi);
gap> g:=Group([[1,3],[0,1]],[[1,0],[3,1]]);
gap> w:=List(GeneratorsOfGroup(g),x->PreImagesRepresentative(psi,x));
[ T^3, S*T^-2*S*T*S^-1*T ]
gap> LoadPackage("kbmag");
Loading  kbmag 1.5.9 by Derek Holt
gap> R:=KBMAGRewritingSystem(F);;
gap> S:=SubgroupOfKBMAGRewritingSystem(R,w);
gap> A:=AutomaticStructureOnCosetsWithSubgroupPresentation(R, S);
true
gap> Index(R,S);
infinity # proof!
```

$$\left\langle \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix} \right\rangle \leq \mathrm{SL}_2(\mathbb{Z})$$

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true
gap> Index(R,S);
infinity # proof!
```

Might `fail`/`false`

Q: Can this work in dimension >2 ?

Example Second Strategy

$R = \mathbb{Z}(\zeta), \zeta^2 + \zeta = -1,$

$\Gamma = GL_2(R) = \langle t, u, j, l, a, w \rangle$ with

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, u = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, j = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, l = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{pmatrix}, a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, w = \begin{pmatrix} -\zeta & 0 \\ 0 & 1 \end{pmatrix}$$

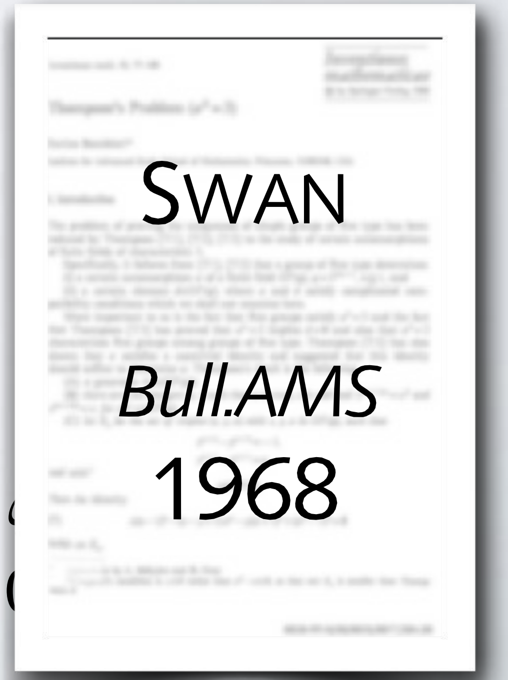
with relations $tu = ut, j^2, tj = jt, uj = ju, lj = jl$, $aj = ja$,$l^3, l^{-1}tl = t^{-1}u^{-1}, l^{-1}ul = t, a^2 = j,$ $(al)^2 = j, (ta)^3 = j, (ual)^3 = j, wj = jw, w^6,$ $wtw^{-1} = u^{-1}, wuw^{-1} = tu, waw^{-1} = jl^2a, wl = lw$

MARGOLIS/BÄCHLE: Is $[\Gamma : G]$ finite for

$G = \langle m_1, m_2, m_3, m_i, m_j, m_t \rangle$ with

$$m_1 = \begin{pmatrix} 97\zeta^2 & -112\zeta - 56\zeta^2 \\ 112\zeta + 56\zeta^2 & 97\zeta^2 \end{pmatrix}, m_2 = \begin{pmatrix} 56\zeta + 41\zeta^2 & 56\zeta + 112\zeta^2 \\ 56\zeta + 112\zeta^2 & -56\zeta + 153\zeta^2 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 56\zeta + 209\zeta^2 & -56\zeta + 56\zeta^2 \\ -56\zeta + 56\zeta^2 & -56\zeta - 15\zeta^2 \end{pmatrix}, m_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, m_j = \begin{pmatrix} \zeta & \zeta^2 \\ \zeta^2 & -\zeta \end{pmatrix}, m_t =$$



