$PSL(2,\mathbb{C})$–representations of knot groups by knot diagrams

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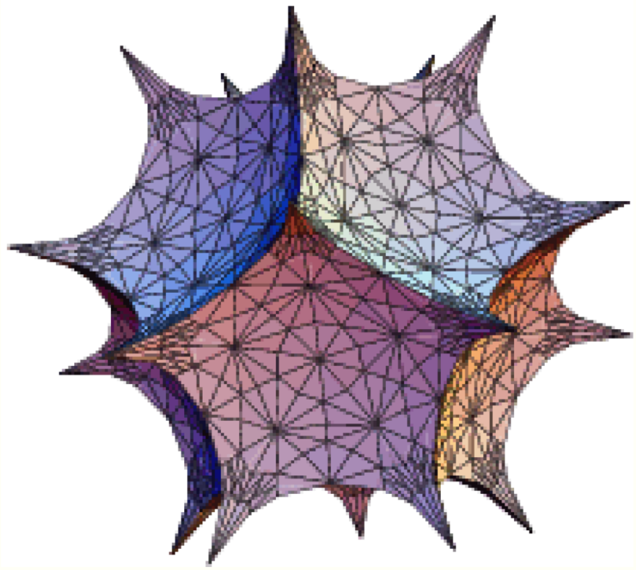
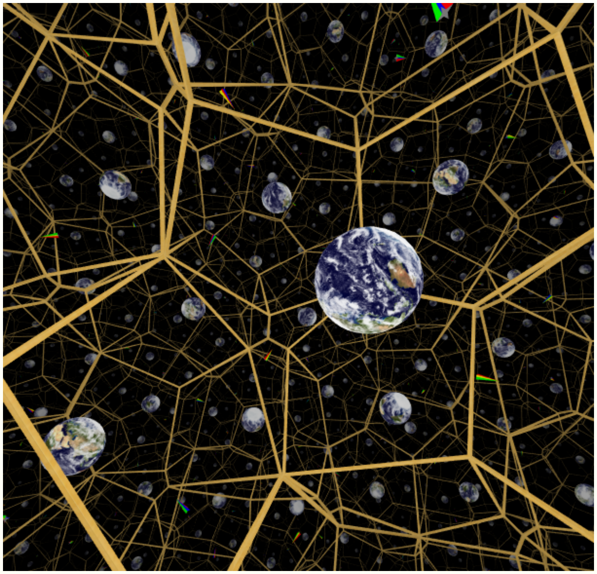
A hyperbolic 3-manifold M is a quotient of $\mathbb{H}^3$ by a discrete group of fixed point free isometries of $\mathbb{H}^3$. This Kleinian group is isomorphic to the fundamental group of the manifold $\pi_1(M)$.

This leads to a tiling of $\mathbb{H}^3$ by hyperbolic polyhedra. M is obtained from a single tile by identifying pairs of faces. M inherits hyperbolic metric and volume from this tile.

