Computational Aspects of Thin Groups I.N.S.

Hyperbolic 3-manifolds with non-integral trace

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jt work with Nicholas Rouse

Non-integral traces: Let $M = \mathbb{H}^3/\Gamma$ be a finite volume orientable hyperbolic 3-manifold (or orbifold). $\Gamma < PSL_2(\mathbb{C})$

Rigidity: implies that $\mathrm{tr}(\gamma)$ is an algebraic number for each $\gamma \in \Gamma$.

ie $\mathrm{tr}(\gamma)$ is a root of an irreducible polynomial equation $p(x) = 0$ where

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_i \in \mathbb{Z}$$

$$a_n \neq 0$$

($\mathrm{tr}(\gamma)$ well-defined up to sign)

So the trace-field $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a finite extension of $\mathbb{Q}$.

Say Γ (or M) has non-integral trace (resp. has integral trace) if $\mathrm{tr}(\gamma)$ is not an algebraic integer for some $\gamma \in \Gamma$ (resp. there is no such γ).

For algebraic integer $a_n = \pm 1$

If $M = S^3 \setminus L$, L a link, say L has non-integral trace.

Topological consequences: (Bass's Theorem) *M is a finite volume hyperbolic 3-manifold with non-integral trace, then M contains a closed embedded essential surface.*

Corollary

1. *If M is non-Haken, then M has integral trace.*
2. *If $K \subset S^3$ is a small knot or link, then $S^3 \setminus K$ has integral trace.*

No closed embedded essential surface

Non-integral trace

$\Rightarrow$ non-trivial splitting of $\pi_1 M$.

Basic example: $SL_2(\mathbb{Z}[\frac{1}{p}]) = SL_2(\mathbb{Z}) \underset{\Gamma_0(p)}{*} G \qquad G \cong SL_2(\mathbb{Z})$

Comments on non-integrality: 1. Let $M = \mathbb{H}^3/\Gamma$ as above with non-integral trace. We know that $\exists$ finitely many elements $\gamma_1, \ldots, \gamma_r \in \Gamma$ s.t. every $tr(\gamma)$ is a $\mathbb{Z}$-polynomial in $tr(\gamma_1), \ldots, tr(\gamma_r)$.

$\therefore$ Non-integrality and "primes certifying" non-integrality are visible in $tr(\gamma_1), \ldots, tr(\gamma_r)$

2. If $\Gamma \supset \Gamma_1$ of finite index then Γ has non-integral traces $\iff$ Γ_1 has non-integral traces

If γ has non-integral trace $\gamma \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda, \frac{1}{\lambda}$ cannot be units. So $\lambda^n, \frac{1}{\lambda^n}$ are not units.

Examples: (1) The SnapPy census manifold `m137` (denoted by M). M has volume $3.6638623767088\ldots$.

Knot complement in $S^2 \times S^1$

From SnapPy, a presentation of $\pi_1(M)$ is

```
<a,b |aaabbABBBAbb=1>.
```

$X = x^{-1}$

The faithful discrete representation is given by:

$$a \mapsto \begin{pmatrix} -\frac{3}{2} + \frac{i}{2} & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{2} - \frac{i}{2} \end{pmatrix}.$$

$\frac{-(2+i)}{1+i}$

Note: $2\mathbb{Z}[i] = \langle (1+i) \rangle^2$

$\frac{-i}{1+i}$

So non-integrality certified by $p=2$.

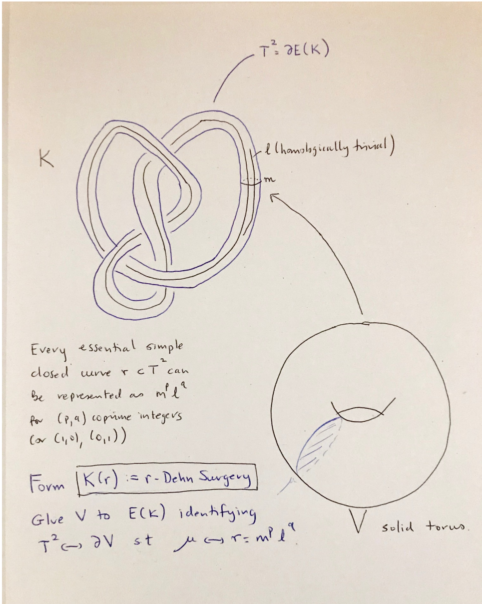
ab is parabolic.

