# Word problems and finite state automata

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# Outline

**(I)** 3-manifolds and the word problem

**(II)** General: Word problem solutions by finite state automata

- Computational view: Regular convergent prefix-rewriting
  - Automatic
  - Autostackable
  - Finite rewrite system
- Geometric/topological views

**(III)** Results

- 3-manifolds:
  - Word Problem solutions using FSA's
  - Computing rewriting systems
- General: Graph of groups closure for rewriting systems

# 3-manifolds and the Word Problem, I

Throughout the talk:

- $M$ = compact, connected 3-dim. manifold with toral boundary
- $G = \pi_1(M)$ = fundamental group of $M$ = 3-manifold group
- $\langle A \mid R \rangle$ = finite presentation for $G$, with $A = A^{-1}$

$A^*$ = set of all words over $A$

**Def.** **Word Problem (WP)** for $G$: Is there a computer program that, upon input of a word $w \in A^*$, decides whether $w = 1$ in $G$?

# 3-manifolds and the Word Problem, II

**History:**

- (Dehn, 1911) states the WP, solves for surface groups

- (Thurston 1982; Perelman 2002-3; Hempel 1987) 3-manifold groups are residually finite, and hence have solvable WP.

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**Questions:** Is there a WP solution...

- in polynomial time?
- in log space?
- by a finite state automaton?

**Goal**: Find WP algorithms by FSA's for 3-manifold groups.

# 3-manifolds and the Word Problem, III

$G$ **residually finite** $\iff$ for all $1 \neq g \in G$, there is a finite group $H$ and homomorphism $\phi : G \to H$ such that $\phi(g) \neq 1$.

**WP Algorithm**

for $G = \langle A \mid R \rangle$ finitely presented and residually finite:

Input $w \in A^*$. Run 2 processes in parallel:

**($w = 1$?)** List all $v \in \langle R \rangle^{normal}$, check if $w = v$ in *FreeGroup*$(A)$.

**($w \neq 1$?)** List all $\phi : G \to H$, check if $\phi(w) \neq 1$.

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# Finite state automata and regular languages

(**Def.**) An **FSA** is a computer with finite memory, recognizing a subset of $A^*$.

**Fact.** $L \subseteq A^*$ is recognized by an FSA $\iff$ $L$ is regular.

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**Def.** **Regular languages** are built from finite sets using
$\cap$, $\quad \cup$, $\quad A^* \setminus (\ )$, $\quad (\ ) \cdot (\ )$, $\quad (\ )^*$

**Def.** $L \cdot M = \{uv \mid u \in L, v \in M\}$,

$$L^* = \{1\} \cup (\cup_{i=1}^{\infty} L^k)$$

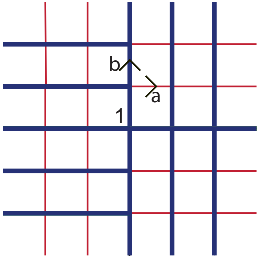
# Normal forms and Cayley graphs

$G = \langle A \rangle$ with $A = A^{-1}$ $\qquad\qquad \pi : A^* \twoheadrightarrow G$

**Def.** $N \subset A^*$ is a set of **normal forms** if $N$ contains exactly one representative for each $g \in G$.

**Def.** The **Cayley graph** $\Gamma = \Gamma(G, A)$ satisfies $V(\Gamma) = G$ and $\vec{E}(\Gamma) := \{g \bullet \xrightarrow{a} \bullet ga \mid g \in G, a \in A\}$

**Ex.** $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$
with
$N = \{a^i b^j \mid i \geq 0\} \cup \{b^j a^i \mid i < 0\}$




**Fact:** Prefix-closed normal forms $\Longleftrightarrow$ maximal tree in $\Gamma$

# Solving the Word Problem using finite automata

**Def.** A **regular convergent prefix-rewriting system (CP-RS)** for $G$ is a finite set $A$ and subset $R \subset A^* \times A^*$ such that

- $G = Mon\langle A \mid R \rangle$.
- The rewritings $uz \to vz$ for all $(u, v) \in R$ and $z \in A^*$ satisfy:
  - There is no infinite sequence $x_1 \to x_2 \to \cdots$
  - $Irr(R) :=$ {irreducible words} $=$ set of normal forms for $G$.
- $R \subset A^* \times A^*$ is regular.

