Lightning Talks Wednesday June 16, 2021 1:45 PM – 2:45 PM

Speaker List:

Jonah Gaster
Nikolay Bogachev
Aleksandr Kolpakov
Max Riestenberg
Julien Paupert

ICERM
6/16

VERTICAL ARCS AND THE MARKOV UNICITY CONJECTURE

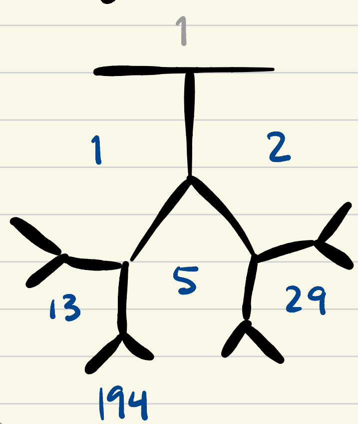
Jonah Gaster
UW-Milwaukee

MUC concerns the Markov numbers $\mathcal{M} = \{1, 2, 5, 13, \dots\}$

i.e. integer solutions to $x^2 + y^2 + z^2 = 3xyz$

The symmetry $(x, y, z) \longleftrightarrow (x, y, 3xy - z)$ leads to

the labelling

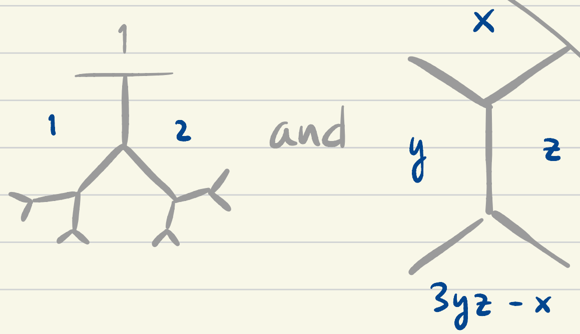


$\lambda_{\mathcal{M}} : \mathcal{H} \longrightarrow \mathcal{M}$

$\mathcal{H}$:

Exerc. $\lambda_{\mathcal{M}}$ is onto.

inductively:

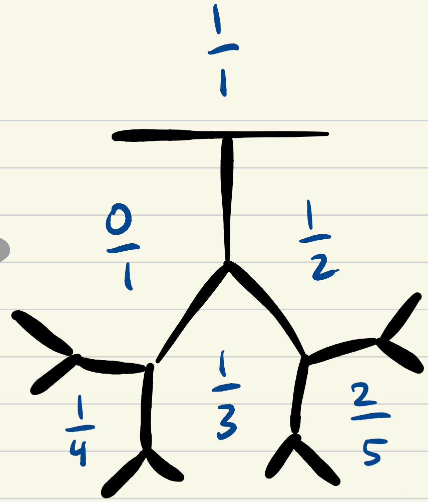


MUC : $\lambda_{\mathcal{M}}$ is 1-1

Manifestations in: ARITHMETIC, COMBINATORICS, GEOMETRY, DIOPHANTINE APPROXIMATION

see "Don't try to solve these problems!" (Guy, 83)

Related:

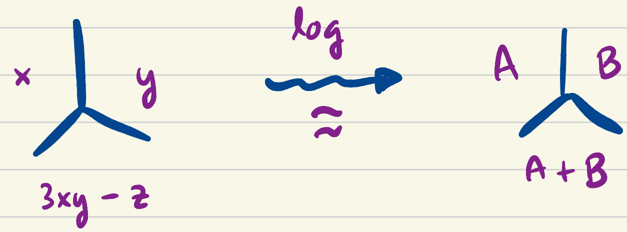


again, inductively!

$\lambda_{\mathcal{F}} : \mathcal{H} \longrightarrow \mathbb{Q} \cap [0, \frac{1}{2}]$.

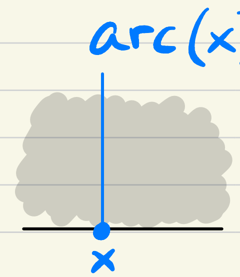
(Zagier) $\operatorname{den}(\lambda_{\mathcal{F}}) \approx \log \lambda_{\mathcal{M}}$

NB.



