Lightning Talks
Wednesday June 16, 2021 1:45 PM – 2:45 PM

Speaker List:

Jonah Gaster
Nikolay Bogachev
Aleksandr Kolpakov
Max Riestenberg
Julien Paupert
MUC concerns the Markov numbers \( M = \{1, 2, 5, 13, \ldots\} \) i.e. integer solutions to \( x^2 + y^2 + z^2 = 3xyz \).

The symmetry \( (x, y, z) \leftrightarrow (x, y, 3xy - z) \) leads to the labelling

\[
\begin{align*}
1 & \quad 1 \\
2 & \quad 13 \\
5 & \quad 29 \\
194 & \\
\end{align*}
\]

The labelling gives \( \lambda_m : H \rightarrow M \).

**Exerc.** \( \lambda_m \) is onto.

**MUC**: \( \lambda_m \) is 1-1

Manifestations in: **ARITHMETIC**, **COMBINATORICS**, **GEOMETRY**, **DIOPHANTINE APPROXIMATION**

see “Don’t try to solve these problems!” (Guy, 83)
Related:

Again, inductively!

\[ \lambda_f : \mathbb{H} \to \mathbb{Q} \cap [0, \frac{1}{2}] . \]

(Zagier) \( \text{den}(\lambda_f) \approx \log \lambda_m \)

\[ \text{NB. } \frac{\log A}{3xy - z} \approx \frac{A}{A + B} \]

**Rmk.** \( \lambda_f \) is 1-1.

Q: Is there a “fractional Markov labelling”?

**Thm (G.):** \( n \in \mathbb{M} \iff \exists k \in \mathbb{N} \text{ s.t. } \text{arc}(\frac{k}{n}) \text{ projects simply to the modular torus } \mathbb{X} \)

**Rmk.** Compare with the well-known:

\[ \mathbb{M} = \{ \text{traces of simple closed geodesics on } \mathbb{X} \} \]

**Cor.** \( \mathbb{M} = \{ \text{“arithmetic heights” of simple proper vertical arcs on } \mathbb{X} \} \)

see Cohn, Series

3M = \{ traces of simple closed geodesics on X \}

\[ \frac{1}{1} \]

\[ 0 \quad \frac{1}{2} \]

\[ \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \]

\[ \frac{3}{4} \quad \frac{1}{2} \quad \frac{3}{5} \]

\[ \frac{5}{4} \quad \frac{3}{2} \quad \frac{5}{3} \]

\[ \frac{7}{4} \quad \frac{5}{2} \quad \frac{7}{3} \]

\[ \frac{9}{4} \quad \frac{7}{2} \quad rac{9}{3} \]

\[ \frac{11}{4} \quad \frac{9}{2} \quad \frac{11}{3} \]

\[ \frac{13}{4} \quad \frac{11}{2} \quad \frac{13}{3} \]

\[ \frac{15}{4} \quad \frac{13}{2} \quad \frac{15}{3} \]

\[ \frac{17}{4} \quad \frac{15}{2} \quad \frac{17}{3} \]

\[ \frac{19}{4} \quad \frac{17}{2} \quad \frac{19}{3} \]

\[ \frac{21}{4} \quad \frac{19}{2} \quad \frac{21}{3} \]

\[ \frac{23}{4} \quad \frac{21}{2} \quad \frac{23}{3} \]

\[ \frac{25}{4} \quad \frac{23}{2} \quad \frac{25}{3} \]

\[ \frac{27}{4} \quad \frac{25}{2} \quad \frac{27}{3} \]

\[ \frac{29}{4} \quad \frac{27}{2} \quad \frac{29}{3} \]

\[ \frac{31}{4} \quad \frac{29}{2} \quad \frac{31}{3} \]

\[ \frac{33}{4} \quad \frac{31}{2} \quad \frac{33}{3} \]

\[ \frac{35}{4} \quad \frac{33}{2} \quad \frac{35}{3} \]

\[ \frac{37}{4} \quad \frac{35}{2} \quad \frac{37}{3} \]

\[ \frac{39}{4} \quad \frac{37}{2} \quad \frac{39}{3} \]

\[ \frac{41}{4} \quad \frac{39}{2} \quad \frac{41}{3} \]

\[ \frac{43}{4} \quad \frac{41}{2} \quad \frac{43}{3} \]

\[ \frac{45}{4} \quad \frac{43}{2} \quad \frac{45}{3} \]

\[ \frac{47}{4} \quad \frac{45}{2} \quad \frac{47}{3} \]

\[ \frac{49}{4} \quad \frac{47}{2} \quad \frac{49}{3} \]