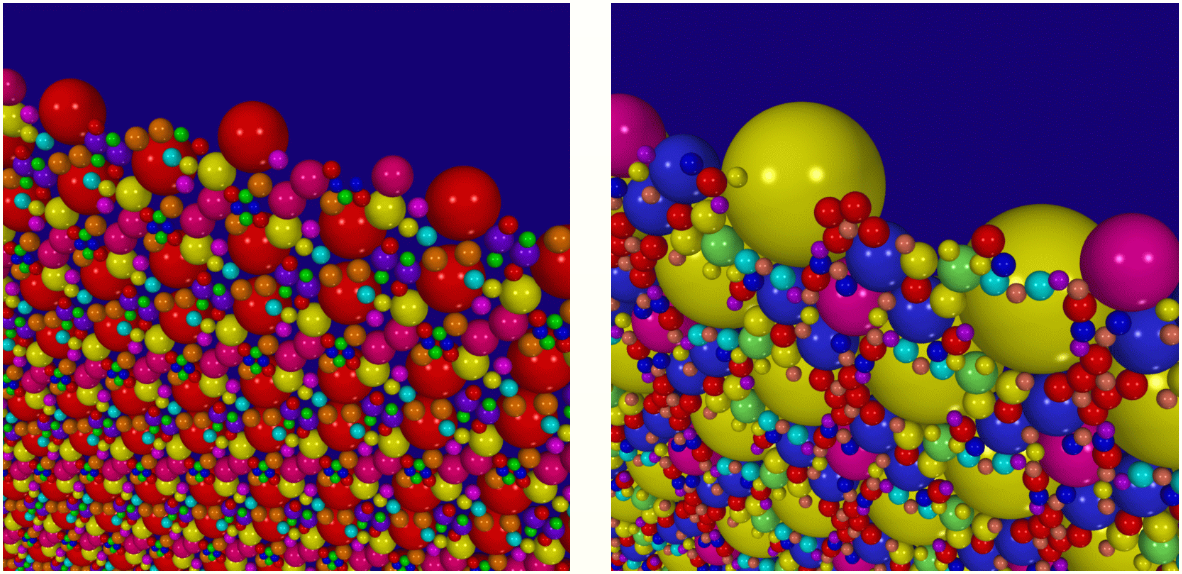




A *hyperbolic knot* is such a knot K that its complement in a 3-sphere S^3 is a hyperbolic manifold. Mostow-Prasad Rigidity Th.: for a finite-volume manifold the hyperbolic metric is unique as long as it is complete.

The set of all representations of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbb{C})$ is the $\mathrm{PSL}(2, \mathbb{C})$-representation variety $R(\Gamma)$.

A *banana* is either a surface which consists of the points at a fixed hyperbolic distance from a geodesic (*axis of the banana*) in $\mathbb{H}^3$, or a horosphere. Bananas are invariant surfaces under isometries of $\mathbb{H}^3$.



Pictures by Morwen Thislethwaite, Jim Belk, Henry Segerman

For many representations, the boundary torus of a knot lifts to closed bananas in $\mathbb{H}^3$. W. Thurston called such representations *geometric*. They correspond to incomplete hyperbolic structures.

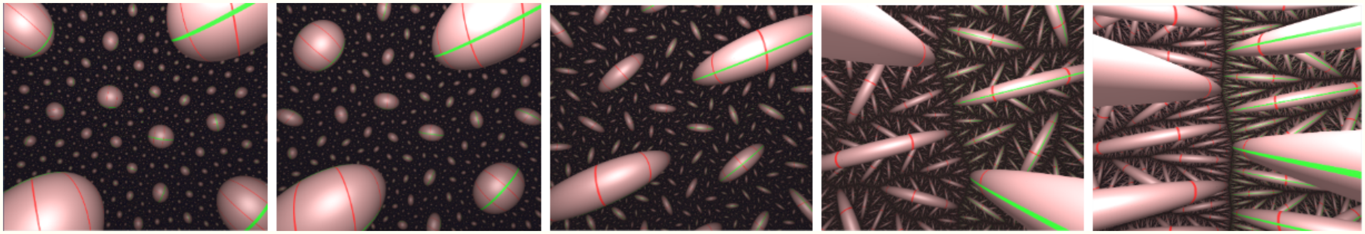
The unique complete hyperbolic structure of M corresponds to the discrete faithful representation.

Conjugate representations correspond to the same structure.

For irreducible representations (not conjugate to an upper triangular one), being conjugate is equivalent to having equal traces. So *character variety* $X(\Gamma)$, the set of all representations up to trace equivalence, is useful.

A component of $X(\Gamma)$ that contains such the unique faithful representation is a *canonical component*. For a hyperbolic knot, this is a complex curve.

For a hyperbolic knot, there are infinitely many geometric representations in any neighborhood of a discrete and faithful representation on the canonical curve.



We obtain equations for geometric representations. The equations define any canonical component. Due to lifting, this will determine $SL_2(\mathbb{C})$ representations as well.

"Heritage" project

2012: Thistlethwaite-T., *An alternative approach to hyperbolic structures on link complements*. Algorithm for computing parabolic representations from a link diagram.

2013: ICERM special program, M. Culler:
Can this be generalized beyond parabolic?