Rmk. $\lambda_{\mathcal{F}}$ is 1-1.

Q: Is there a "fractional Markov labelling"?



Thm (G.): $n \in \mathcal{M} \iff \exists\, k \in \mathbb{N}$ s.th. $\operatorname{arc}(k/n)$ projects simply to the modular torus X

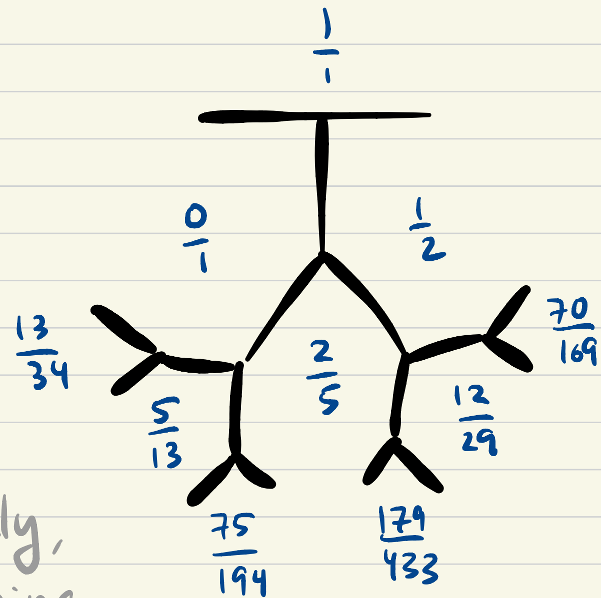
Rmk. Compare with the well-known: see Cohn, Series

$$3\mathcal{M} = \{\text{traces of simple closed geodesics on } X\}$$

COR. $\mathcal{M} = \{$"arithmetic heights" of simple proper vertical arcs on $X\}$

some consequences:

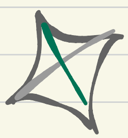
(i) $\exists$ fractional Markov labelling $\lambda_{FM} : \mathcal{H} \longrightarrow \mathbb{Q} \cap [0, \frac{1}{2}]$



- λ_{FM} is 1-1. "pf": like λ_F

- Suppose $\dfrac{k}{n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \frac{1}{a_r}}}$

inductively, via flipping



Then $\lambda_F \circ \lambda_{FM}^{-1}\left(\frac{k}{n}\right) = \dfrac{\#\{a_i = 2\}}{1 + \frac{1}{2}\sum a_i}$

(ii) MUC for $n = p^r$: Lemma. $k^2 \equiv -1 \pmod{n}$

NB. Not new! see Baragar, Lang-Tan, Zhang ...

Pf idea: simple vert. arcs $\subsetneq$ elliptic vert. arcs $\subsetneq$ vert. arcs

Q: Who is "k"? Why $\left(\mathbb{Z}/n\mathbb{Z}\right)^*$?

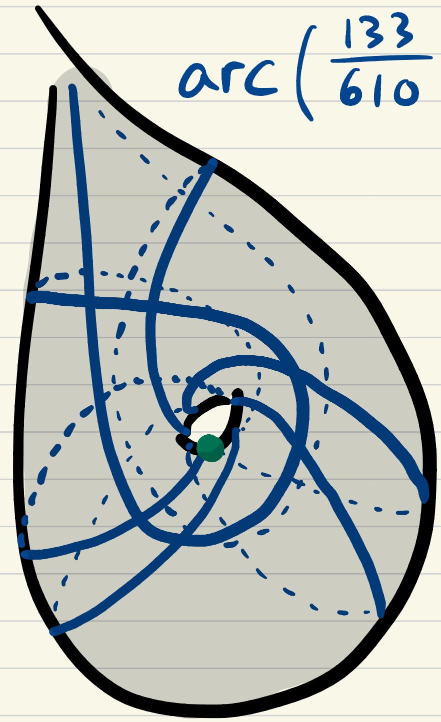
For instance, $\mathrm{arc}\left(\frac{k}{n}\right)$ projects simply to X iff $\exists\, x, y \in \mathbb{Z}$ s. th.

$$n x^2 - (2k + 3n)\, xy + \left(3k + \frac{1+k^2}{n}\right) y^2 = -n$$

using hyperbolic geometry

Observe: the "fake k's" have topological meaning as non-simple proper geodesic arcs on X preserved by the elliptic involution.