Remark: Lattices with non-integral trace is a rank 1 phenomena.

High rank all lattices are arithmetic. Same for $Sp(n,1)$ $n \geq 2$ and $\mathrm{Isom}(\text{Cayley}\,\mathbb{H}^2)$. [Baldi-Ullmo] all lattices in $SU(n,1)$ $(n \geq 2)$ have integral traces.

$\therefore$ Only $SO(n,1)$!

(2) $K = 5_2$ and $N = K(10/1)$ then N has non-integral trace (note $10/1$ is a boundary slope). The trace of the image of the meridian satisfies: $2Z^8 - 17Z^6 + 46Z^4 - 40Z^2 + 8 = 0$.



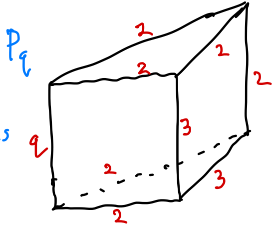
Dehn Filling

$m^d = 1$

~~[illegible]~~ $^d = 1$

$\ell^d = 1$

(3) Reflection orbifolds



$q \geq 7$

Γ_q = group generated by reflections in the faces of P_q

Γ_q^+ = orientation-preserving subgroup of index 2.

Take $q = 6p$

p prime $\equiv 1 \bmod 3$

Γ_q^+ has non-integral trace

Certified by the prime p.

(4) There are infinitely many 2 component links with non-integral trace [Chesebro-Deblois].

(5) Some knots in the tables were known to have non-integral trace, e.g.: 9_{29}, 9_{38}, 10_{96}, 10_{97}, 10_{99}.

Using Snap 21 knots through 12 crossings were identified as having non-integral trace.
[Coulsen-Goodman-Hodgson-Neumann]

There are 2977 prime knots with $\leq$ 12 crossings

10_{88} - trace field has degree 50

In the work with Rouse we pushed this further, we identify $\boxed{170}$ knots through 12 crossings with non-integral trace.

Note that we only need consider those hyperbolic knots through 12 crossings that contain a closed embedded essential surface. These were enumerated by [Burton-Coward-Tillmann]. (Bass's Thm)

Of the 2977, 1019 are "large".

Of these 1019 we are able to certify integral or non-integral for 450

Note: There are large knots with integral trace.

Questions: (1) Are there infinitely many knots with non-integral trace?

(2) What happens to non-integrality on Dehn surgery?

Note: [Culler-Gordon-Luecke-Shalen] implies that a closed embedded essential surface in a 1-cusped hyperbolic 3-manifold remains essential in "most" Dehn surgeries.

Exploration of persistence and lack there-of of non-integral trace.

An example: Back to the SnapPy census manifold $M =$ `m137`.

Set $\lambda = (ba)^{-1}$, then $\pi_1(M)$ can be generated by $\{b, \lambda\}$ and using this, a description for the canonical component of M is given as the curve in $\mathbb{C}^2$ obtained as the vanishing set of the polynomial:

$$\boxed{P(s,t) = (-2 - 3s + s^3)t^4 + (4 + 4s - s^2 - s^3)t^2 - 1,}$$

where $s = \chi_\rho(\lambda)$, $t = \chi_\rho(b)$ and $\chi_\rho(b\lambda) = t - \frac{1}{t(s+1)}$.

Note that $(-2 - 3s + s^3) = (s+1)^2(s-2)$ and $(4 + 4s - s^2 - s^3) = (s+1)(s+2)(s-2)$.

Thus, understanding the behavior of $t = \chi_\rho(b)$ (i.e. integral versus non-integral) is reduced to understanding when $(s+1)$ and $(s-2)$ are units in the number fields arising from Dehn filling representations.

ie set $\lambda^d = 1$

For example, if we consider $(0, d)$ Dehn fillings (with respect to the framing (m, λ)) with d odd, we are led to consideration of when $(2\cos(2\pi/d) + 1)$ and $(2\cos(2\pi/d) - 2)$ are and are not units.

FACT: $(2\cos(2\pi/d) - 2)$ is never a unit for d a power of a prime.

Thus, modulo checking irreducibility of $P(2\cos(2\pi/d), t)$ we see that for d odd prime power $(0, d)$ filling has non-integral trace.