SWAN

Bull.AMS

1968

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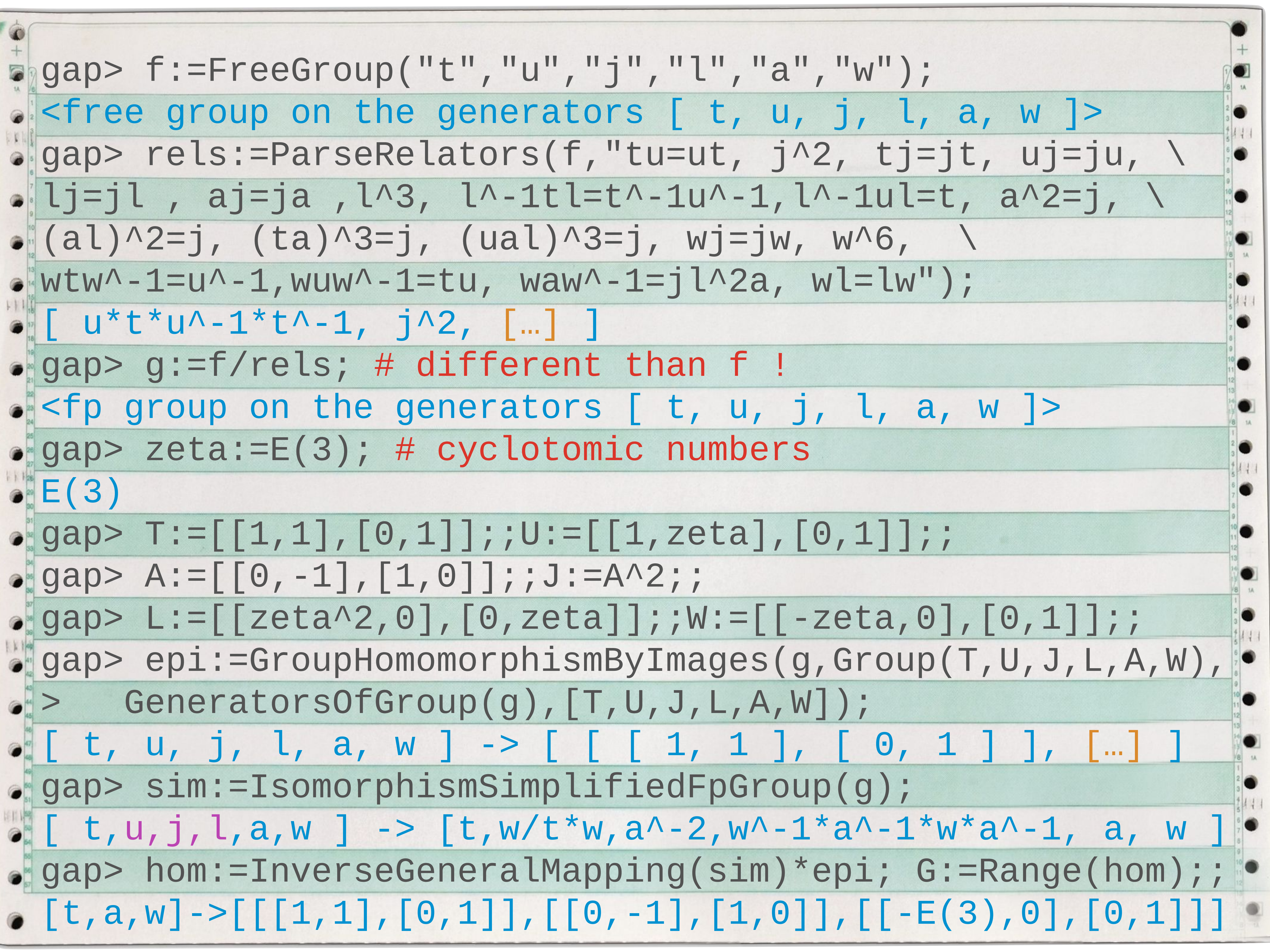
$aj = ja$,$l^3, l^{-1}tl = t^{-1}u^{-1}, l^{-1}ul = t, a^2 = j,$
$(al)^2 = j, (ta)^3 = j, (ual)^3 = j, wj = jw, w^6,$
$wtw^{-1} = u^{-1}, wuw^{-1} = tu, waw^{-1} = jl^2a, wl = lw$