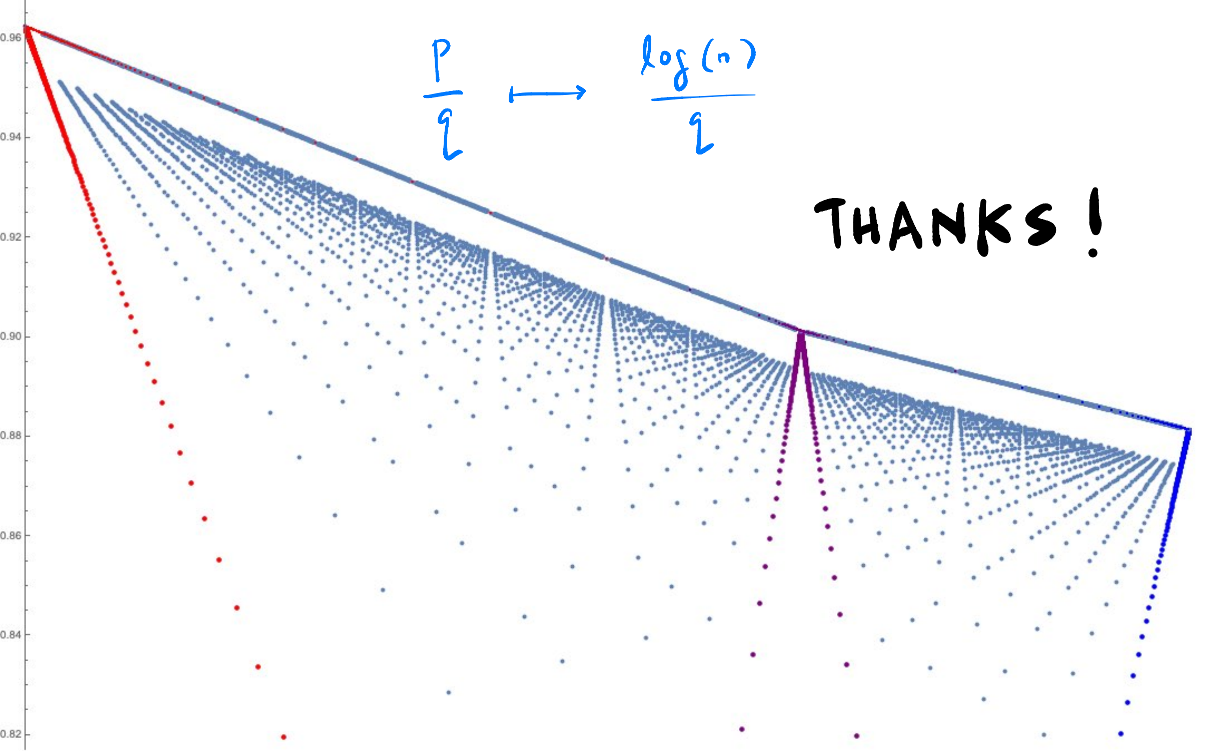
EX:



Is there some RECIPE to produce those ?

GOAL. For each p, q s. th. $\exists\ x^2 \equiv -1 \pmod{p \cdot q}$, produce a non-simple elliptic arc with height $p \cdot q$

(!) As a corollary, you would obtain MUC for $n = p \cdot q$



0.96
0.94
0.92
0.90
0.88
0.86
0.84
0.82
$\frac{p}{q} \longmapsto \frac{\log(n)}{q}$
THANKS!

Geometric and arithmetic properties of hyperbolic orbifolds, and the Vinberg algorithm

Nikolay Bogachev (Skoltech & MIPT)

Vinberg
1967 Theory of hyperbolic reflection gps
1972 Algorithm
1981 No compact Coxeter polytopes in $\mathbb{H}^{\geq 30}$.

Hard problems:
1) Constructing new Coxeter polytopes
2) Classification of arithmetic hyperbolic refl gps
3) Efficient methods and tools for 1) и 2).
(and convenient/user friendly).