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**Idea:** Input word $w$, rewrite $w$ to $normal\_form(w)$.

# Solving the Word Problem using finite automata

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**Ex.** $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ $\qquad N = \{a^i b^j \mid i \geq 0\} \cup \{b^j a^i \mid i < 0\}$

$$\begin{aligned} R = \ & \{x\ell\ell^{-1} \to x \mid x \in A^*,\ \ell \in A\} \\ & \cup \{xa^{-1}b^{\nu} \to xb^{\nu}a^{-1} \mid x = b^j a^i,\ i \leq 0,\ \nu = \pm 1\} \\ & \cup \{xb^{\nu}a^{\sigma} \to xa^{\sigma}b^{\nu} \mid x = a^i b^j;\ \nu, \sigma \in \{\pm 1\};\ \nu j, i, i + \sigma \geq 0\} \end{aligned}$$

$$a^{-1}ba^2a^{-1} \to ba^{-1}aaa^{-1} \to baa^{-1} \to aba^{-1} \to aa^{-1}b \to b$$

# Computational view, II

**Prop.** Regular CP-RS $\Rightarrow$ word problem solution using a FSA,
prefix-closed regular normal forms.

*Proof.* $w =_G 1$ if and only if $w \to \cdots \to 1$.

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**Special cases of regular CP-RS's:**

**(Prefix-closed) Automatic** $=$ Interreduced regular CP-RS:

For all $(u, v) \in R$: $u = \tilde{u}a$ with $a \in A$, $\tilde{u}, v \in Irr(R)$ (Otto 1999)

**Autostackable** $=$ bounded regular CP-RS:

There is a $k > 0$ such that for all $(u, v) \in R$: $(u, v) = (xu', xv')$ for some $x, u', v' \in A^*$ with $l(u') + l(v') \leq k$.

**Finite rewrite system** $=$ prefix-free $+$ bounded regular CP-RS:
For all $(xu, xv) \in R$: $(wu, wv) \in R$ for all $w \in A^*$

# Computational view, III

**Thm.** (Brittenham, H, Holt, 2014) Prefix-closed automatic $\Rightarrow$ autostackable.

**Note.** Finite rewrite system $\Rightarrow$ autostackable.

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**More specific goal:** Solve the WP for 3-manifold groups using FSA's by finding - and effectively computing - automatic or autostackable structures.

# Comparing the special cases

{**P.c. automatic or finite rewrite**} $\subsetneq$ { **autostackable**}

| **Closure properties** | Automatic | Finite rewrite | Autostackable |
|---|---|---|---|
| Fin. ind. supergroup | ✓ [1] | ✓ [2] | ✓ [3] |
| Extension | X [1] | ✓ [2] | ✓ [3] |
| Graph/free/dir. product | ✓ [4] | ✓ [4] | ✓ [3] |
| Amalg. prod., HNN, $\pi_1$(graph of gps) | some | some | some |

(Epstein+ '92)[1], (H,Meier '95)[4]

**Examples:**

*P.c. automatic:* Word hyperbolic groups (ECHLPT 1992),
Relatively hyperbolic groups (rel $\mathbb{Z}^n$s) (Antolin, Ciobanu 2016)

*Finite rewrite system:*
f.g. nilpotent, polycyclic groups (Groves, Smith 1993)[2]

*Autostackable:*
Stallings' non-$FP_3$ group (Brittenham, H, Johnson 2016)[3]
Thompson's group $F$ (Corwin, Golan, H, Johnson, Šunić 2020)

# Automatic groups: Geometric view

$G = \langle A \rangle$ with $A = A^{-1}$ and $|A| < \infty$. $\Gamma$ = Cayley graph

**Thm.** (ECHLPT, 1992) $G$ is **automatic** iff there exist

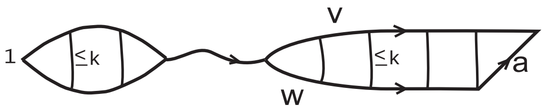
- $L \subset A^*$ a regular language of normal forms for $G$, and
- a constant $k > 0$

  such that

- for all $v, w \in L$ and $a \in A$ with $v = wa$ in $G$,

the paths from 1 labeled $v$ and $w$ $k$-**fellow travel** in $\Gamma$.