MARGOLIS/BÄCHLE: Is $[\Gamma : G]$ finite for

$G = \langle m_1, m_2, m_3, m_i, m_j, m_t \rangle$ with

$m_1 = \begin{pmatrix} 97\zeta^2 & -112\zeta - 56\zeta^2 \\ 112\zeta + 56\zeta^2 & 97\zeta^2 \end{pmatrix}, m_2 = \begin{pmatrix} 56\zeta + 41\zeta^2 & 56\zeta + 112\zeta^2 \\ 56\zeta + 112\zeta^2 & -56\zeta + 153\zeta^2 \end{pmatrix},$

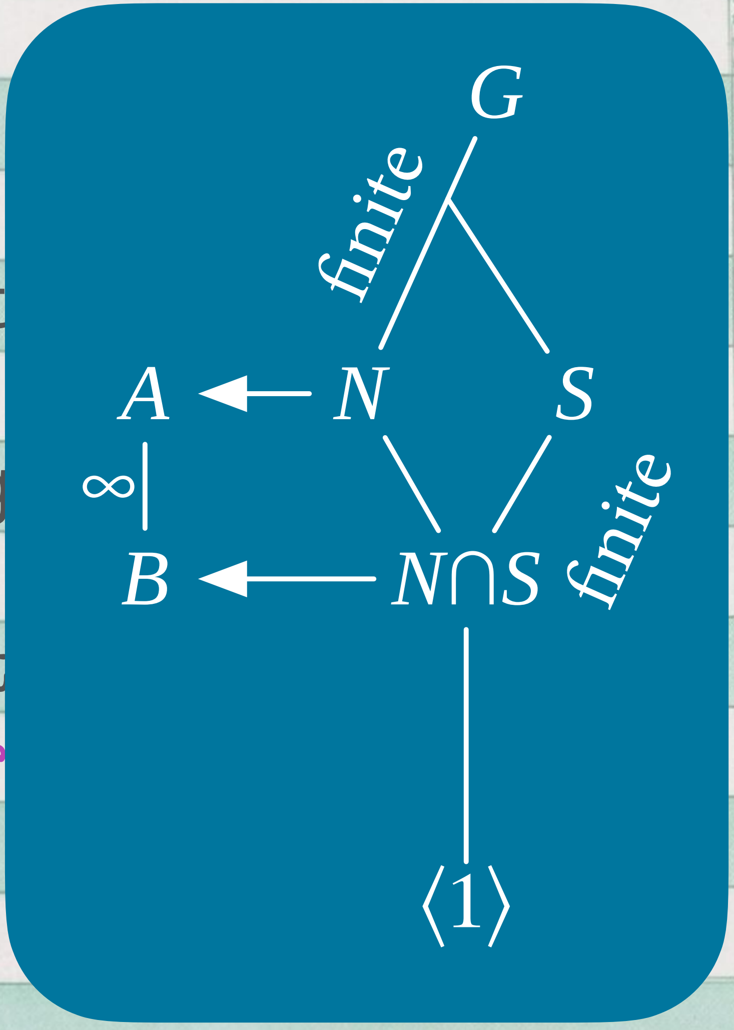
$m_3 = \begin{pmatrix} 56\zeta + 209\zeta^2 & -56\zeta + 56\zeta^2 \\ -56\zeta + 56\zeta^2 & -56\zeta - 15\zeta^2 \end{pmatrix}, m_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, m_j = \begin{pmatrix} \zeta & \zeta^2 \\ \zeta^2 & -\zeta \end{pmatrix}, m_t = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}$

