Geometric algorithms for discrete subgroups of Lie groups

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The problems

- P1. Given a hyperbolic group Γ given by its finite presentation and a homomorphism $\rho : \Gamma \to G$ to a semisimple Lie group, determine if ρ is discrete and faithful.
- P2. Given $A_1, ..., A_n \in G$ determine if the subgroup $\langle A_1, ..., A_n \rangle$ generated by these elements is discrete.
- I will discuss two geometric (semi) algorithms, one is based on the Poincaré Fundamental Polyhedron Theorem,
- the the other based on ideas from Geometric Group Theory, more precisely, Morse quasigeodesics.
- Since our time is limited, I will mostly restrict to the case when $G = SO(n, 1)$.
- In fact, both algorithms check if the subgroup of G is satisfies a condition which is stronger than discreteness, namely, some form of geometric finiteness.

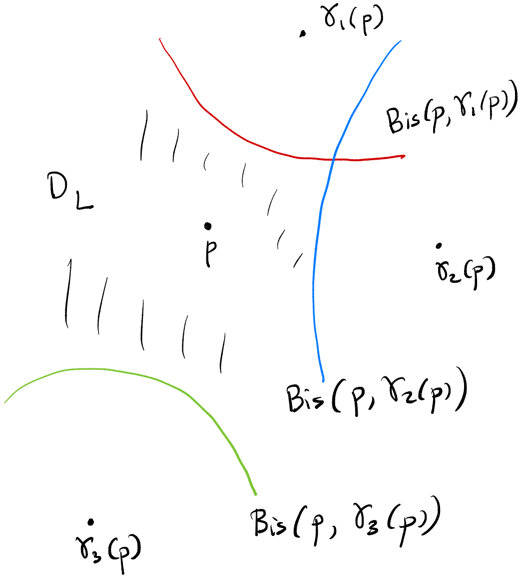
Remarks

- My lectures aim for what works (potentially) in higher rank Lie groups.
- There are other geometric algorithms for discreteness, such as algorithms of Jane Gilman and Bernard Maskit, and Jane Gilman, which are historically the first rigorous algorithms for discreteness of subgroups of $SL(2, \mathbb{R})$ with two generators. Their algorithm is different from the ones I will discuss. Jane Gilman also analyzed of the computational complexity (it's linear for their algorithm for subgroups of $SL(2, \mathbb{Q})$ and exponential for Riley's).
- There is yet another certifying maximal representations, e.g. discrete and faithful representations of surface groups to $SL(2, \mathbb{R})$; done by approximately computing the 1st Chern class of a representation.
- The discreteness problem for 2-generated subgroups of $SL(2, \mathbb{C})$ is undecidable (in a suitable sense). A proof is based on our extensive knowledge of discrete subgroups of $SL(2, \mathbb{C})$. Undecidability is unknown for subgroups of $SL(3, \mathbb{R})$.

Geometrically finite subgroups of $SO(n,1)$

- **Definition.** A subgroup Γ of $SO(n,1)$ is called geometrically finite in the sense of Ahlfors if for some $p \in \mathbb{H}^n$ the Dirichlet fundamental polyhedron $D_{\Gamma,p}$ in $\mathbb{H}^n$ has finitely many faces.
- Geometric finiteness implies discreteness.
- Dirichlet fundamental polyhedron:
$$D = D_{\Gamma,p} = \{x \in \mathbb{H}^n : d(x,p) \leq d(x,\gamma p), \forall \gamma \in \Gamma, \gamma \neq 1\},$$
p not fixed by $\gamma \in \Gamma \setminus \{1\}$.
- Below is a simplified version of the Poincaré algorithm for constructing D (Jorgensen; Riley;...).
- For each $L \in \mathbb{N}$, enumerate elements w of Γ of word length $\ell(w) \leq L$.
- Compute $D_L := \{x \in \mathbb{H}^n : d(x,p) \leq d(x,\gamma p), \ell(w) \leq L\}$. Check if D_L meets the conditions of Poincaré Fundamental Polyhedron Theorem.
- Algorithm terminates if and only if Γ is geometrically finite.
- **Question.** How to generalize this to Lie groups of higher rank, e.g. $SL(n,\mathbb{R})$?