Non-integrality certified by the prime p if $d = p^r$.

Comments:

1. Experimenting it seems that for $d = 10k$ $(0,d)$ filling is integral.

2. $(0,14)$ filling also has integral trace.

3. $(1,n)$ filling for $n \in [-7,-3] \cup [2,6]$ has integral trace.

Upshot: Challenge to understand exactly what happens to non-integrality upon filling.

$\boxed{\begin{array}{l}\textbf{Theorem (R-Rouse)} \\ \textit{There are infinitely many distinct knots with non-integral trace.}\end{array}}$

Basic idea: $L = J \cup K$ a 2 component hyperbolic link with $S^3 \setminus L = \mathbb{H}^3/\Gamma$. Assume J is an unknot and Linking Number between J and K $= \pm 2$.

J unknot $\Rightarrow$ d-fold branched cyclic cover of S^3 branched over J is S^3.

Linking number $= \pm 2$ $\Rightarrow$ when d is odd preimage of K in the branched cover is connected. ie a knot K_d

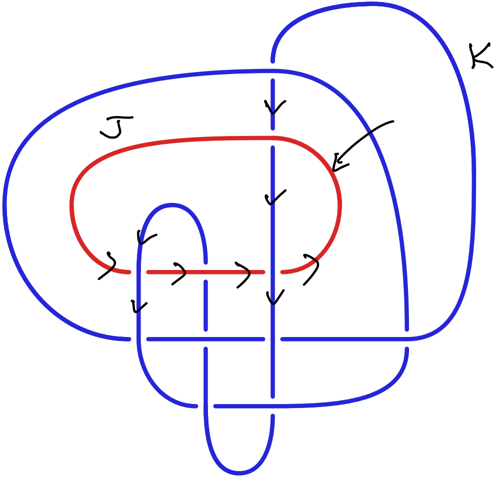
d large enough $S^3 \setminus K_d$ is hyperbolic: Can think of $S^3 \setminus K_d$ as d-fold cyclic cover of $(d,0)$ filling on J. (Q_d)

$\exists\, \alpha \in \Gamma$ with $\mathrm{tr}(\alpha)$ algebraic non-integer. Control $\chi_\rho(\alpha)$ on canonical cpt. Show Q_d has non-integral trace.

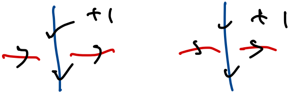
$\Rightarrow S^3 \setminus K_d$ has non-integral trace.

The link: Let L be the 2 component link `L11n106` from Thistlethwaite's table of 2 component links through 11 crossings shown below.

K is the knot 7_6



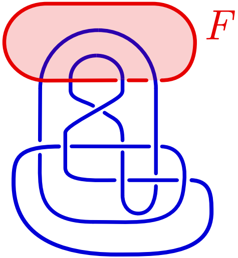
Where J goes under K



Sum is 2.

The volume of $S^3 \setminus L$ is approximately 10.666979133796239.

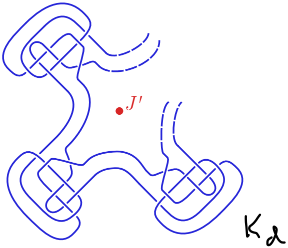
The knots K_d obtained as branched covers of J.



Perform isotopy of L

Cut along Seifert Surface F

Glue d copies together.



J' = lift of J

Glue in solid torus to get K_d

[Gordon-Litherland : $S^3 \setminus K_d$ contains a closed embedded essential surface]

From SnapPy a presentation for $\Gamma = \pi_1(S^3 \setminus L)$ is given as follows. Generators are a and b with relation:

$abbbaBAbaabABaBAbaabABabbbaBAbaabABBBAbaBAABab$
$AbaBAABabABBBAbaBAAB = 1$

Also from SnapPy meridians for J and K are given by

$J : baabABabbbaBAABabABBBAbaBAABabbbaBA$ (call this m_0)

$K : ba$

Using SnapPy (or Snap) it can be checked that the trace-field of Γ is $\mathbb{Q}(\sqrt{-7})$ and that $\mathrm{tr(a)} = \pm(13 + 7\sqrt{-7})/8$ and $\mathrm{tr(b)} = \pm(17 + 3\sqrt{-7})/8$ and so both are algebraic non-integers.

