Hyperbolic 3-manifolds with non-integral trace

Alan Reid

Rice University

jt work with Nicholas Rouse
Non-integral traces: Let $M = \mathbb{H}^3 / \Gamma$ be a finite volume orientable hyperbolic 3-manifold (or orbifold). $\Gamma \leq \text{PSL}_2(\mathbb{C})$

Rigidity: implies that $\text{tr}(\gamma)$ is an algebraic number for each $\gamma \in \Gamma$.

ie $\text{tr}(\gamma)$ is a root of an irreducible polynomial equation $p(x) = 0$ where

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad a_i \in \mathbb{Z}$$

$$a_n \neq 0 \quad (\text{tr}(\gamma) \text{ well-defined up to sign})$$

So the trace-field $\mathbb{Q}(\text{tr}\Gamma)$ is a finite extension of $\mathbb{Q}$.

Say $\Gamma$ (or $M$) has non-integral trace (resp. has integral trace) if $\text{tr}(\gamma)$ is not an algebraic integer for some $\gamma \in \Gamma$ (resp. there is no such $\gamma$).

For algebraic integer $a_n > \pm 1$

If $M = S^3 \setminus L$, $L$ a link, say $L$ has non-integral trace.
Topological consequences: (Bass’s Theorem) \( M \) is a finite volume hyperbolic 3-manifold with non-integral trace, then \( M \) contains a closed embedded essential surface.

Corollary

1. If \( M \) is non-Haken, then \( M \) has integral trace.
2. If \( K \subset S^3 \) is a small knot or link, then \( S^3 \setminus K \) has integral trace.

Basic example: \( SL_2(\mathbb{Z}[\frac{1}{p}]) = SL_2(\mathbb{Z}) \rtimes G \), \( G \in SL_2(\mathbb{Z}) \)
Comments on non-integrality: 1. Let $M = H^3/\mu$ as above with non-integral trace. We know that $\exists$ finitely many elements $\eta_1, \ldots, \eta_r \in \Gamma$ s.t. every $\text{tr}(\eta_i)$ is a $\mathbb{Z}$-polynomial in $\text{tr}(\eta_1), \ldots, \text{tr}(\eta_r)$.

Non-integrality and "primes certifying non-integrality" are visible in $\text{tr}(\eta_1), \ldots, \text{tr}(\eta_r)$.

2. If $\Gamma \supset \Gamma_1$ of finite index then $\Gamma$ has non-integral traces $\Rightarrow \Gamma_1$ has non-integral traces.

If $\sigma$ has non-integral trace $\sigma \in \mathfrak{o}$, then $1, 2$ cannot be units. So $\eta^n, \eta^n$ are not units.
**Examples:** (1) The SnapPy census manifold $m137$ (denoted by $M$). $M$ has volume $3.6638623767088\ldots$. From SnapPy, a presentation of $\pi_1(M)$ is

$$\langle a, b \mid aaabbABBBAb=1 \rangle.$$ 

The faithful discrete representation is given by:

$$a \mapsto \begin{pmatrix} -\frac{3}{2} + \frac{i}{2} & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{2} - \frac{i}{2} \end{pmatrix}.$$ 

Note: $2\mathbb{Z}[i] = \langle (1+i)^2 \rangle$ so non-integrality certified by $p=2$.

**Remark:** Lattices with non-integral trace is a rank 1 phenomena. High rank all lattices are arithmetic. Same for $Sp(n,1) \nexists$ and $Isom(Cayley^{1,n})$. [Baldi-Ullmo] all lattices in $\text{SU}(n,1)$ ($n>2$) have integral traces. Only $\text{SO}(n,1)$!
(2) $K = 5_2$ and $N = K(10/1)$ then $N$ has non-integral trace (note 10/1 is a boundary slope). The trace of the image of the meridian satisfies: $2Z^8 - 17Z^6 + 46Z^4 - 40Z^2 + 8 = 0$. 

Dehn Filling

$m^d = 1$

$\lambda^d = 1$

$t^d = 1$
(3) Reflection orbifolds

\[ \Gamma_q \leq \text{group generated by reflections in the faces of } P_q \]

\[ \Gamma_q^+ \leq \text{orientation-preserving subgroup of index } 2. \]

(4) There are infinitely many 2 component links with non-integral trace [Chesbro-Deblois].

(5) Some knots in the tables were known to have non-integral trace, e.g.: \(9_{29}, 9_{38}, 10_{96}, 10_{97}, 10_{99}.\)

Using Snap 21 knots through 12 crossings were identified as having non-integral trace.

