Main results

Van Kampen Diagrams

Decoration on the boundaries of diagrams

Canonical representatives

Counting diagrams and decorations

First-order sentences in random groups

O. Kharlampovich (CUNY, Grad Center and Hunter College)
joint results with R. Sklinos (Stevens Institute)

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Random groups

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The idea of random groups draws its origins in Gromov's seminal paper introducing hyperbolic groups.

On a different line of thought, Sela and Kharlampovich-Myasnikov proved that the first-order theories of free groups of different ranks > 2 are the same. Some hyperbolic groups, called towers, also have the same theory, but not all hyperbolic groups.

This brings us to the subject of this talk.

Density model of random group

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We will use Gromov's density model of randomness.

Definition

Let $\mathbb{F}_n := \langle e_1, \ldots, e_n \rangle$ be a free group of rank n. Let S_ℓ be the set of reduced words on $e_1, \ldots, e_n$ of length ℓ.
Let $0 \leq d \leq 1$. Then a random set of relators of density d at length ℓ is a subset of S_ℓ that consists of $(2n-1)^{d\ell}$-many elements picked randomly (uniformly and independently) among all elements of S_ℓ.
A group $G := \langle e_1, \ldots, e_n \mid \mathcal{R} \rangle$ is called random of density d at length ℓ if R is a random set of relators of density d at length ℓ.

A random group of density d satisfies some property (of groups) P, if the probability of occurrence of P tends to 1 as ℓ goes to infinity.

Properties of random groups

Theorem

(Gromov, 93) Let $0 < a < 1$ and $d < a/2$. Then a random group of density d satisfies the small cancellation property $C'(a)$.

Theorem

(Gromov 93, Ollivier 2005) Let G be a random group of density d.

- *If $d < 1/2$. Then G is infinite torsion-free hyperbolic.*
- *if $d > 1/2$. Then G is either trivial or $\mathbb{Z}_2$.*

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Conjecture[J. Knight] Let σ be a first-order sentence in the language of groups. Then σ is true in a nonabelian free group $\mathbb{F}$ if and only if it is almost surely true in a random group.

Theorem

Let Γ be "the random group" of some fixed density $d < 1/16$. Let σ be a universal sentence in the language of groups. Then Γ almost surely satisfies σ if and only if $\mathbb{F} \models \sigma$.

We can prove this for $\forall\exists$-sentences too (at $d < 1/16$).

Corollary

Given an equation one can decide if it has a non-trivial solution in a random group of fixed density $d < 1/16$.

Limit groups

A f. g. group G is a **limit group** is for any finite set of elements there is a homomorphism $G \to \mathbb{F}$ that is injective on this set of elements.

Limit groups are exactly the f. g. models of the universal theory of $\mathbb{F}$.

Limit groups

Proposition

(Kozma, Lubotzky) Let $0 < d < 1$ and G be a random group of density d. Then, for any fixed k, there is no nontrivial degree k representation of G over any field.

Proposition

A random group for $d < 1/2$ is not a limit group.

Proof.

Sanov proved that $\mathbb{F}_2$ embeds into $SL_2(\mathbb{Z})$ and consequently into $SL_2(\mathbb{R})$. Since, a limit group is universally free, it embeds into an ultrapower of $\mathbb{F}_2$, which in turn embeds into an ultrapower of $SL_2(\mathbb{R})$. The latter is isomorphic to $SL_2(\mathbb{R}^*)$, where $\mathbb{R}^*$ is an ultraproduct of $\mathbb{R}$. Hence, a limit group admits a nontrivial representation over $\mathbb{R}^*$. By Proposition a random group of any density cannot be a limit group. □

Comments

For a few relators model this was shown by Turbo Ho.

Even for $d < 1/2$ the random group contains a nonabelian free group, hence it satisfies the existential theory of the free group.

We fix $n \geq 2$, the free group $\mathbb{F}_n$ of rank n, and a random group $\Gamma := \langle e_1, \ldots, e_n \mid \mathcal{R} \rangle$ of density d at length ℓ with the canonical epimorphism $\pi : \mathbb{F}_n \to \Gamma$.

In order to prove that each universal axiom of the theory of the free group holds in the random group (for some fixed density), it is enough to prove that each axiom of the form

$$\sigma = \forall \bar{x}(w_1(\bar{x}) = 1 \vee \ldots \vee w_k(\bar{x}) = 1 \vee v_1(\bar{x}) \neq 1 \vee \ldots \vee v_m(\bar{x}) \neq 1).$$

holds in such random group.
We need to prove that the probability that Γ satisfies $\neg\sigma$ goes to 0 as ℓ goes to infinity. Γ satisfies $\neg\sigma$ only if there exists a tuple $\bar{b}$ in Γ such that $v_1(\bar{b}) = 1 \wedge \ldots \wedge v_m(\bar{b}) = 1$, while for any pre-image $\bar{c}$ of $\bar{b}$ via π we have $\vee_{i=1}^{m} v_i(\bar{c}) \neq 1$.