Methods and tools
1) Vinberg's algorithm
2) Nikulin's methods (see also Allcock)
3) Scharlau's approach
4) Faces of higher-dim polytopes (Borcherds)

Results:

1) Recent software implementations of the Vinberg algorithm (AlVin 2016 Guglielmetti
VinAl 2017–20 Bogachev, Perepechko
VinAlNF 2021) Bottinelli

2) Method of small ridges (Bogachev '2018-20)

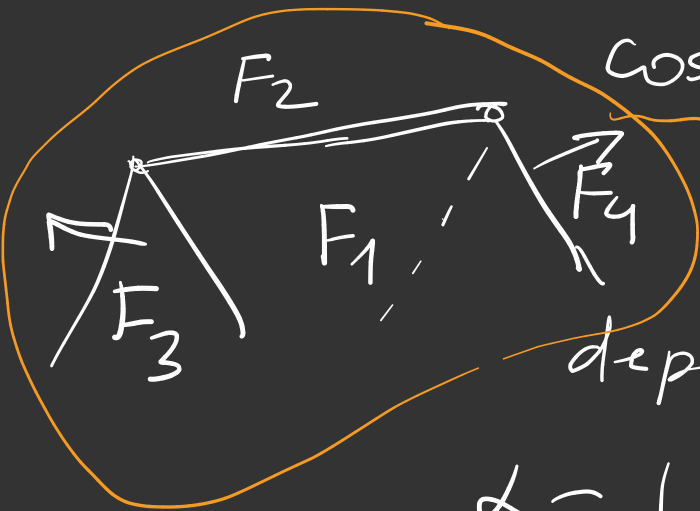
3) Faces of quasi-arith Coxeter polytopes (Bogachev, Kolpakov 2020)

The key idea: geometry and arithmetic help to each other!

4) Totally geodesic subspaces of hyperbolic orbifolds (Belolipetsky, Bogachev, Kolpakov, Slavich - 2021)

Thm (Bogachev, 2018-2020).

(1) Let P be a compact Coxeter polytope in H^3. Then P has an edge:



a ridge

$$\cosh \rho(F_3, F_4) \le t_\alpha < 5{,}75$$

where t_α is a number depending on the set

$$\alpha = (\alpha_{12}, \dots, \alpha_{34}) \text{ only,}$$

where $\alpha_{ij} = \angle(F_i, F_j)$.

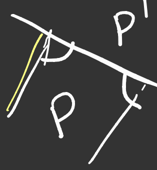
(2) If P is a compact Coxeter polytope in H^n, $n \ge 4$, and it has a 3-dim Coxeter face, then it has a ridge of width $\le t_\alpha < 5{,}75$

(3) If a small ridge of a Coxeter polytope in H^n is right-angled, then its width < 2.

Thm (Bogachev & Kolpakov, 2020)

1) Let P be a quasi-arithmetic (over k) Coxeter polytope in $\mathbb{H}^n$ and let P' be a k-dim face of P ($2 \le k \le n-1$). If P' is a Coxeter polytope, then P' is also quasi-arith and also over k.

2) If P is an arithmetic Coxeter polytope and P' is a facet (codim $= 1$) of P, and supporting hyperplane

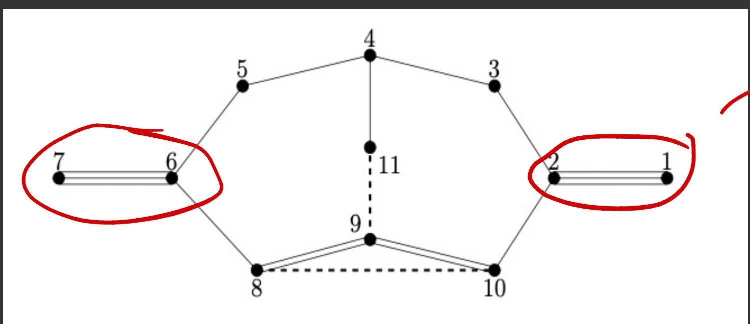


$H_e \supset P'$ meets with adjacent facets at "even" angles (of the form $\frac{\pi}{2m}$),