[Coulsen-Goodman-Hodgson-Neumann]

There are 2977 prime knots with \( \leq 12 \) crossings.
In the work with Rouse we pushed this further, we identify 170 knots through 12 crossings with non-integral trace. Note that we only need consider those hyperbolic knots through 12 crossings that contain a closed embedded essential surface. These were enumerated by [Burton-Coward-Tillmann].

\[\text{loose - true field has degree 50}\]

Of the 2977, 1019 are "large". Of these 1019 we are able to certify integral or non-integral for 450.

Note: There are large knots with integral trace.
Questions: (1) Are there infinitely many knots with non-integral trace?

(2) What happens to non-integrality on Dehn surgery?

Note: [Culler-Gordon-Luecke-Shalen] implies that a closed embedded essential surface in a 1-cusped hyperbolic 3-manifold remains essential in ”most” Dehn surgeries.
Exploration of persistence and lack there-of of non-integral trace.

An example: Back to the SnapPy census manifold $M = m137$.

Set $\lambda = (ba)^{-1}$, then $\pi_1(M)$ can be generated by $\{b, \lambda\}$ and using this, a description for the canonical component of $M$ is given as the curve in $\mathbb{C}^2$ obtained as the vanishing set of the polynomial:

$$P(s, t) = (-2 - 3s + s^3)t^4 + (4 + 4s - s^2 - s^3)t^2 - 1,$$

where $s = \chi_\rho(\lambda)$, $t = \chi_\rho(b)$ and $\chi_\rho(b\lambda) = t - \frac{1}{t(s+1)}$.

Note that $(-2 - 3s + s^3) = (s + 1)^2(s - 2)$ and $(4 + 4s - s^2 - s^3) = (s + 1)(s + 2)(s - 2)$.

Thus, understanding the behavior of $t = \chi_\rho(b)$ (i.e. integral versus non-integral) is reduced to understanding when $(s + 1)$ and $(s - 2)$ are units in the number fields arising from Dehn filling representations.
For example, if we consider \((0, d)\) Dehn fillings (with respect to the framing \((m, \lambda)\)) with \(d\) odd, we are led to consideration of when \((2 \cos(2\pi/d) + 1)\) and \((2 \cos(2\pi/d) - 2)\) are and are not units.

**FACT:** \((2 \cos(2\pi/d) - 2)\) is never a unit for \(d\) a power of a prime.

Thus, modulo checking irreducibility of \(P(2 \cos(2\pi/d), t)\) we see that for \(d\) odd prime power \((0, d)\) filling has non-integral trace.

*Non-integrality certified by the prime \(p\) if \(d = p^r\).*
Comments:
1. Experimenting it seems that for \(d=10k\) (old) filling is integral.
2. \((0,14)\) filling also has integral trace.
3. \((\frac{1}{n})\) filling for \(n \in [-7,-3] \cup [2,6]\) has integral trace.

Upshot: Challenge to understand exactly what happens to non-integrality upon filling.
Theorem (R-Rouse)
There are infinitely many distinct knots with non-integral trace.

Basic idea: $L = J \cup K$ a 2 component hyperbolic link with $S^3 \setminus L = \{4^3/1\}$. Assume $J$ is an unknot and

**Linking Number** between $J$ and $K = \pm 2$.

If $\text{Unknot} \Rightarrow d$-fold branched cyclic cover of $S^3$ branched over $J = S^3$.

When $d$ is odd

Linking number $= \pm 2 \Rightarrow$ preimage of $K$ in the branched cover is connected, i.e., a knot $K_d$.

If large enough $S^3 \setminus K_d$ is hyperbolic: Can think of $S^3 \setminus K_d$ as $d$-fold cyclic cover of $(d,0)$ filling on $J$. ($C_{2n}$)

$\exists d \in \Gamma$ with $\text{tr}(\alpha)$ algebraic non-integer. Control $X_p(\alpha)$ on canonical cpt. Show $Q_d$ has non-integral trace.

$\Rightarrow S^3 \setminus K_d$ has non-integral trace.
The link: Let $L$ be the 2 component link $L_{11n106}$ from Thistlethwaite’s table of 2 component links through 11 crossings shown below.

The volume of $S^3 \setminus L$ is approximately $10.666979133796239$. 

$K$ is the knot $7_6$

Where $J$ goes under $K$

\[ +1 \quad -5 \quad +1 \]

Sum is 2.
The knots $K_d$ obtained as branched covers of $J$.

Perform isotopy of $L$.
Cut along Seifert Surface $F$.
Glue $d$ copies together.

$J'$ = lift of $J$.
Glue in solid torus to get $K_d$.