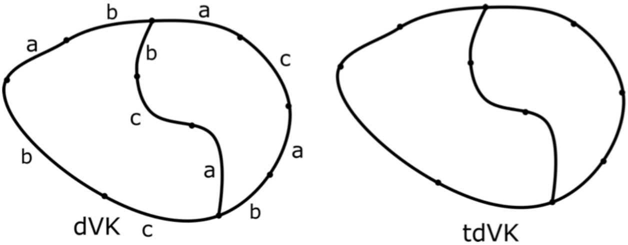
Theorem

For any system of equations $V(\bar{x}) = 1$, the probability that there exists a tuple $\bar{b}$ in Γ such that $V(\bar{b}) = 1$ while for any pre-image $\bar{c}$ of $\bar{b}$ via π we have $V(\bar{c}) \neq 1$, goes to 0 as ℓ goes to infinity.

Equations

Our main goal is to show that for the random group Γ of density $d < 1/16$ and any fixed system of equations $V(\bar{x}) = 1$, the probability of the existence of some $\bar{b}$ in Γ such $V(\bar{b}) = 1$, while for any pre-image $\bar{c} \in \pi^{-1}(\bar{b})$, is not a solution of $V(\bar{x}) = 1$ in $\mathbb{F}$ goes to 0 as ℓ goes to infinity.

van Kampen Diagrams



van Kampen Diagrams

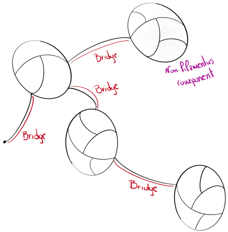


Figure: Filamentous van Kampen Diagrams

Intuition

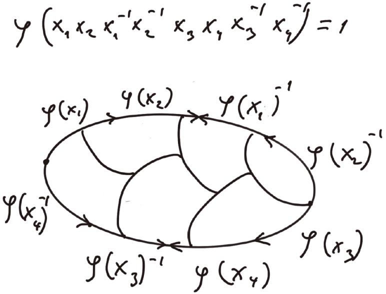
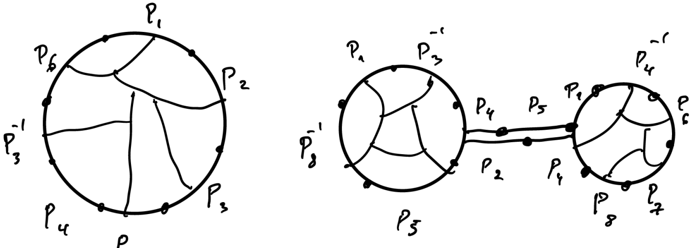


Figure: Let $\phi : F(\bar{x}) \to \Gamma$ be a solution of the equation $[x_1, x_2][x_3, x_4] = 1$. **Suppose $\phi([x_1, x_2][x_3, x_4])$ is written on the boundary of a non-filamentous diagram**.

Decoration on the boundaries of diagrams

Definition

We fix a family, DVK, of van Kampen diagrams. Then a set of variables $\{p_1, \ldots, p_k\}$ is called a decoration of the family DVK if the boundary word of each diagram is the evaluation of some group word $w_i(p_1, \ldots, p_k)$ (for the i-th diagram) such that there is no cancellation between the consecutive values of p_i's and each variable p_i occurs in the union of words $w_1(p_1, \ldots, p_k), \ldots,$ $w_q(p_1, \ldots, p_k)$ at least twice (as p_i or p_i^{-1}).



Decoration on the boundaries of diagrams

Proposition

Suppose we have a tDVK with rigid decoration P (the position and length of each p_i is fixed) with total number of f faces. Then the number of letters that can be chosen arbitrarily in the relators to fulfill the tDVK is bounded from above by $\frac{1}{2}f\ell$.

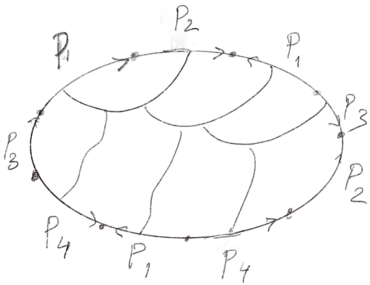


Figure: Decoration on the boundaries

Decoration on the boundaries of diagrams

Proposition

Suppose we fix a tDVK with f faces, such that the boundary of each face has length ℓ and fix a rigid decoration P on the boundary of the tDVK. Then the probability that this tDVK with P can be fulfilled by relations of the group with $(2m-1)^{d\ell}$ random relations of length ℓ is bounded by

$$(2m-1)^{-f\ell(1/2-d)}.$$

Proof

The total number of possibilities to select f words of length ℓ (non necessarily different) is $(2m-1)^{f\ell}$. We can only select not more than $(2m-1)^{f\ell/2}$ letters arbitrarily in the relators that fulfill the tDVK with the rigid decoration P. Therefore the probability that f given words of length ℓ (not necessarily different) can label such tDVK that satisfies the rigid decoration P is less than

$$\frac{(2m-1)^{f\ell/2}}{(2m-1)^{f\ell}} = (2m-1)^{-f\ell/2}.$$

There are $(2m-1)^{d\ell}$ relators in a random presentation. Since the probability of the union of events is not more than the sum of their probabilities, the probability that we can find f of relators fulfilling the tDVK with the rigid decoration is at most $((2m-1)^{d\ell})^f(2m-1)^{-f\ell/2} = (2m-1)^{-\ell f(1/2-d)}$

Strategy

We wish to deal with non-filamentous diagrams.