2021: preprint Petersen-T., prove that it can be generalized for knots. Not yet links: to simplify, assume all meridianal curves are homotopic.

Upcoming, by a team of postdocs and students: SnapPy-based code that takes a knot and gives equations for canonical component using this method.

Computing geometric representations: prior work

SnapPea: method for computing the complete hyperbolic structure of 3–manifolds by Thurston, Weeks. SnapPy by Culler, Dunfield, Goerner. Generalized to compute geometric representations: gluing variety. Also generalized to compute boundary-unipotent representations into $\mathrm{SL}(n, \mathbb{C})$ by Garoufalidis, D. Thurston, Zickert, project CURVE.

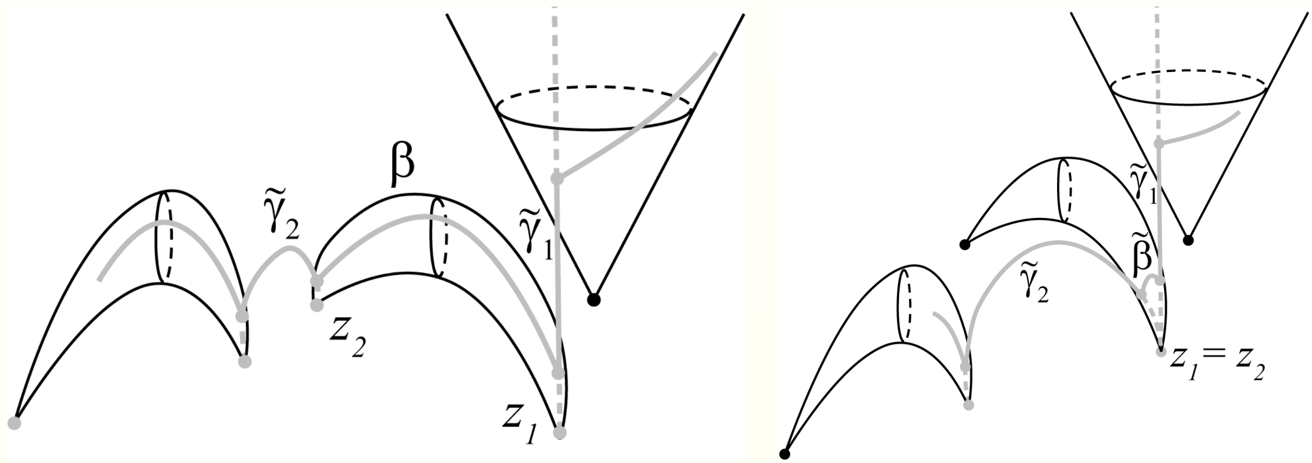
This relies on a suitable triangulation that is being found heuristically using the software.

Our method only uses regions of a knot diagram D; no triangulation or polyhedral decomposition is involved. Different equations with different degrees are useful for investigating and simplifying. Easy to use for infinite families of knots with similar diagrams.

Computing *parabolic* representations only: Riley (1972), Thislethwaite-T.(2012) and Kim-Kim-Yoon (2018) with similarities. Computations of varieties for families of knots that admit nice representations: Macasieb-Peterson-Lujik, Chehn, Tran, ...

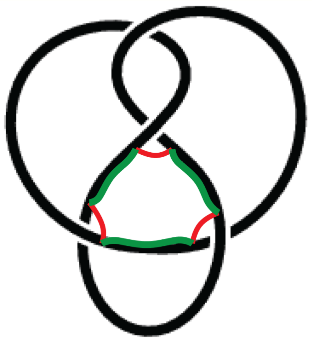
A topological accidental parabolic is a curve that is not boundary parallel in a manifold, but is homotopic to a peripheral arc. We work with taut diagrams: each associated checkerboard surface must be incompressible and boundary incompressible in the link complement, and not contain any simple closed curve representing a topological accidental parabolic.