[ Gordon-Litherland: $S^3 \setminus K_d$ contains a closed embedded essential surface ]
From SnapPy a presentation for $\Gamma = \pi_1(S^3 \setminus L)$ is given as follows. Generators are $a$ and $b$ with relation:

$$abbaBAbaabABaBAbaabABabbbBAbaabABBBAbaBAAAbab AbaBAABBabABBBAbaBAABAbabABBBAbaBAAB = 1$$

Also from SnapPy meridians for $J$ and $K$ are given by

$J : baabABabbbBAABaBabABBBBAbaBAAAbabbbBA$  

$K : ba$  

Using SnapPy (or Snap) it can be checked that the trace-field of $\Gamma$ is $\mathbb{Q}(\sqrt{-7})$ and that $\text{tr}(a) = \pm (13 + 7\sqrt{-7})/8$ and $\text{tr}(b) = \pm (17 + 3\sqrt{-7})/8$ and so both are algebraic non-integers. 

$$\text{tr}(ab) = -2$$
We will consider how $\chi_\rho(b)$ varies on that part of the canonical component $X_0$ of $\pi_1(S^3 \setminus L)$ where

$$\chi_\rho(ba) = -2 \ (\rho(ba) \text{ is kept parabolic}), \text{ and}$$

$$\chi_\rho(m_0) = 2 \cos(2\pi/d), \ d \text{ odd} \ (\rho(m_0) \text{ is elliptic of order } d).$$

Setting $X = \chi_\rho(a)$ and $Y = \chi_\rho(b)$ we find that $X$ and $Y$ satisfies $P(X, Y) = 0$ where $P(X, Y)$ is given by:

$$X^8Y + 7X^7Y^2 - 2X^7 + 21X^6Y^3 - 7X^6Y + 35X^5Y^4 - 3X^5Y^2 - 8X^5 + 35X^4Y^5 + 20X^4Y^3 - 29X^4Y + 21X^3Y^6 + 40X^3Y^4 - 39X^3Y^2 - 7X^3 + 7X^2Y^7 + 33X^2Y^5 - 23X^2Y^3 - 17X^2Y + XY^8 + 13XY^6 - 5XY^4 - 14XY^2 + X + 2Y^7 - 4Y^3$$

$P(X, Y)$ is irreducible over $\mathbb{Q}$ (indeed over $\mathbb{C}$).

Set $t = \chi_\rho(m_0)$. Computing gives a polynomial $Q(t, X, Y)$ and eliminating $X$ using $P(X, Y)$ results in the following polynomial $R(t, Y)$:
\[669124t - 2t^7 - 498002t^5 - 5223073t^3 - 16tY^{24} +
(120t^2 + 176)Y^{23} + (- t^5 - 344t^3 - 1595t)Y^{22} +
(-t^7 - 265t^5 - 8323t^3 - 5017t) Y^{20} + (31t^7 - 820t^5 + 45501t^3 +
26034t) Y^{18} + (-428t^7 + 34065t^5 - 60100t^3 - 223825t) Y^{16} +
(3393t^7 - 229701t^5 - 1671221t^3 - 1389221t) Y^{14} +
(-16709t^7 + 392665t^5 + 4196073t^3 + 3978713t) Y^{12} +
(51769t^7 + 613384t^5 + 1570051t^3 + 257774t) Y^{10} + (-
97592t^7 - 3180386t^5 - 27592720t^3 - 28733690t) Y^8 +
(102474t^7 + 3256419t^5 + 42551766t^3 + 53431661t) Y^6 +
(-49677t^7 + 1658479t^5 - 6346815t^3 - 21240713t) Y^4 +
(6945t^7 - 5819870t^5 - 50037327t^3 - 50675755t) Y^2 +
(2t^6 + 466t^4 + 5400t^2 + 1265) Y^{21} + (8t^6 + 5340t^4 - 4891t^2 -
551) Y^{19} + (246t^6 - 65918t^4 - 71499t^2 + 10156) Y^{17} + (-
8510t^6 + 292550t^4 + 1114568t^2 + 263159) Y^{15} +
(62972t^6 + 480016t^4 + 532043t^2 - 387) Y^{13} + (-184968t^6 -
4075296t^4 - 11015955t^2 - 1827985) Y^{11} + (148666t^6 +
7363350t^4 + 27163139t^2 + 4743016) Y^9 +
(389244t^6 + 2024132t^4 - 5822600t^2 + 2654010) Y^7 + (-
959338t^6 - 19599946t^4 - 57150066t^2 - 22718115) Y^5 +
(659693t^6 + 19149660t^4 + 77616992t^2 + 31164769) Y^3 +}
\[(t^8 + 235110t^6 + 11747029t^4 + 26741431t^2 - 669124)Y\]

Note that the highest degree term as a polynomial in \(\mathbb{Z}[t]\) is \(16tY^{24}\).