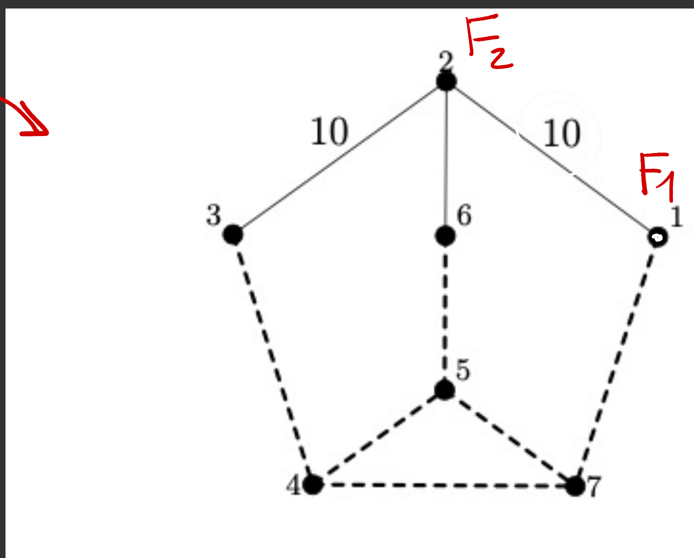
then P' is a Coxeter polytope and is arithmetic.

Example (by PLoF) $n=2$ Vinberg 2012

Bugaenko's polytope $P \subset \mathbb{H}^7$, $k(P) = \mathbb{Q}(\sqrt{5})$. It has a 3-dim arithmetic Coxeter face P':



P'



A 3-dimensional Coxeter face P' of P.

And this 3-dim P' has facets F_1 and F_2.

By our Thm 1) F_1 and F_2 are quasi-arithmetic, and by Thm 2) F_1 is arithmetic.

But one can check that F_2 is properly quasi-arithmetic

Belolipetsky, Bogachev, Kolpakov, and Slavich

"Subspace stabilisers in hyperbolic lattices"

Thm

A finite volume hyperbolic orbifold $M = \mathbb{H}^n/\Gamma$ is arithmetic if and only if all its totally geod. subspaces are fc-subspaces (i.e. correspond to finite subgps of $\mathrm{Comm}(\Gamma)$) and there are infinitely many of them.

If M is (quasi-) arithmetic over k then all its fc-subspaces are of the same type.

There are more results and examples....

Computing reflection centralisers in hyperbolic reflection groups

A. Kolpakov (Université de Neuchâtel), joint with N. Bogachev (Skoltech)

Howlett, 1980; Brink, 1996

Structure of centralisers in Coxeter groups

Let (W, S) = Coxeter system, and $s \in S$ a simple reflection.

$C_W(s) = \langle s \rangle \times \langle W_\Omega, \Gamma_\Omega \rangle$ = the centraliser of s in W.

Here W_Ω is generated by the reflections in $C_W(s)$ other than s, and $\Gamma_\Omega = \pi_1$("odd" Coxeter diagram of (W, S)).

Allcock, 2013

Structure of centralisers in Coxeter groups

Explicit set of generators: we only need to use linear algebra to get them, once (W, S) is given: either $W < \mathrm{Isom}(\mathbb{H}^n)$ or W has geometric representation (Jacques Tits).

If the "odd" diagram of (W, S) has no cycles, then $C_W(s) = \langle s \rangle \times W_\Omega$.

Now we can …

Project underway

Program (in SageMath) a fairly general version of the algorithm. Play around with it (soon on GitHub).

Prove that once Γ_Ω is trivial then W_Ω gives a new (quasi-)arithmetic reflection lattice once W is a (quasi-)arithmetic lattice (have more general results in Belolipetsky, Bogachev, K., Slavich, arXiv:2105.06897).

Thank you !