$\mathrm{tr}(ba) = -2$

We will consider how $\boxed{\chi_\rho(b)}$ varies on that part of the canonical component X_0 of $\pi_1(S^3 \setminus L)$ where

$\chi_\rho(ba) = -2$ ($\rho(ba)$ is kept parabolic), and

$\chi_\rho(m_0) = 2\cos(2\pi/d)$, d odd ($\rho(m_0)$ is elliptic of order d).

Setting $X = \chi_\rho(a)$ and $Y = \chi_\rho(b)$ we find that X and Y satisfies $P(X, Y) = 0$ where $P(X, Y)$ is given by:

Using Mathematica

$$X^8Y + 7X^7Y^2 - 2X^7 + 21X^6Y^3 - 7X^6Y + 35X^5Y^4 - 3X^5Y^2 - 8X^5 + 35X^4Y^5 + 20X^4Y^3 - 29X^4Y + 21X^3Y^6 + 40X^3Y^4 - 39X^3Y^2 - 7X^3 + 7X^2Y^7 + 33X^2Y^5 - 23X^2Y^3 - 17X^2Y + XY^8 + 13XY^6 - 5XY^4 - 14XY^2 + X + 2Y^7 - 4Y^3$$

$P(X, Y)$ is irreducible over $\mathbb{Q}$ (indeed over $\mathbb{C}$).

Using Mathematica

Set $t = \chi_\rho(m_0)$. Computing gives a polynomial $Q(t, X, Y)$ and eliminating X using $P(X, Y)$ results in the following polynomial $R(t, Y)$:

$$
\begin{aligned}
&669124t - 2t^7 - 498002t^5 - 5223073t^3 - \boxed{16tY^{24}} + \\
&(120t^2 + 176)\,Y^{23} + (-t^5 - 344t^3 - 1595t)Y^{22} + \\
&(-t^7 - 265t^5 - 8323t^3 - 5017t)\,Y^{20} + (31t^7 - 820t^5 + 45501t^3 + \\
&26034t)Y^{18} + (-428t^7 + 34065t^5 - 60100t^3 - 223825t)\,Y^{16} + \\
&(3393t^7 - 229701t^5 - 1671221t^3 - 1389221t)Y^{14} + \\
&(-16709t^7 + 392665t^5 + 4196073t^3 + 3978713t)\,Y^{12} + \\
&(51769t^7 + 613384t^5 + 1570051t^3 + 257774t)\,Y^{10} + (- \\
&97592t^7 - 3180386t^5 - 27592720t^3 - 28733690t)Y^8 + \\
&(102474t^7 + 3256419t^5 + 42551766t^3 + 53431661t)\,Y^6 + \\
&(-49677t^7 + 1658479t^5 - 6346815t^3 - 21240713t)\,Y^4 + \\
&(6945t^7 - 5819870t^5 - 50037327t^3 - 50675755t)Y^2 + \\
&(2t^6 + 466t^4 + 5400t^2 + 1265)\,Y^{21} + (8t^6 + 5340t^4 - 4891t^2 - \\
&551)Y^{19} + (246t^6 - 65918t^4 - 71499t^2 + 10156)\,Y^{17} + (- \\
&8510t^6 + 292550t^4 + 1114568t^2 + 263159)Y^{15} + \\
&(62972t^6 + 480016t^4 + 532043t^2 - 387)\,Y^{13} + (-184968t^6 - \\
&4075296t^4 - 11015955t^2 - 1827985)Y^{11} + (148666t^6 + \\
&7363350t^4 + 27163139t^2 + 4743016)Y^9 + \\
&(389244t^6 + 2024132t^4 - 5822600t^2 + 2654010)\,Y^7 + (- \\
&959338t^6 - 19599946t^4 - 57150066t^2 - 22718115)Y^5 + \\
&(659693t^6 + 19149660t^4 + 77616992t^2 + 31164769)Y^3 +
\end{aligned}
$$

$$\left(t^8 + 235110t^6 + 11747029t^4 + 26741431t^2 - 669124\right) Y$$

Note that the highest degree term as a polynomial in $\mathbb{Z}[t]$ is $\boxed{16tY^{24}}$.

If at algebraic integer specializations of t, the polynomial $R(t, Y)$ remains irreducible, then Y is an algebraic non-integer.

Remarks: (1) As a check, Mathematica shows that $R(-2, (17 + 3\sqrt{-7})/8) = 0$ (i.e. at the faithful discrete representation).