We reduce a system of equations in Γ to a system of triangular equations, use Rips-Sela's **canonical representatives**, that were used to reduce equations in hyperbolic groups to equations in free groups and solve the isomorphism problem for hyperbolic groups. Then we use Kh-Miasnikov's **Generalized equations with parameters** that were used in the proof of Tarski's conjectures on the theory of free groups, to go from a decoration of filamentous diagrams to decorations on their non-filamentous components.

Reduction to triangular equations

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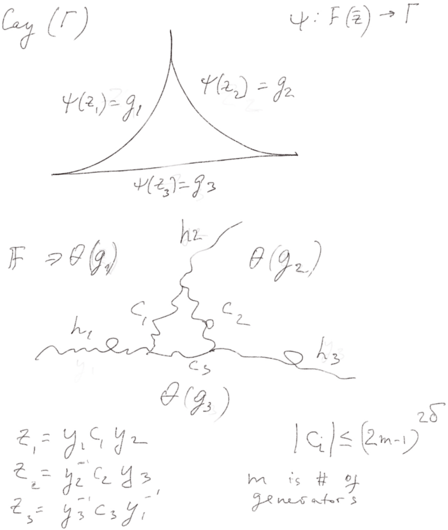
$V(\bar{x}) = 1$ is equivalent to a triangular system $U(\bar{z}) = 1$ in some n variables $z_1, \ldots, z_l$. Suppose it consists of q equations, i.e.

$$U(\bar{z}) = \{z_{\sigma(j,1)} z_{\sigma(j,2)} z_{\sigma(j,3)} = 1, j = 1, \ldots, q\}$$

where $\sigma(j,k) \in \{1, \ldots, l\}$. Indeed, any equation $x_1 x_2 w(\bar{x}) = 1$ is equivalent to the system of shorter equations
$x_1 x_2 z^{-1} = 1, zw(\bar{x}) = 1$ where z is a new variable.
We assign to each element $g \in \Gamma$ its Rips-Sela canonical representative it is a specially selected reduced word $\theta(g) \in F$ satisfying

$$\theta(g) = g \text{ in } \Gamma.$$

Using canonical representatives



Using canonical representatives

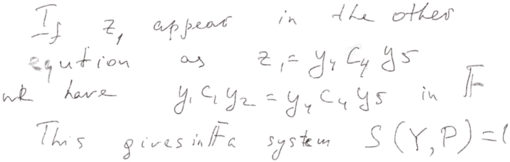


Figure: Reduction to a system with parameters in F plus diagrams

Generalized equations

Each solution of a system of equations in a nonabelian free group $\mathbb{F}$ corresponds to a cancellation scheme and the latter can be combinatorially captured by a *generalized equation* (which is, actually, a system of equations in a free monoid). In a generalized equation each variable of the original system of equations can be split in pieces called *bases*. An assignment of elements of $\mathbb{F}$ to the bases produces a solution of the system. If a system of equations has parameters we can cover all variables by pieces that either appear at least twice in the parameters or are free, in the sense that they do not appear in the parameters and can be assigned arbitrary values in F.

Counting diagrams and decorations

Lemma

The number of planar graphs without degree 2 vertices, with f faces is less than 2^{10f}.

Proof.

If a planar graph without vertices of valency 2 has f faces, then by Euler's formula, it has at most $2f - 4$ vertices and at most $3f - 2$ edges.

The estimate was given in (N. Bonichon, C. Gavoille, N. Hanusse, D. Poulalhon, G. Schaeffer, Planar Graphs, via Well-Orderly Maps and Trees, Graphs Combin. 22 (2006), 185- 202, inspired by A. Vdovina) on the number $p(n)$ of planar graphs with n vertices, $p(n) \leq 2^{5n}$. Therefore the number of planar graphs without degree 2 vertices is no more than 2^{10f}. □

Counting diagrams and decorations

Proposition

The total number of tDVK with f faces (with boundary length ℓ) and the length of the bridges bounded by a polynomial of degree 3 $p(\ell)$, is bounded by $(2^{10}\ell^4)^f 2^f p(\ell)^f$.

Proposition

The total number of differently rigidly decorated tDVK with f faces (with boundary length ℓ) and the length of the bridges bounded by $p(\ell)$ is bounded by $B_1(f\ell p(\ell))^t(2^{12}\ell^5 p(\ell))^f$, where t is a global constant and $p(\ell)$ is a polynomial of degree 3.

Counting diagrams and decorations

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Proposition

The probability that for a random presentation $G := \langle e_1, \ldots, e_n \mid \mathcal{R} \rangle$ with relations of length ℓ and $|\mathcal{R}| = (2m-1)^{d\ell}$, there exists a non-trivial DVK with decoration P corresponding to the solution of one of the finite number of generalized equations (therefore, solving $U(\bar{z}) = 1$) approaches 0 as $\ell \to \infty$.