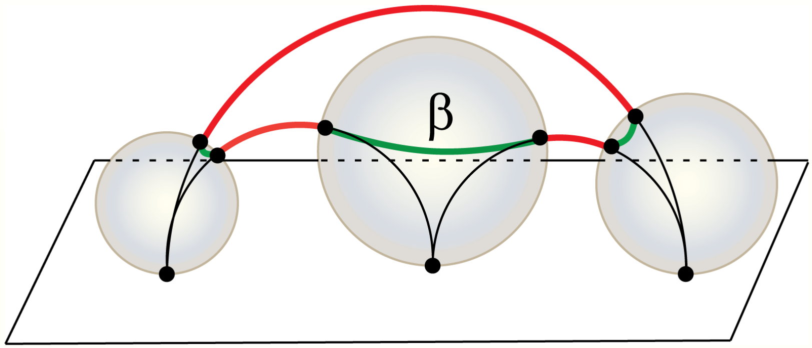
Lemma. If D is taut, then for infinitely many geometric representations, the lift of a crossing arc in $\mathbb{H}^3$ is homotopic to a unique geodesic.



Lemma. Consider two consecutive crossing arcs in a region of a link diagram. Their preimages in $\mathbb{H}^3$ share an ideal point. I.e. the situation on the right occurs. The situation on the left does not occur.

A region of D is bounded by red arcs from an overpass to an underpass, and green arcs on the boundary torus. Its preimage in $\mathbb{H}^3$ for infinitely many geometric representations is now unique. It is a cyclic sequence of bananas in $\mathbb{H}^3$, connected by geodesic arcs.



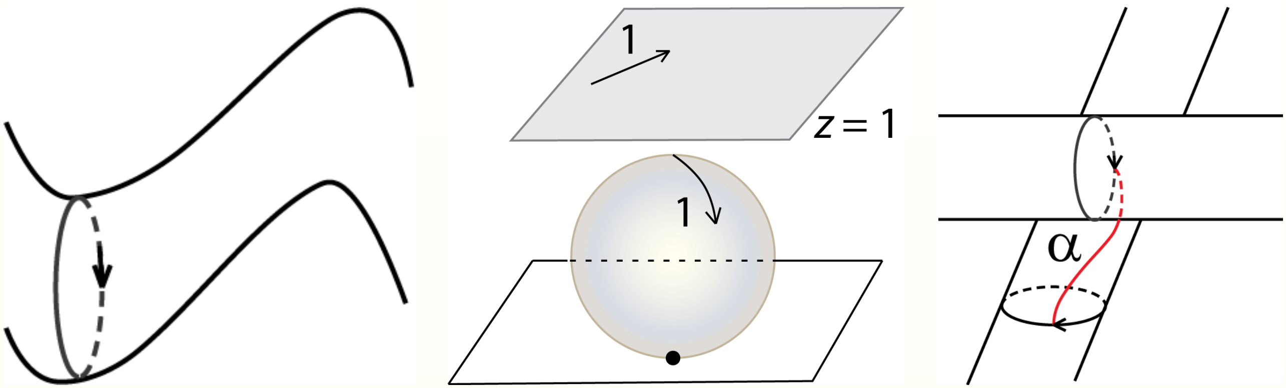


But a representation is naturally defined for all elements of $\pi_1(M)$: loops. We prove that infinitely many representations on the canonical component extend to red and green arcs as well.

Hence we can assign an element of $\mathrm{PSL}(2,\mathbb{C})$ to every arc. This corresponds to an isometry that fixes the geodesic as a set in $\mathbb{H}^3$ and sends a lift of the base point to the next lift of this point along the geodesic.

Finally we prove that any set-up of this form determines a representation of the knot group to $\mathrm{PSL}(2,\mathbb{C})$.

A simple closed curve μ traveling once around the boundary torus of a link is a *meridian*. Its preimage in $\mathbb{H}^3$ lies on a banana. There is an isometry that corresponds traveling along a meridian. Up to conjugation, it is $\rho(\mu) = M = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}$.