**Def.** Let $v, w \in A^*$. The paths from 1 labeled $v$ and $w$ **$k$-fellow travel** if for all $t \in \mathbb{N}$ we have $d_\Gamma(v(t), w(t)) \leq k$.



# Autostackable groups: Topological view

$G = \langle A \rangle$ with $A = A^{-1}$ and $|A| < \infty$. $\Gamma =$ Cayley graph

$\vec{E} =$ directed edges; $\vec{P} =$ directed paths

**Thm.** (Brittenham, H, Holt, 2014) $G$ is **autostackable** iff there is

- $L \subset A^*$ prefix-closed regular language of normal forms for $G$, and
- a partition $L = \coprod_{j=1}^{n} L_j$ with $L_j$ regular and $a_j \in A$, $v_j \in A^*$, such that
- the function $\Phi : \vec{E} \to \vec{P}$ defined by

  $[\Phi(g \xrightarrow{a_j} ga_j) = g \xrightarrow{v_j} ga_j$ if $normal\_form(g) \in L_j]$

  is a **flow function** to the maximal tree $T$ (assoc. to $L$) of $\Gamma$:
  - $\Phi$ fixes $T$ and
  - the extension of $\Phi$ to $\widehat{\Phi} : \vec{P} \to \vec{P}$ satisfies:
    For each $p \in \vec{P}$ there is a $n_p \in \mathbb{N}$ such that
    $\widehat{\Phi}^{n_p}(p)$ is a path in $T$ between the same endpoints.

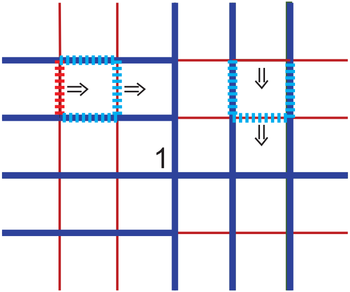
# Example

**Ex.** $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ $\quad N = \{a^i b^j \mid i \geq 0\}) \cup \{b^j a^i \mid i < 0\}$

**Automatic:**

$R_1 = \{a^i b^j b^{\pm 1} \to a^i b^{j \pm 1} \mid i \geq 0\} \cup \{b^j a^i a^{\pm 1} \to b^j a^{i \pm 1} \mid i < 0\}$
$\cup \{a^i b^j a^{\pm 1} \to a^{i \pm 1} b^j \mid i, i \pm 1 \geq 0\} \cup \{b^j a^i b^{\pm 1} \to b^{j \pm 1} a^i \mid i < 0\}$

**Autostackable:** $R_2 = \{x \ell \ell^{-1} \to x \mid x \in A^*,\ \ell \in A\}$
$\cup \{x a^{-1} b^{\nu} \to x b^{\nu} a^{-1} \mid x = b^j a^i,\ i \leq 0,\ \nu = \pm 1\}$
$\cup \{x b^{\nu} a^{\sigma} \to x a^{\sigma} b^{\nu} \mid x = a^i b^j;\ \nu, \sigma \in \{\pm 1\};\ \nu j, i, i + \sigma \geq 0\}$




$R_1$: Normal forms 2-fellow travel.

$R_2$: Iterating the rewriting rules replaces nontree edges by a path in the tree between the same endpoints after finitely many steps.

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# 3-manifolds - solution to the Word Problem using FSA's

Let $M$ be a compact, connected 3-manifold with toral boundary.

**Thm.** (Epstein, Cannon, Holt, Levy, Paterson, Thurston 1992)
*If $M$ has no Nil or Sol prime factors, then $\pi_1(M)$ is automatic.*

**Thm.** (Thurston 1992; N. Brady 2001)
*If $M$ has Nil or Sol geometry, then $\pi_1(M)$ is* **not** *automatic.*

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**Thm.** (Brittenham, H, Susse 2018)
*Let $M$ be a compact, connected 3-manifold with toral boundary. Then $\pi_1(M)$ is autostackable.*

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**Nil, Sol geometries:** If $M$ is a closed 3-manifold and its universal cover has Nil or Sol geometry, then $\pi_1(M) =$ a finite index supergroup of an **extension** of groups built from $\mathbb{Z}$, $\mathbb{Z}^2$.