(2) $R(-2, Y)$ is reducible, factoring as

$R(-2, Y) =$
$(Y^9 + 15Y^8 + 104Y^7 + 435Y^6 + 1205Y^5 + 2285Y^4 + 2956Y^3 + 2506Y^2 + 1257Y + 283)^2 (2Y^2 - 5Y + 4) \boxed{(4Y^2 - 17Y + 22)} (4Y^2 - 11Y + 8)$

with the factor corresponding to the complete structure being $4Y^2 - 17Y + 22$.

For d odd, perform $(d, 0)$-Dehn filling on J, which amounts to setting $t = 2\cos(2\pi/d)$ in $R(t, Y)$.

FACT: For d odd, $2\cos(2\pi/d)$ is a unit.

Result will now follow from:

Proposition: *For infinitely many odd $d > 1$, the polynomial $R(2\cos(2\pi/d), Y)$ is irreducible over $\mathbb{Q}(\cos(2\pi/d))$.*

$\Rightarrow$ Specializing $t = 2\cos 2\pi/d$, d odd Y is not an algebraic integer $\Rightarrow$ Q_d has non-integral trace.

Ideas in the proof:

It is convenient to change to the polynomial
$S(X,Y) = X^8 R(X + X^{-1}, Y)$.

WHY: Let $\zeta_d = \exp(2\pi i/d)$, and note that
$S(\zeta_d, Y) = \zeta_d^8 R(2\cos(2\pi/d), Y)$, so $S(\zeta_d, Y)$ is irreducible in $\mathbb{Q}(\zeta_d)[Y]$ if and only if $R(2\cos(2\pi/d), Y)$ is. That $S(\zeta_d, Y)$ is irreducible in $\mathbb{Q}(\zeta_d)[Y]$ will be established using the following result.

Theorem (Dvornicich and Zannier)

Let k be a number field and k^c the field obtained by adjoining all roots of unity to k. If $f \in k^c[X,Y]$ and $f(X^m, Y)$ is irreducible in $k^c[X,Y]$ for all positive integers $m \le \deg_Y f$, then $f(\zeta, Y)$ is irreducible in $k^c[Y]$ for all but finitely many roots of unity ζ.

Thus need to check $S(X^m, Y)$ is irreducible over $\mathbb{Q}^c$ for all $m \le 24$. Indeed we prove it over $\overline{\mathbb{Q}}$.

Theorem (Bertone, Chéze, Galligo)

Let k be a field and $f(X,Y) \in k[X,Y]$ be an irreducible polynomial. Let $\{(i_1, j_1), \ldots, (i_l, j_l)\} \subset \mathbb{Z}^2$ be the vertex set of its Newton polygon. If $\gcd(i_1, j_1, \ldots, i_l, j_l) = 1$, then $f(X,Y)$ is irreducible over $\overline{k}$.

Need to check:

1. *$S(X^m, Y)$ is irreducible over $\mathbb{Q}$;* (using Mathematica)

2. *the Newton polygon of $S(X^m, Y)$ satisfies the conditions of the Theorem.*

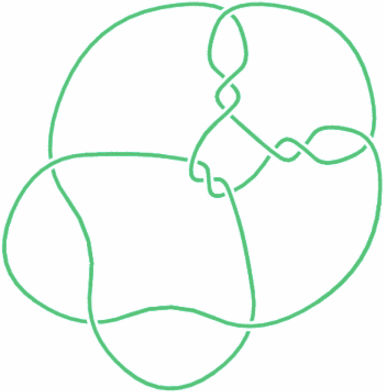
Question: *Is the figure-eight knot the only knot in S^3 with quadratic imaginary invariant trace-field?*

By arithmeticity, if there is such a knot it would have non-integral trace!

Conjecture: YES.

Question: Does there exist a closed hyperbolic 2-, 3-manifold $\mathbb{H}^*/\Gamma$ with all non-trivial elements of Γ having the property that $tr(\gamma)$ is an algebraic non-integer.

THANK YOU



Knotscape diagram: 12n_888

Non-integral knot.

Last of the 12 crossing non-alternating knots.

Volume $\approx$ 13.64075....

Trace field generated by a root of

$$x^4 - x^3 - 2x^2 + 3x + 1 = 0$$

(Discriminant -2068).

Non-integrality certified by $p = 2$