\[ \frac{51}{4} \quad \frac{49}{2} \quad \frac{51}{3} \]

\[ \frac{53}{4} \quad \frac{51}{2} \quad \frac{53}{3} \]

\[ \frac{55}{4} \quad \frac{53}{2} \quad \frac{55}{3} \]

\[ \frac{57}{4} \quad \frac{55}{2} \quad \frac{57}{3} \]

\[ \frac{59}{4} \quad \frac{57}{2} \quad \frac{59}{3} \]

\[ \frac{61}{4} \quad \frac{59}{2} \quad \frac{61}{3} \]

\[ \frac{63}{4} \quad \frac{61}{2} \quad \frac{63}{3} \]
some consequences:

(i) \exists fractional Markov labelling \( \lambda_\text{FM} : \mathcal{H} \to \mathbb{Q} \cap [0, \frac{1}{2}] \)

\[ \begin{array}{c}
1 \\
0 \\
\frac{1}{2} \\
\frac{13}{34} \ rac{5}{13} \\
\frac{2}{5} \ rac{70}{169} \\
\frac{12}{29} \\
\end{array} \]

- \( \lambda_\text{FM} \) is 1-1. "Pf" : like \( \lambda_T \)

\[ \begin{align*}
\text{inductively, via flipping} \\
\text{via flipping} \\
\end{align*} \]

Then \( \lambda_\text{FM} \cdot \lambda_\text{FM}^{-1}(\frac{k}{n}) = \frac{\# \{ a_i = 2 \}}{1 + \frac{1}{2} \sum a_i} \)

(ii) MUC for \( n = p^r \):

\( \text{Lemma.} \ k^2 \equiv -1 \pmod{n} \)

\( \text{NB.} \) Not new!

Pf idea: simple vert. \( \subseteq \) elliptic vert. \( \subseteq \) vert. arcs + arcs

see Baragar, Lang-Tan, Zhang ...

\[ Q: \ \text{Who is "} k \text{"? Why } (\mathbb{Z}/n\mathbb{Z})^*? \]
For instance, $\text{arc}(\frac{k}{n})$ projects simply to $X$ iff $\exists x, y \in \mathbb{Z}$ s.t.

$$nx^2 - (2k + 3n)x y + (3k + \frac{1+k^2}{n})y^2 = -n$$

Using hyperbolic geometry.

Observe: the \textit{"fake $k$'s"} have topological meaning as \underline{non-simple proper geodesic arcs} on $X$ preserved by the \underline{elliptic involution}.

Is there some \underline{RECIPE} to produce those?

\underline{GOAL}. For each $p,q$ s.t.

$\exists x^2 \equiv -1 \pmod{p,q}$, produce a \underline{non-simple elliptic arc}

with height $p,q$

(1) As a corollary, you would obtain $\text{MUC}$ for $n = p \cdot q$
\[ \frac{P}{q} \implies \frac{\log(n)}{q} \]

THANKS!
Geometric and arithmetic properties of hyperbolic orbifolds, and the Vinberg algorithm

Nikolay Bogachev (Skoltech & MIPT)

Vinberg
1967
1972
1981
- Theory of hyperbolic reflection gps
- Algorithm
- No compact Coxeter polytopes in \( \mathbb{H}^n \)

Hard problems:
1) Constructing new Coxeter polytopes
2) Classification of arithmetic hyperbolic reflgps
3) Efficient methods and tools for 1) & 2)
   (and convenient/user friendly)

Methods and tools:
1) Vinberg’s algorithm
2) Nikulin’s methods (see also Allcock)
3) Scharlau’s approach
4) Faces of higher-dim polytopes (Borchards)

Results:
1) Recent software implementations of the Vinberg algorithm
   (AlViin 2016, Guglielmetti, VinAl 2017-20, Bogachev, Perepeiko, VinAllNFT 2021, Bottinelli)
2) Method of small ridges (Bogachev 2018-20)
3) Faces of quasi-arith Coxeter polytopes (Bogachev, Kolpakov 2020)

The key idea: geometry and arithmetic help to each other!

4) Totally geodesic subspaces of hyperbolic orbifolds
   (Belolipetsky, Bogachev, Kolpakov, Slavich - 2021)
**Thm (Bogadchev, 2018-2020).**

1. Let $P$ be a compact Coxeter polytope in $H^3$. Then $P$ has an edge $e$ such that

   $\cosh \rho(F_3, F_4) \leq t_2 < 5.75$

   where $t_2$ is a number depending on the set $\mathbf{d} = (d_{12}, \ldots, d_{34})$ only,

   where $d_{ij} = \angle(F_i, F_j)$.