Partial Dirichlet domain



Convex-cocompact subgroups

- **Definition.** A subgroup Γ of G, a rank 1 Lie group (say, $G = PO(n, 1)$) is called convex-cocompact if it is geometrically finite and has no parabolic elements.
- A better definition is via Geometric Group Theory.
- Let Γ be a finitely-generated group, $|\gamma|$ denotes the distance to $e \in \Gamma$ in the given Cayley graph.
- A representation $\rho : \Gamma \to G$ is called undistorted if there exists a constant $L \geq 1$ such that
- for each $\gamma \in \Gamma$
$$L^{-1}|\gamma| \leq d(1_G, \rho(\gamma))$$
- where d is a left-invariant metric on G.
- Note that such representation is automatically discrete and has finite kernel.
- **Fact of life:** It is hard to construct discrete and faithful representations of a finitely generated group whose image is not geometrically finite.

Undistorted subgroups

- **Theorem (Bowditch?).** A finitely-generated subgroup $\Gamma < G$ (say, $G = PO(n,1)$) is convex-cocompact if and only if the inclusion map $\Gamma \to G$ is undistorted.
- I will now describe a *local-to-global* test for non-distortion.
- **Definition.** Let $c : [0,N] \to (X,d)$ (a metric space) be a piecewise-geodesic path such that each $c([i,i+1])$ is a geodesic segment of length $\leq L$. Then c is an L-quasigeodesic if
- $L^{-1}|i-j| \leq d(c(i),c(j)), \quad i,j \in [0,N] \cap \mathbb{Z}.$
- Let $\rho : \Gamma \to G = Isom(X)$ be a representation. Fix $p \in X$.
- Then we have the orbit map $o_p : \Gamma \to X$, $o_p(\gamma) = \rho(\gamma)(p) \in X$.
- **Easy Observation.** ρ is undistorted if and only if there exists L such that o_p sends geodesics in the Cayley graph of Γ to L-quasigeodesics in X.

Local-to-global test for quasigeodesics

- **Theorem (Folklore?).** Suppose that c is a piecewise-geodesic path in $\mathbb{H}^n$ (or any rank 1 symmetric space X of curvature ≤ -1) whose angles at the vertices are $\geq \alpha > 0$ and whose sides are longer than a, where α and a satisfy the inequality

$$(\star) \qquad \cosh(a/2)\sin(\alpha/2) \geq \nu$$

  where $\nu > 1$ is some fixed constant, say, $\sqrt{2}$. Then c is an L-quasigeodesic. The constant L depends only on ν.
- Thus, if the orbit map sends geodesics in the Cayley graph of Γ to paths c satisfying the condition $(\star)$, then ρ is undistorted.
- Note that we just need to test the above condition $(\star)$ only on broken paths of the form

$$[\rho\gamma_i^{\pm 1}(p)p] \cup [p, \rho\gamma_j^{\pm 1}(p)]$$

  where $[\gamma_i^{\pm 1}, e] \cup [e, \gamma_j^{\pm 1}]$ are geodesic in Γ and γ_i, γ_j are generators.

Midpoint modification

- The good news: We got a truly local-to-global principle.
- The bad news: The chances that an undistorted representation satisfies the condition ($\star$) are quite slim.
- Hence, we will look for a modification procedure that converts each quasigeodesic path to one satisfying ($\star$).
- Consider a broken geodesic path in X which is the union of at least three geodesic segments:

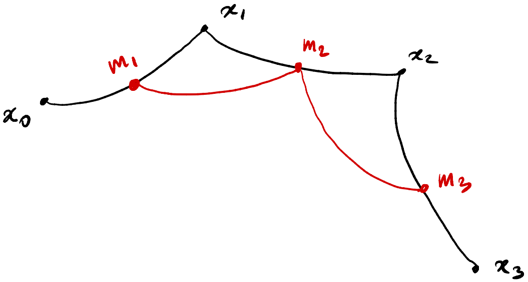
$$c = ...[x_0 x_1] \cup [x_1 x_2] \cup [x_2 x_3]...$$

- Let m_i denote the midpoint of $[x_{i-1} x_i]$.
- I'll say that c satisfies the **midpoint condition** if the associated midpoint path

$$...[m_1 m_2] \cup [m_2 m_3]...$$

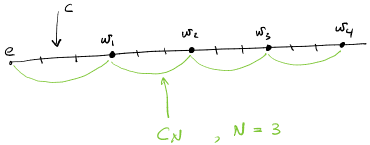
- satisfies the quasigeodesic condition ($\star$) from the page before. (As before, the condition is local.)

Midpoint modification



The KLP algorithm

- Now, given a geodesic line σ in the Cayley graph of Γ (containing e) and a natural number N,
- we define the *N-skip* biinfinite sequence σ_N by using the sequence of vertices (w_i) in it such that $w_0 = e$ and $|w_i w_{i+1}^{-1}| = N$, $i \in \mathbb{Z}$.