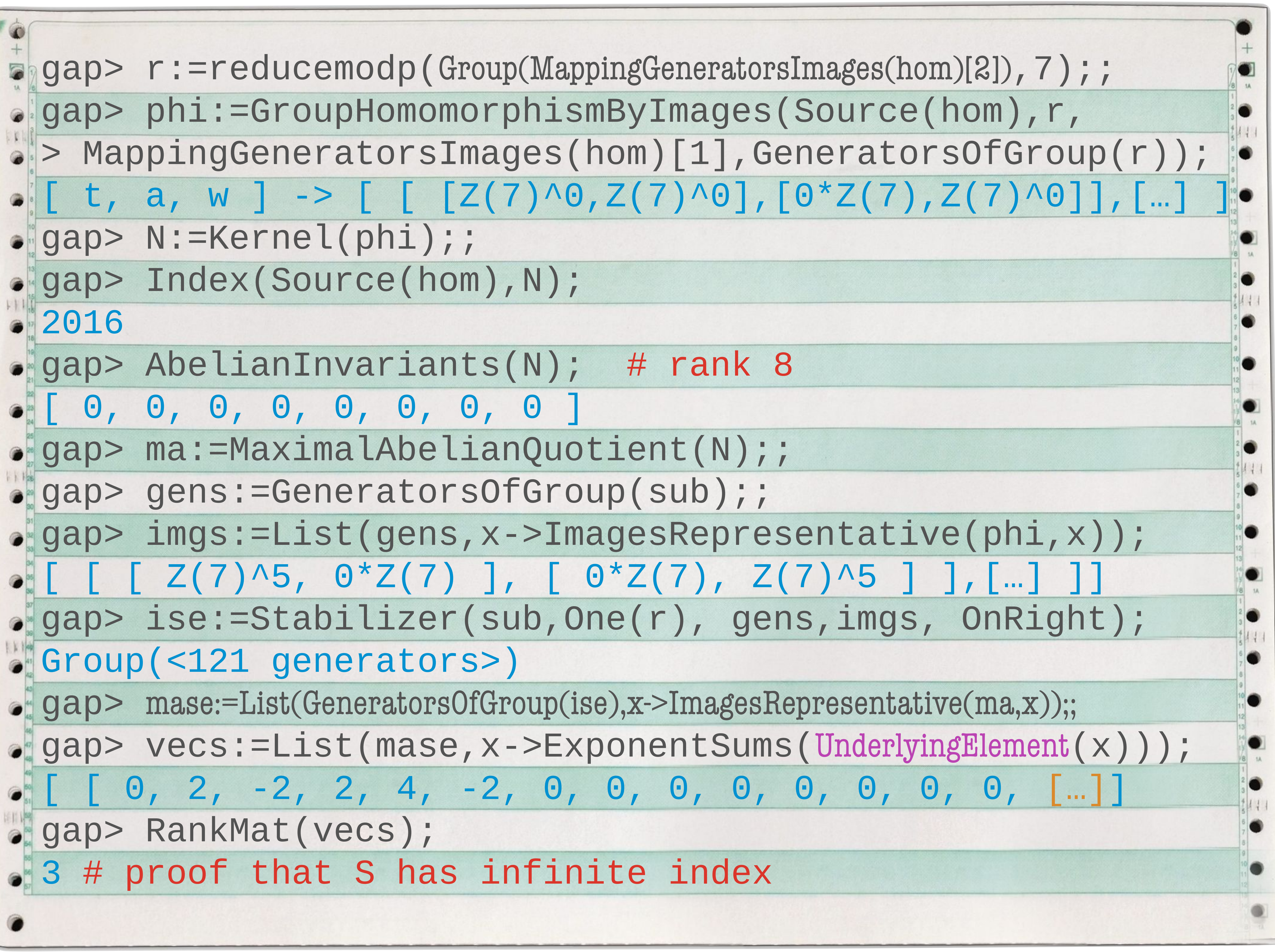


```
gap> f:=FreeGroup("t","u","j","l","a","w");
<free group on the generators [ t, u, j, l, a, w ]>
gap> rels:=ParseRelators(f,"tu=ut, j^2, tj=jt, uj=ju, \
lj=jl , aj=ja ,l^3, l^-1tl=t^-1u^-1,l^-1ul=t, a^2=j, \
(al)^2=j, (ta)^3=j, (ual)^3=j, wj=jw, w^6,   \
wtw^-1=u^-1,wuw^-1=tu, waw^-1=jl^2a, wl=lw");
[ u*t*u^-1*t^-1, j^2, […] ]
gap> g:=f/rels; # different than f !
<fp group on the generators [ t, u, j, l, a, w ]>
gap> zeta:=E(3); # cyclotomic numbers
E(3)
gap> T:=[[1,1],[0,1]];;U:=[[1,zeta],[0,1]];;
gap> A:=[[0,-1],[1,0]];;J:=A^2;;
gap> L:=[[zeta^2,0],[0,zeta]];;W:=[[-zeta,0],[0,1]];;
gap> epi:=GroupHomomorphismByImages(g,Group(T,U,J,L,A,W),
>    GeneratorsOfGroup(g),[T,U,J,L,A,W]);
[ t, u, j, l, a, w ] -> [ [ [ 1, 1 ], [ 0, 1 ] ], […] ]
gap> sim:=IsomorphismSimplifiedFpGroup(g);
[ t,u,j,l,a,w ] -> [t,w/t*w,a^-2,w^-1*a^-1*w*a^-1, a, w ]
gap> hom:=InverseGeneralMapping(sim)*epi; G:=Range(hom);;
[t,a,w]->[[[1,1],[0,1]],[[0,-1],[1,0]],[[-E(3),0],[0,1]]]
```

```
gap> normred:=function(phi,mat) […] end;;
gap> w1:=normred(hom,m1);
((t^-1*w^-1*a^-1*t^-1)^8)^(w^-1)
gap> w2:=normred(hom,m2);; w3:=normred(hom,m3);;
gap> wi:=normred(hom,mi);; wj:=normred(hom,mj);;
gap> wt:=normred(hom,mt);;
gap> rg:=Source(hom);;
gap> sub:=Subgroup(rg,[w1,w2,w3,wi,wj,wt]);;# fp version
gap> reducemodp:=function(g,p) […] end;;
gap> prd:=3*7*31*97*169*361;  # found by trying out
3852535323
gap> h:=reducemodp(G,prd);
Group([[[ZmodnZObj( 1, 3852535323 ), […]  )
gap> FittingFreeLiftSetup(h);;Size(h);
295803518081960107436094259200
gap> hs:=reducemodp(S,prd);;FittingFreeLiftSetup(hs);;
gap> Size(h)/Size(hs);
142580874240
```

```