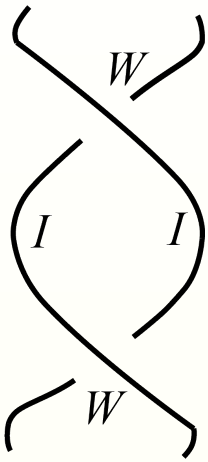
We then show that we can conjugate infinitely many geometric representations as follows. Green arcs correspond to an <u>edge matrix</u> U. Red arcs correspond a <u>crossing matrix</u> W, where $U = (\pm)\begin{pmatrix} v & u \\ 0 & v^{-1} \end{pmatrix}$, $W = (\pm)\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}$.

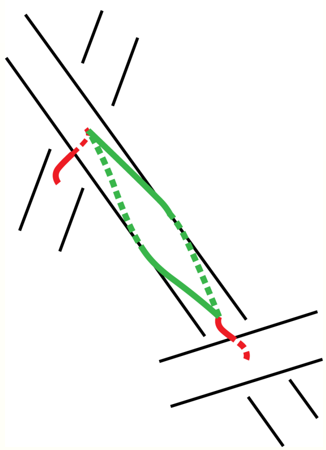
The matrix entries have geometric meaning. Once the scale of the bananas is fixed, the modulus of the complex number captures hyperbolic distance, and argument captures the dihedral angle between meridians on bananas.

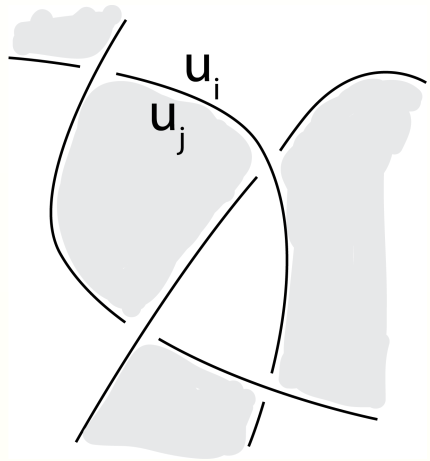
Relations

Use symmetry of the diagram to assign matrices.

For bigons/twists, edge matrices are I, and crossing matrices are identical.







Color the regions of the diagram as a checkerboard. Each edge gives rise to two arcs: on the boundary of the black region, labeled U_j, and of the white one, U_i. Traveling along one arc and returning along another makes a meridian: $U_i U_j^{-1} = M$.

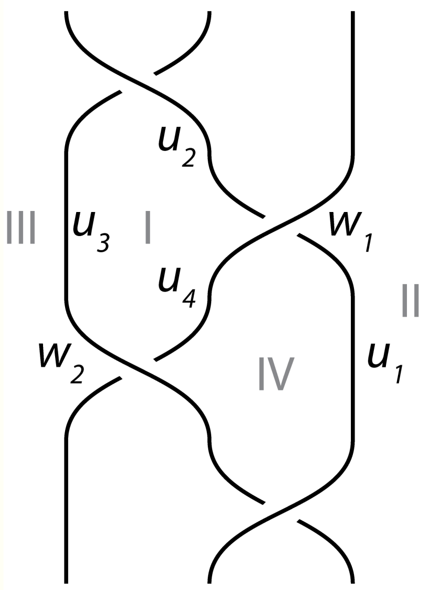
Every region polygon closes up, so the composite of all the corresponding isometries is I. Hence the product of the respective matrices is a scalar multiple of I.

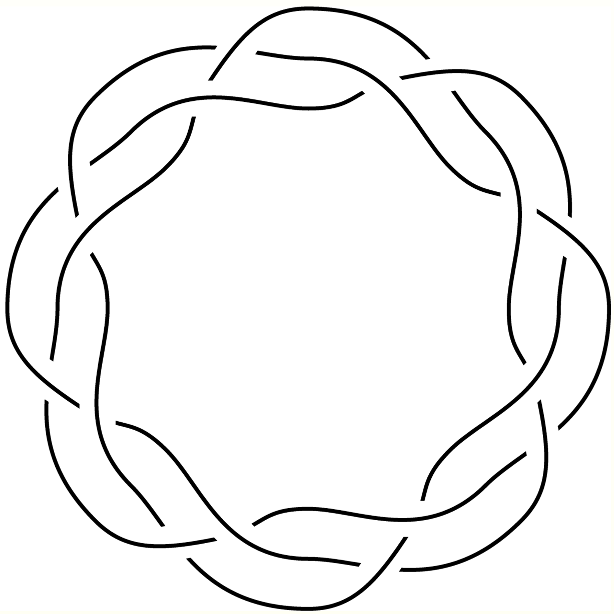
From the matrix entries read off independent polynomial relations for every region.