$X$ automatic ✓ autostackable

# Automatic 3-manifolds: Computing automaticity, I

**Goal:** Find/implement procedure to construct automatic structures for 3-manifolds with no Nil or Sol prime factors. And use normal forms that reflect the manifold structure.

| | Automatic | Finite rewrite | Autostackable |
|---|---|---|---|
| Procedure to find structure | KBMAG * | Knuth-Bendix * | Limited... * |
| Derivation fcn | $\leq$ quadratic | ? | ? |

*: All packages restrict the ordering

**Motivation:** Quadratic time word problem algorithm using FSA's.

**Difficulty:** GAP/KBMAG software can fail to find automatic structures for automatic 3-manifolds:

- Software uses "wrong" normal forms
- Automaticity proof (ECHLPT) not amenable to implementation

# Decomposing $M$:

(1) $\pi_1(M) =$ finite index supergroup of $\pi_1(M_1) * \cdots * \pi_1(M_k)$, where each $M_i$ is an orientable prime manifold.

✓ automatic ✓ autostackable

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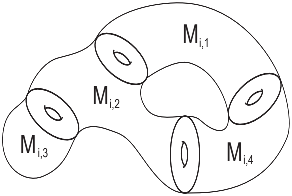
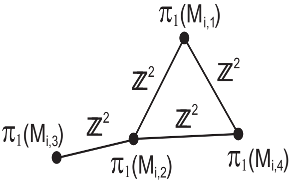
(2) **Geometrization and JSJ decomposition:**

**Topology:** For prime 3-manifold $M_i$, either:

- $M_i$ is geometric (admits one of Thurston's 8 geometries); or
- cut $M_i$ along embedded, incompressible tori; each resulting 3-manifold $M_{i,j}$ is either • Seifert fibered or • hyperbolic

**Group Theory:** $\pi_1(M_i) =$ a fundamental group of a graph of groups, with • vertex groups: $\pi_1(M_{i,j})$, • edge groups: $\mathbb{Z}^2$



# Automatic 3-manifolds: Computing automaticity, II

By finding new automaticity closure results, we obtained a new proof of automaticity - but with normal forms reflecting the JSJ decomposition and with implementable construction - for many 3-manifolds:

**Cor.** (H, Holt, Rees, Susse, 2019) Let $M$ be an orientable, connected, compact 3-manifold with incompressible toral boundary whose prime factors have JSJ decompositions containing only hyperbolic pieces. Then $\pi_1(M)$ is automatic with respect to Higgins' normal forms for graphs of groups.

# Automaticity and closure for graphs of groups, I

**Idea**: For graph of groups with

- vertex groups: $\pi_1(M_{i,j})$,
- edge groups: $\mathbb{Z}^2$

want automaticity closure.

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**Thm.** (Epstein, Cannon, Holt, Levy, Paterson, Thurston, 1992) An amalgamated product or HNN extension of automatic groups along a finite edge subgroup is automatic.

**Thm.** (Baumslag, Gersten, Shapiro, Short, 1991) An amalgamated product of two finitely generated abelian vertex groups over any subgroup is automatic.

**Thm.** (Shapiro, 1992) A tree of negatively curved vertex groups with cyclic edge groups is automatic.

Amalgamated product: 1 edge, 2 vertices;
HNN extension: 1 edge, 1 vertex

# Automaticity and closure for graphs of groups, II

Obtained several new closure theorems...

**Results**:
(H, Holt, Rees, Susse, 2019)
Automaticity closure results for graphs of groups in which

• (Vertex group, edge group) pairs are either coset automatic pairs or automatic and admissible (geodesics concatenate).

• Geometry is controlled across edges and between intersecting edge subgroups within the same vertex group.

# Open questions

• Construct software that upon input of the graph of groups decomposition for $\pi_1(M)$ constructs an automatic or autostackable structure for the group.

• What bounds can be found for the time/space complexity of the word problem solution by FSA's from the autostackable structures for 3-manifold groups?

**Thank you!**