- **Fact.** If a representation ρ is undistorted, then there exists N (depending only on the qi constant L) such that for each geodesic σ in Γ the N-skip path σ_N is mapped via the orbit map o_p to a path satisfying the midpoint condition.
- Now, I can describe in the case of representations to $G = Isom(X)$ (e.g. $PO(n,1)$) the KLP algorithm (Kapovich-Leeb-Porti, 2014) testing if the given representation ρ is undistorted, assuming that Γ has generators $A_1, \ldots, A_m$.

The KLP algorithm

- For each natural number N, consider all geodesic N-quadruples $(1, w_1, w_2, w_3)$ of (reduced) words in $A_1^{\pm 1}, ..., A_m^{\pm 1}$,

$$d_\Gamma(e, w_1) = N, d_\Gamma(w_1, w_2) = N, d_\Gamma(w_2, w_3) = N.$$

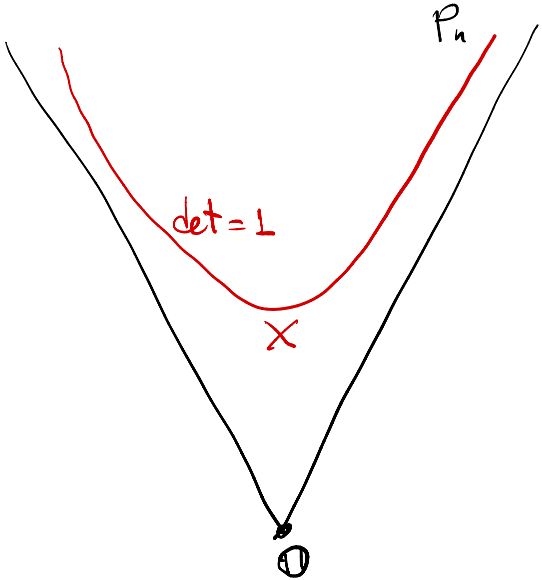
  Here, $d(e, w_2) = 2N, d(e, w_3) = 3N$.
- For every such N-quadruple, check if it's image under the orbit map satisfies the **midpoint condition** as defined above. If all such N-quadruples pass the midpoint test, the algorithm stops: This means that ρ is undistorted.
- If one of the N-quadruples $(1, w_1, w_2, w_3)$ fails the test, stop the analysis of N-quadruples, increase N by 1 and repeat.
- As a bonus, we also get an estimate from above of the quasiisometry constant of the orbit map $o_p : \gamma \to \rho(\gamma)p \subset X$ (if the algorithm stops).
- Note: The algorithm stops if and only if ρ is undistorted.

Higher rank

- In fact, the KLP algorithm was designed to works for representations to higher rank Lie groups and test for whether the given representation ρ is *Anosov*, which is one of higher rank notions replacing convex-cocompactness.
- I will not go into details, the algorithm follows the same idea but is much more complex since the inequalities replacing $(\star)$ are considerably more complicated.
- Explicit estimates for the replacement of the inequality $(\star)$ were made by Max Riestenberg in his PhD thesis this/last year.

Selberg's higher rank generalization of Dirichlet domain

- From now on, I will consider subgroups of $G = SL(n, \mathbb{R})$.
- The associated symmetric space $X = G/K = SL(n, \mathbb{R})/SO(n)$ can be identified with
- $\{M \in Sym_{n\times n}(\mathbb{R}) : M > 0, \det(M) = 1\}$.
- The group G acts on matrices M by $M \mapsto g^T M g$.
- The trouble is that it's hard to compute with the G-invariant Riemannian distance function d on X.
- Also, bisectors $Bis_d(A, B) = \{M : d(A, M) = d(B, M)\}$ are not totally geodesic.
- I will describe a 2-point invariant $s(A, B)$ due to Selberg (1960), which, while not a metric, can be used in lieu of one to define Dirichlet domains in X, more precisely, in $P_n = \{A \in Sym_{n\times n}(\mathbb{R}) : A > 0\}$, the convex cone of positive-definite matrices in the space of symmetric matrices.
- The advantage of $s(A, B)$ is that it's easy to compute and the corresponding bisectors are linear.