ICERM Lightning Talk

Max Riestenberg

June 2021

A quantified local-to-global principle for Anosov representations

Max Riestenberg
Universität Heidelberg
June 2021

Discrete subgroups of Lie groups

closed surface groups in $PSL(2,\mathbb{R})$

undistorted subgroups in $Isom(\mathbb{H}^n)$

Anosov representations in semisimple G

$\pi_1 S \to PSL(2,\mathbb{R})$

discrete & faithful

a.k.a. convex cocompact

$\Gamma \to G$

Gromov hyperbolic

real semisimple Lie group

Anosov

Undistorted subgroups in negative curvature

Def: A finitely generated subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is **undistorted** if any orbit map is a quasi-isometric embedding: $\exists\, p \in \mathbb{H}^n$, $c_1, c_2, c_3, c_4 > 0$ such that $\forall \gamma \in \Gamma$,

$$\frac{1}{c_1}|\gamma| - c_2 \leq d_{\mathbb{H}^n}(p, \gamma p) \leq c_3 |\gamma| + c_4 .$$

word length of γ

Facts:

① $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is undistorted if and only if any orbit map sends geodesics to quasigeodesics

② $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ undistorted $\Rightarrow$ Γ discrete and stable

Semisimple Lie groups & higher rank symmetric spaces

$$SL(3,\mathbb{R}) \curvearrowright X = \left\{ \begin{matrix} \text{real } 3\times 3 \text{ symmetric positive definite} \\ \text{matrices with determinant } 1 \end{matrix} \right\}$$

$$g \cdot x = g x g^T$$

$$F = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} a_1, a_2, a_3 > 0 \\ a_1 a_2 a_3 = 1 \end{matrix} \right\} \subseteq X$$

maximal flat

$$\text{higher rank} \iff \mathbb{R}^2 \xhookrightarrow{\text{totally geodesic}} X$$

In $\mathbb{H}^n$:

Undistorted subgroups are stable

$\Uparrow$

The local-to-global principle

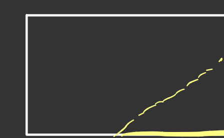
$\Uparrow$

The Morse Lemma

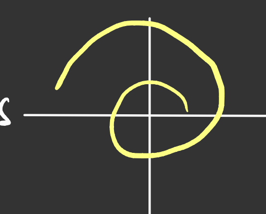
Challenges in higher rank:

Undistorted subgroups are no longer stable

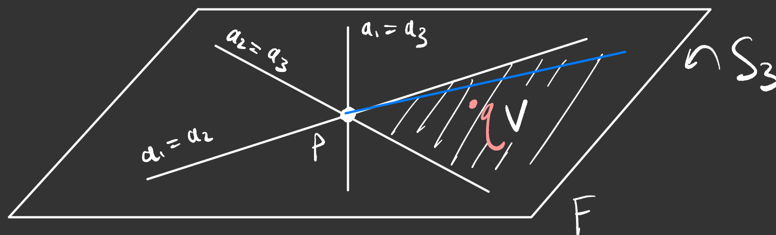
The local-to-global principle fails



The Morse Lemma fails



Weyl chambers and regularity



$$F = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} a_1, a_2, a_3 > 0 \\ a_1 a_2 a_3 = 1 \end{array} \right\}$$

$$\cup$$

$$V = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} a_1 \geq a_2 \geq a_3 > 0 \\ a_1 a_2 a_3 = 1 \end{array} \right\}$$

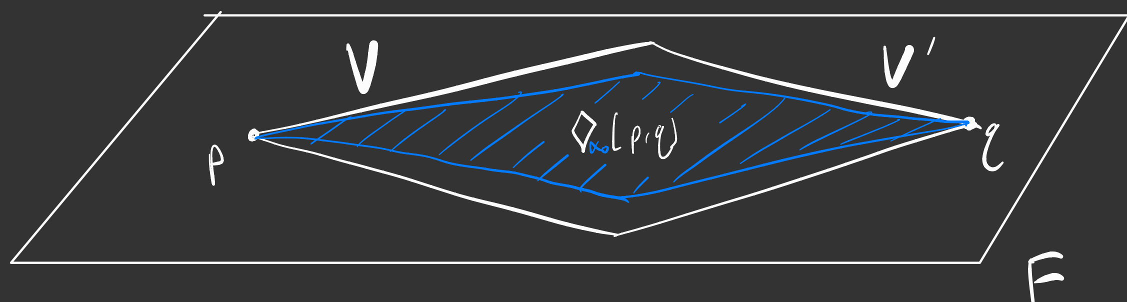
Weyl chamber

Def A point $q \in V$ is <u>α_0-regular</u> if $\sin \angle_p (q, \text{any wall}) \geq \alpha_0 > 0$.