Example: character variety of an infinite family of braids

$(\sigma_1(\sigma_2)^{-1})^n$, $n > 2$

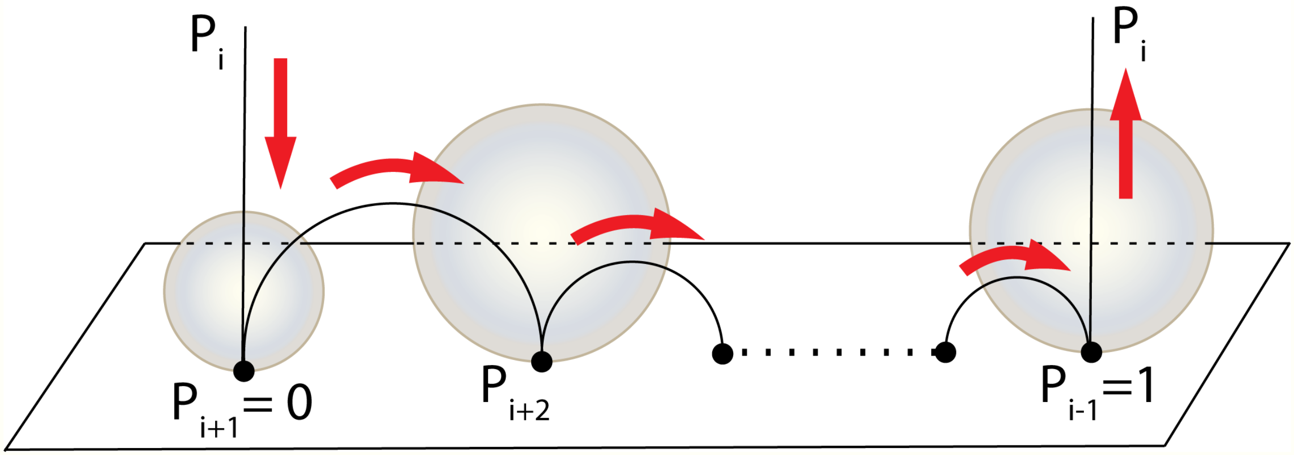
Region I: $W_1 U_2 W_2 U_3^{-1} W_2 U_4 = k_1 I$.
Region II: $(W_1 U_1^{-1})^n = k_2 I$.
Region III: $(W_2 M^{-1} U_3)^n = k_3 I$.
Region IV: $W_1 M^{-1} U_1 W_1 (M^{-1} U_4)^{-1} W_2 (M^{-1} U_2)^{-1} = k_4 I$.
Here I is the identity matrix and $k_j, j = 1, 2, 3, 4$, is a scalar multiple.





As a result, we obtain formulas for equations in terms of matrix entries for an arbitrary n. These are equations for the character variety of the canonical component.

The equations can be simplified further. There is an isometry of $\mathbb{H}^3$ which maps three consecutive banana endpoints P_{i-1}, P_i, P_{i+1} to P_i, P_{i+1}, P_{i+2}. It is $z \to \frac{-\xi_i}{z-1}$, where $\xi_i = \frac{|P_{i-1}P_i||P_{i+1}P_{i+2}|}{|P_{i-1}P_{i+1}||P_iP_{i+2}|}$ is the cross-ratio of distances between 4 points, called a shape parameter. One can write it in terms of edge and crossing matrix entries.



Consider $X_i = \xi_i^{-1/2} \begin{pmatrix} 0 & -\xi_i \\ 1 & -1 \end{pmatrix}$. For a non-degenerate region S, which is not a bigon, with edges labeled from 1 to n, we have

$$\prod_{i=1}^{n} X_i = \pm I.$$

This leads to multiplying k matrices rather than $2k$ for a k-sided region.