gap> r:=reducemodp(Group(MappingGeneratorsImages(hom)[2]),7);;
gap> phi:=GroupHomomorphismByImages(Source(hom),r,
> MappingGeneratorsImages(hom)[1],GeneratorsOfGroup(r));
[ t, a, w ] -> [ [ [Z(7)^0,Z(7)^0],[0*Z(7),Z(7)^0]],[…] ]
gap> N:=Kernel(phi);;
gap> Index(Source(hom),N);
2016
gap> AbelianInvariants(N);  # rank 8
[ 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> ma:=MaximalAbelianQuotient(N);;
gap> gens:=GeneratorsOfGroup(sub);;
gap> imgs:=List(gens,x->ImagesRepresentat
[ [ [ Z(7)^5, 0*Z(7) ], [ 0*Z(7), Z(7)^5
gap> ise:=Stabilizer(sub,One(r), gens,img
Group(<121 generators>)
gap> mase:=List(GeneratorsOfGroup(ise),x->ImagesRepresent
gap> vecs:=List(mase,x->ExponentSums(Under
[ [ 0, 2, -2, 2, 4, -2, 0, 0, 0, 0, 0, 0,
gap> RankMat(vecs);
3 # proof that S has infinite index
```





```
gap> r:=reducemodp(Group(MappingGeneratorsImages(hom)[2]),7);;
gap> phi:=GroupHomomorphismByImages(Source(hom),r,
> MappingGeneratorsImages(hom)[1],GeneratorsOfGroup(r));
[ t, a, w ] -> [ [ [Z(7)^0,Z(7)^0],[0*Z(7),Z(7)^0]],[…] ]
gap> N:=Kernel(phi);;
gap> Index(Source(hom),N);
2016
gap> AbelianInvariants(N);  # rank 8
[ 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> ma:=MaximalAbelianQuotient(N);;
gap> gens:=GeneratorsOfGroup(sub);;
gap> imgs:=List(gens,x->ImagesRepresentative(phi,x));
[ [ [ Z(7)^5, 0*Z(7) ], [ 0*Z(7), Z(7)^5 ] ],[…] ]]
gap> ise:=Stabilizer(sub,One(r), gens,imgs, OnRight);
Group(<121 generators>)
gap> mase:=List(GeneratorsOfGroup(ise),x->ImagesRepresentative(ma,x));;
gap> vecs:=List(mase,x->ExponentSums(UnderlyingElement(x)));
[ [ 0, 2, -2, 2, 4, -2, 0, 0, 0, 0, 0, 0, 0, 0, […]]
gap> RankMat(vecs);
3 # proof that S has infinite index
```