Diamonds

Def: For $q \in V$ regular

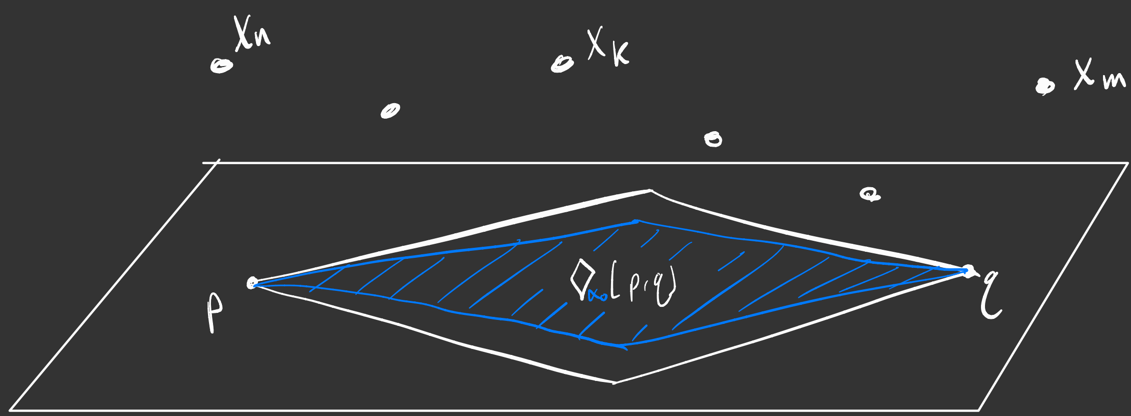
$$\Diamond(p,q) := V \cap V'$$



Morse quasigeodesics

Def: A (c_1, c_2, c_3, c_4)-quasigeodesic (x_n) in X is <u>(α_0, D)-Morse</u> if $\forall\ n < m$, $\exists\ p, q \in X$ so that

$$d(x_n, p) \leq D, \quad \forall\ n \leq k \leq m,\ d(x_k, \Diamond_{\alpha_0}(p,q)) \leq D, \quad d(x_m, q) \leq D$$



The local-to-global principle for Morse quasigeodesics

<u>Def / Thm</u> (Kapovich-Leeb-Porti 2014) (Labourie, Guichard-Wienhard)

A representation $\rho: \Gamma \to G$ is <u>Anosov</u> if any orbit map $\Gamma \to X$ sends geodesics to (α_0, D)-Morse (c_1, c_2, c_3, c_4)-quasigeodesics.

<u>Theorem</u> (Kapovich-Leeb-Porti 2014) (local-to-global principle for Morse quasigeodesics)

$\forall\ \alpha_0 > \alpha_0'$, D, c_1, c_2, c_3, c_4 $\exists$ a scale L so that:

Every L-local (α_0, D)-Morse (c_1, c_2, c_3, c_4)-quasigeodesic is an (α_0', D')-Morse (c_1', c_2', c_3', c_4')-quasigeodesic.

The quantified local-to-global principle for Morse quasigeodesics

(R. 2020)

<u>Theorem</u> (Kapovich-Leeb-Porti 2014) (local-to-global principle for Morse quasigeodesics)

$\forall\, \alpha_0 > \alpha_0'$, D, c_1, c_2, c_3, c_4 $\exists$ an explicit scale L so that:

Every L-local (α_0, D)-Morse (c_1, c_2, c_3, c_4)-quasigeodesic is an (α_0', D')-Morse (c_1', c_2', c_3', c_4')-quasigeodesic.

Explicit perturbation neighborhoods for Anosov representations

Let $S = \left\{ \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}\right\} \right\}$

with $\log\lambda = \cosh^{-1}(\cot \pi/8)$ generate a subgroup Γ of $SL(3,\mathbb{R})$.

Theorem (R. 2020)

If $\rho : \Gamma \to SL(3,\mathbb{R})$ satisfies $|\rho(s) - s| \leq 10^{-3,698,433}$ for all $s \in S$, then ρ is Anosov.