2. If $P$ is a compact Coxeter polytope in $H^n$, $n \geq 4$, and it has a 3-dim Coxeter face, then it has a ridge of width $\leq t_2 < 5.75$.

3. If a small ridge of a Coxeter polytope in $H^n$ is right-angled, then its width $< 2$. 
Thm (Bogachev & Kolpakov, 2020) (over $k$)

1) Let $P$ be a quasi-arithmetic Coxeter polytope in $H^n$ and let $P'$ be a $k$-dim facet of $P$ ($2 \leq k \leq n-1$).

If $P'$ is a Coxeter polytope, then $P'$ is also quasi-arithmetic and also over $k$.

2) Let $P$ be a quasi-arithmetic Coxeter polytope and $P'$ is a facet ($\text{codim} = 1$) of $P$, and supporting hyperplane $P' \cap P$ meets with adjacent facets at the "even" angles (of the form $\frac{\pi}{2m}$).

Then $P'$ is a Coxeter polytope and is arithmetic.

Example (by PLLoF) $n=2$ Vinberg 2012

Bogachev's polytope $P \subset H^7$, $k(P) = Q(\sqrt{5})$. It has a 3-dim arithmetic Coxeter face $P'$:

![Diagram showing a 3-dimensional Coxeter face $P'$ of $P$.]

And this 3-dim $P'$ has facets $F_1$ and $F_2$. By our Thm 1) $F_1$ and $F_2$ are quasi-arithmetic, and by Thm 2) $F_1$ is arithmetic. But one can check that $F_2$ is properly quasi-arithmetic.
A finite volume hyperbolic orbifold $M = \mathbb{H}^n / \Gamma$ is arithmetic if and only if all its totally geodesic subspaces are fc-subspaces (i.e. correspond to finite subgroups of $\text{Comm}(\Gamma)$) and there are infinitely many of them.

If $M$ is (quasi-)arithmetic over $\mathbb{R}$, then all its fc-subspaces are of the same type.

There are more results and examples....
Computing reflection centralisers in hyperbolic reflection groups

A. Kolpakov (Université de Neuchâtel), joint with N. Bogachev (Skoltech)
Let \((W, S) = \) Coxeter system, and \(s \in S\) a simple reflection.

\[ C_W(s) = \langle s \rangle \times \langle W_\Omega, \Gamma_\Omega \rangle = \text{the centraliser of } s \text{ in } W. \]

Here \(W_\Omega\) is generated by the reflections in \(C_W(s)\) other than \(s\), and \(\Gamma_\Omega = \pi_1(\text{"odd" Coxeter diagram of } (W, S)).\)
Explicit set of generators: we only need to use linear algebra to get them, once $(W, S)$ is given: either $W < \text{Isom} (\mathbb{H}^n)$ or $W$ has geometric representation (Jacques Tits).

If the “odd” diagram of $(W, S)$ has no cycles, then

$$C_W(s) = \langle s \rangle \times W_\Omega.$$
Now we can ...

Program (in SageMath) a fairly general version of the algorithm. Play around with it (soon on GitHub).

Prove that once $\Gamma_\Omega$ is trivial then $W_\Omega$ gives a new (quasi-)arithmetic reflection lattice once $W$ is a (quasi-)arithmetic lattice (have more general results in Belolipetsky, Bogachev, K., Slavich, arXiv:2105.06897).
Thank you!
ICE RM Lightning Talk

Max Riesenberg
June 2021
A quantified local-to-global principle for Anosov representations

Max Riestenberg
Universität Heidelberg
June 2021
Discrete subgroups of Lie groups

- Closed surface groups in $\text{PSL}(2, \mathbb{R})$
- Undistorted subgroups in $\text{Isom}(\mathbb{H}^n)$
- Anosov representations in semisimple $G$

$\pi_1 S \rightarrow \text{PSL}(2, \mathbb{R})$

- Discrete & faithful
- a.k.a. convex cocompact

$\Gamma \rightarrow G$

- Gromov hyperbolic
- Real semisimple Lie group
- Anosov
Undistorted subgroups in negative curvature

Def: A finitely generated subgroup $P \subset \text{Isom}(H^n)$ is **undistorted** if any orbit map is a quasi-isometric embedding:

$$\exists p \in H^n, c_1, c_2, c_3, c_4 \geq 0 \text{ such that } \forall x \in P,$$

$$\frac{1}{c_1} |x| - c_2 \leq d_{H^n}(p, \varphi p) \leq c_3 |x| + c_4,$$

word length of $\sigma$

Facts:

1. $P \subset \text{Isom}(H^n)$ is undistorted if and only if any orbit map sends geodesics to quasigeodesics
2. $P \subset \text{Isom}(H^n)$ undistorted $\Rightarrow P$ discrete and stable
Semisimple Lie groups & higher rank symmetric spaces

$SL(3, \mathbb{R}) \sim X = \{ \text{real } 3 \times 3 \text{ symmetric positive definite matrices with determinant } 1 \}$

$g \cdot x = gxg^T$

$F = \{ \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_3 \end{pmatrix} \mid a_1, a_2, a_3 > 0, a_1a_2a_3 = 1 \} \subset X$

maximal flat

higher rank $\iff \mathbb{R}^2 \hookrightarrow X$
Challenges in higher rank:

- Undistorted subgroups are no longer stable
  - $\times$
  - $\circlearrowleft$

- The local-to-global principle fails

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In $H^n$:

- Undistorted subgroups are stable
  - $\uparrow$

- The local-to-global principle
  - $\uparrow$

- The Morse Lemma

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Weyl chambers and regularity

\[ F = \{ (a_1, a_2, a_3) \mid a_1, a_2, a_3 > 0 \} \]

\[ V = \{ (a_1, a_2, a_3) \mid a_1, a_2, a_3 > 0, a_1 a_2 > a_3 \} \]

Weyl chamber

**Def** A point \( q \in V \) is \( \alpha_0 \)-regular if \( \sin \angle_{\rho}(q, \text{any wall}) \neq \alpha_0 > 0 \).
\textbf{Diamonds}

\textbf{Def.} For $q \in V$ regular

$$\Diamond(p,q) := V \cap V'$$
Morse quasi-geodesics

Def: A \((c_1, c_2, c_3, c_4)\)-quasi-geodesic \((x_n)\) in \(X\) is \((x_0, D)\)-Morse if

\[
d(x_n, p) \leq D, \quad \forall n \leq m, \quad d(x_k, \bigtriangleup_{x_0}(p, q)) \leq D, \quad d(x_m, q) \leq D
\]
The local-to-global principle for Morse quasigeodesics

**Definition/Theorem** (Kapovich-Leeb-Ponti 2014) (Labouvie, Guichard-Wienhard)

A representation $\rho : P \to G$ is **Anosov** if any orbit map $P \to X$ sends geodesics to $(x_0, D)$-Morse $(c_1, c_2, c_3, c_4)$-quasigeodesics.

**Theorem** (Kapovich-Leeb-Ponti 2014) (local-to-global principle for Morse quasigeodesics)

For $x_0 > x_0', D, c_1, c_2, c_3, c_4$ there exists a scale $L$ so that:

Every $L$-local $(x_0, D)$-Morse $(c_1, c_2, c_3, c_4)$-quasigeodesic is an $(x_0', D')$-Morse $(c_1', c_2', c_3', c_4')$-quasigeodesic.
The quantified local-to-global principle for Morse quasigeodesics

Theorem (Kapovich–Leeb–Porti 2014) (local-to-global principle for Morse quasigeodesics)

\[
\forall \, \lambda_0 > \lambda_0', \, \delta, \, c_1, \, c_2, \, c_3, \, c_4 \exists \text{ an explicit scale } L \text{ so that:}
\]

Every \(L\)-local \((\lambda_0, \delta, (c_1, c_2, c_3, c_4))\)-Morse \((c_1, c_2, c_3, c_4)\)-quasigeodesic is an \((\lambda_0', \delta')\)-Morse \((c_1, c_2, c_3, c_4)\)-quasigeodesic.
Explicit perturbation neighborhoods for Anosov representations

Let $S = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \bigg| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \Theta \cos \theta \\ 0 & 0 & -\sin \theta \\ \sin \theta & \cos \theta & 0 \end{pmatrix} \bigg| \begin{pmatrix} 0 & 0 & \Theta \cos \theta \\ 0 & 0 & -\sin \theta \\ \sin \theta & \cos \theta & 0 \end{pmatrix} \right\}$ for $\Theta \not\in \mathbb{Z}$

with $\log \lambda = \cosh^{-1}(\cot \pi/8)$ generate a subgroup $\Gamma$ of $\text{SL}(3, \mathbb{R})$.

**Theorem (R. 2020)**

If $\rho : \Gamma \to \text{SL}(3, \mathbb{R})$ satisfies $|\rho(s) - s| \leq 10^{-3.698.433}$ for all $s \in S$, then $\rho$ is Anosov.
Thanks!