P_n
det = 1
X
0

The invariant

- For $A, B \in P_n$, $s(A, B) := \log(\frac{1}{n} tr(A^{-1}B))$.
- Then s is G-invariant because trace is conjugacy-invariant.
- The normalization is chosen so that $s(A, B) \geq 0$ with equality if and only if $A = B$.
- However, $s(A, B) \neq s(B, A)$ and s fails the triangle inequality.
- The s-bisectors:

  $$Bis(A_1, A_2) = \{B : tr(A_1^{-1}B) = tr(A_2^{-1}B)\}$$

  are defined by equation linear in the variable B. Hence, s-bisectors are linear.
- Another useful property: The function $B \mapsto s(I, B)$ is proper when restricted to X, because of the ...
- comparison to the invariant Finsler metric d_{max} on X: $s(I, A) \leq d_{max}(I, A) \leq s(I, A) + \log(n)$.

Selberg-Dirichlet domain

- Selberg-Dirichlet domain for a discrete subgroup $\Gamma < G$ such that $\Gamma \cap O(n) = I$.
- $D(\Gamma, I) = \{M \in P_n : s(I, M) \leq s(\gamma, M) \quad \forall \gamma \in \Gamma \setminus \{I\}\}$.
- This is indeed a fundamental domain of Γ if it is discrete.
- The key property to check (follows from properness of $s(I, M)$ on X):

$$\lim_{||A|| \to \infty} d(I, Bis(I, A) \cap X) = \infty,$$

- whenever $A \in X = \{\det = 1\} \cap P_n$, our symmetric space.
- Linearity of bisectors implies that $D(\Gamma, I)$ is a convex polyhedral cone.

Questions

- The definition suggests a slew of questions, all open. For instance:
- Which discrete subgroups admit finitely-sided Selberg-Dirichlet domains?
- Uniform lattices do, but what about non-uniform ones? Anosov subgroups?
- Is there a simple condition on matrices $A, B \in X$ which is necessary and sufficient for disjointness of the bisectors $Bis(I, A), Bis(I, B)$ in P_n?
- There is an analogue of the Poincaré Fundamental polyhedron theorem for intersections of half-spaces defined by $s(I, \cdot)$. But the "ridge cycle condition" is more complex than the one for the hyperbolic space. Is there an analogue of the angle between bisectors so that one can formulate a replacement of the condition

$$\sum \angle_i = 2\pi$$

for each ridge-cycle defined by face-pairing transformations?

Poincare Algorithm and faithfulness

- The/A Poincare algorithm is useful even if you dealing with a subgroup which you already know is discrete, e.g. a subgroup of $GL(n, \mathbb{Z})$.
- Namely, a finite-sided fundamental domain gives you not only generators but also relators.
- For instance, given two matrices $A_1, A_2 \in GL(n, \mathbb{Z})$, the algorithm gives you a new set of generators $B_1 = w_1(A_1, A_2), ..., B_k = w_k(A_1, A_2)$ and defining relators $S_1, .., S_r$ in the new generators.
- Now, rewrite the relators S_i in terms of A_1, A_2, we get $R_1, ..., R_s$.
- Then the homomorphism $F(A_1, A_2) \to \Gamma = \langle A_1, A_2 \rangle$ is faithful if and only if $R_1, ..., R_s$ are all trivial.
- Aparently, Igor Rivin tested a version of the Selberg-Poincare Algorithm for subgroups of $GL(n, \mathbb{Z})$, $n = 3, 4$.
- The only known to me implementations are for subgroups of $SL(2, \mathbb{C})$ (Snappy) and $PU(2, 1)$ (Deraux; Cartwright–Steger).

Poincare Algorithm for finding a finite presentation of a uniform lattice

- Let $\Gamma < SL(n, \mathbb{R})$ be a uniform lattice given by its arithmetic data.
- For each $N \in \mathbb{N}$ find the subset Γ_N matrices $A \in \Gamma$ such that $s(I, A) \leq N$ — a finite search.
- Compute the intersection C_N of closed half-spaces $\{s(I, x) \leq s(A, x)\}$ for $A \in \Gamma_N$.
- Check if this intersection is contained in the cone P_n of positive-definite matrices.
- If it is not, then increase N to $N + 1$ and repeat.
- If yes, then compute the Selberg-radius $\delta = \max\{s(I, x) : x \in D_N\}$ of $D_N := C_N \cap X$ (imposing the condition $\det = 1$).
- Find all $A \in \Gamma$ such that $N \leq s(I, A) \leq 2\delta + 2\log(n)$.
- $D_{2\delta+2\log(n)}$ will be the Selberg-Dirichlet domain of Γ.