Thanks!

Presentations for cusped arithmetic hyperbolic lattices

Alice Mark (Vanderbilt University),
Julien Paupert (Arizona State University)

We present a method to compute in principle a presentation for any cusped, arithmetic hyperbolic lattice, and implement it for:

- the Picard modular groups $\mathrm{PU}(2,1,\mathcal{O}_d)$ with $d = (1,3)$, 7 ($d = 3, 1$: Falbel–Parker '06, Falbel–Francsics–Parker '11)
- the Hurwitz modular group $\mathrm{PSp}(2,1,\mathcal{H})$ with $\mathcal{H} = \mathbb{Z}[i, j, k, \frac{1+i+j+k}{2}]$ the ring of Hurwitz integers
- David Polletta successfully applied this method to the Picard modular groups with $d = 2, 11$.

Rough idea:

Let X be a negatively curved symmetric space, Γ a non-cocompact lattice in $G = \mathrm{Isom}(X)$, $\infty \in \partial_\infty X$ a parabolic fixed point of Γ and $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$.
Assume that Γ has a single cusp, and let $B = B_u$ denote the (open) horoball at height $u > 0$ based at ∞.

- For $u > 0$ small enough, X is covered by ΓB_u
- Then by Macbeath's theorem, Γ admits a presentation whose generators correspond to pairwise intersections $B \cap \gamma B$, and whose relations correspond to triple intersections $B \cap \gamma B \cap \gamma' B$.
- Both of these sets are infinite, but they are finite modulo the action of Γ_∞
- (... make this effective)

Example: a Γ-adapted horoball covering for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$.

Questions: Assume Γ has a single cusp.

- How can we find "the" optimal height u_0 such that ΓB_{u_0} covers X?
- How can we understand/control the corresponding horoball intersections?

Idea: Use levels/depths coming from arithmetic structure (and geometry!)

Computational output

(#generators,#relations)

- For $d = 1, 3$ the covering depth is 4; we obtain (8,247), resp. (8,583). Magma reduces this to (2,6) for $d = 1$ and $(3, 19)$ for $d = 3$.
- For $d = 7$ the covering depth is 7; we obtain (18,406); Magma reduces this to (3,13).
- (Polletta) For $d = 2$ the covering depth is 16; he obtains (54,$\sim$ 5800) (!); Magma reduces this to $(3, 29)$.
  For $d = 11$ the covering depth is 43; he obtains (263,$\sim$ 24,000) (!!); Magma reduces this to $(5, 26)$.
- For the Hurwitz modular group the covering depth is (at most) 4. We obtain $(33, \sim 10^6)$:(Magma is not happy either, but we can cajole it into giving some results about the group, eg abelianization and nice generating sets.

$$\Gamma(7) = \langle T, R, I \mid$$
$$R^2 = \mathrm{Id}$$
$$I^2 = \mathrm{Id}$$
$$(RI)^2 = \mathrm{Id}$$
$$RTRT = TRTR$$
$$(TIT^{-1}R)^4 = \mathrm{Id}$$
$$(T^{-1}ITR)^4 = \mathrm{Id}$$
$$T^{-1}IT^{-1}ITITIT^{-3}ITITIT^{-1}IT^{-1} = \mathrm{Id}$$
$$(T^{-1}ITITIT^{-1}IT^{-1}I)^2 = \mathrm{Id}$$
$$(IT^{-1}R)^7 = \mathrm{Id}$$
$$T^{-1}ITITIT^{-2}IT^{-1}ITIT^2IT^{-1}IT^{-1}ITI = \mathrm{Id}$$
$$T^{-1}ITITIRTIRTITIT^{-1}IT^{-1}ITRT^{-1}IRT^{-1}I = \mathrm{Id}$$
$$RTIRTITIT^{-1}IT^{-1}IRT^{-1}IRT^{-1}IT^{-1}ITITIT^{-1} = \mathrm{Id}$$
$$RTIRTRT^{-1}ITITIRTITIT^{-1}RTRITRT^{-1}ITITITIT^{-1} = \mathrm{Id}\rangle$$