If at algebraic integer specializations of \(t\), the polynomial \(R(t,Y)\) remains irreducible, then \(Y\) is an algebraic non-integer.

Remarks: (1) As a check, Mathematica shows that \(R(-2, (17 + 3\sqrt{-7})/8) = 0\) (i.e. at the faithful discrete representation).

(2) \(R(-2, Y)\) is reducible, factoring as

\[R(-2, Y) = (Y^9 + 15Y^8 + 104Y^7 + 435Y^6 + 1205Y^5 + 2285Y^4 + 2956Y^3 + 2506Y^2 + 1257Y + 283)^2(2Y^2 - 5Y + 4)(4Y^2 - 17Y + 22)(4Y^2 - 11Y + 8)\]

with the factor corresponding to the complete structure being \(4Y^2 - 17Y + 22\).
For $d$ odd, perform $(d, 0)$-Dehn filling on $J$, which amounts to setting $t = 2 \cos(2\pi/d)$ in $R(t, Y)$.

**FACT:** For $d$ odd, $2 \cos(2\pi/d)$ is a unit.

Result will now follow from:

**Proposition:** For infinitely many odd $d > 1$, the polynomial $R(2 \cos(2\pi/d), Y)$ is irreducible over $\mathbb{Q}(\cos(2\pi/d))$.

$$\Rightarrow$$ Specializing $t = 2 \cos\frac{2\pi}{d}, d \text{ odd}$, $Y$ is not an algebraic integer $\Rightarrow \mathbb{Q}_2$ has non-integral trace.
Ideas in the proof:

It is convenient to change to the polynomial
\( S(X, Y) = X^8 R(X + X^{-1}, Y). \)

**WHY:** Let \( \zeta_d = \exp(2\pi i/d), \) and note that
\( S(\zeta_d, Y) = \zeta_d^8 R(2 \cos(2\pi/d), Y), \) so \( S(\zeta_d, Y) \) is irreducible in \( \mathbb{Q}(\zeta_d)[Y] \) if and only if \( R(2 \cos(2\pi/d), Y) \) is. That \( S(\zeta_d, Y) \) is irreducible in \( \mathbb{Q}(\zeta_d)[Y] \) will be established using the following result.

**Theorem (Dvornicich and Zannier)**

Let \( k \) be a number field and \( k^c \) the field obtained by adjoining all roots of unity to \( k. \) If \( f \in k^c[X, Y] \) and \( f(X^m, Y) \) is irreducible in \( k^c[X, Y] \) for all positive integers \( m \leq \deg_Y f, \) then \( f(\zeta, Y) \) is irreducible in \( k^c[Y] \) for all but finitely many roots of unity \( \zeta. \)

Thus need to check \( S(X^m, Y) \) is irreducible over \( \mathbb{Q}^c \) for all \( m \leq 24. \) Indeed we prove it over \( \mathbb{Q}. \)
Theorem (Bertone, Chéze, Galligo)

Let $k$ be a field and $f(X, Y) \in k[X, Y]$ be an irreducible polynomial. Let $\{(i_1, j_1), \ldots, (i_l, j_l)\} \subset \mathbb{Z}^2$ be the vertex set of its Newton polygon. If $\gcd(i_1, j_1, \ldots, i_l, j_l) = 1$, then $f(X, Y)$ is irreducible over $\overline{k}$.

Need to check:

1. $S(X^m, Y)$ is irreducible over $\mathbb{Q}$; (using Mathematica)

2. the Newton polygon of $S(X^m, Y)$ satisfies the conditions of the Theorem.
**Question:** Is the figure-eight knot the only knot in $S^3$ with quadratic imaginary invariant trace-field? 

By arithmeticity, if there is such a knot it would have non-integral trace!

**Conjecture:** YES.

**Question:** Does there exist a closed hyperbolic 2-$\mathbb{Z}/3\mathbb{Z}$-manifold $\tilde{Y}/\Gamma$ with all non-trivial elements of $\Gamma$ having the property that $tr(\gamma)$ is an algebraic non-integer?
THANK YOU

Non-integral knot.

Last of the 12 crossing non-alternating knots.

Volume \( \approx 13.64075 \ldots \)

Trace field generated by a root of

\[ x^4 - x^3 - 2x^2 + 3x + 1 = 0 \]

(Discriminant \(-2068\)).

Non-integrality certified by \( p = 2 \).