Arithmetic and rigidity beyond lattices:

Examples from hyperbolic geometry

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I.

Rigidity in flexible 3-manifolds

(with Amir Mohammadi, Hee Oh and Yongquan Zhang)

Planes in 3-manifolds

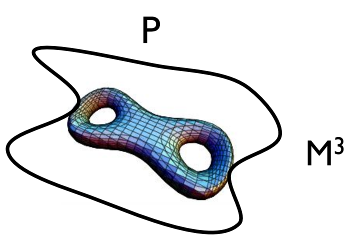
Hyperbolic plane $\mathbb{H}^2 \rightarrow$ M $= \mathbb{H}^3/\Gamma$

image = immersed plane P $\subset$ M

Shah, Ratner

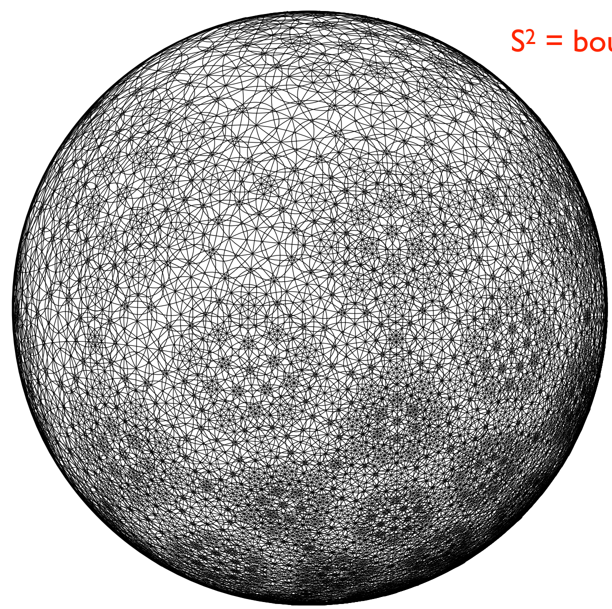
M is compact $\Rightarrow$ P is dense (typical) or

P is a closed surface (miracle)

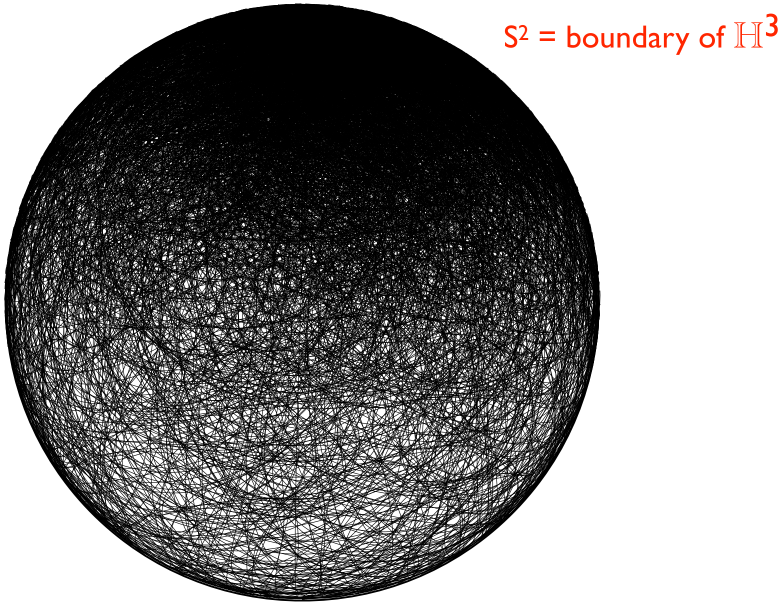


Orbit of a circle, $\Gamma \cdot C$

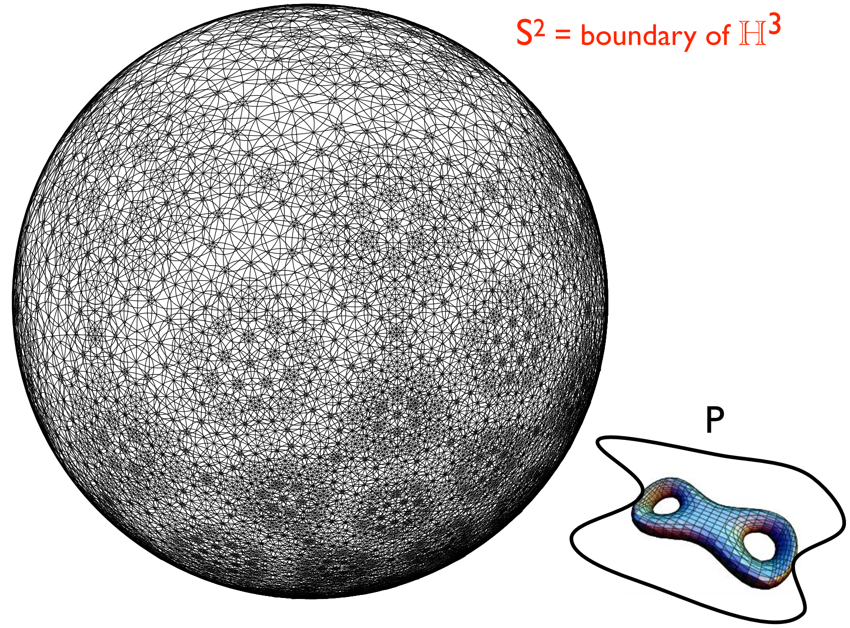
S^2 = boundary of $\mathbb{H}^3$



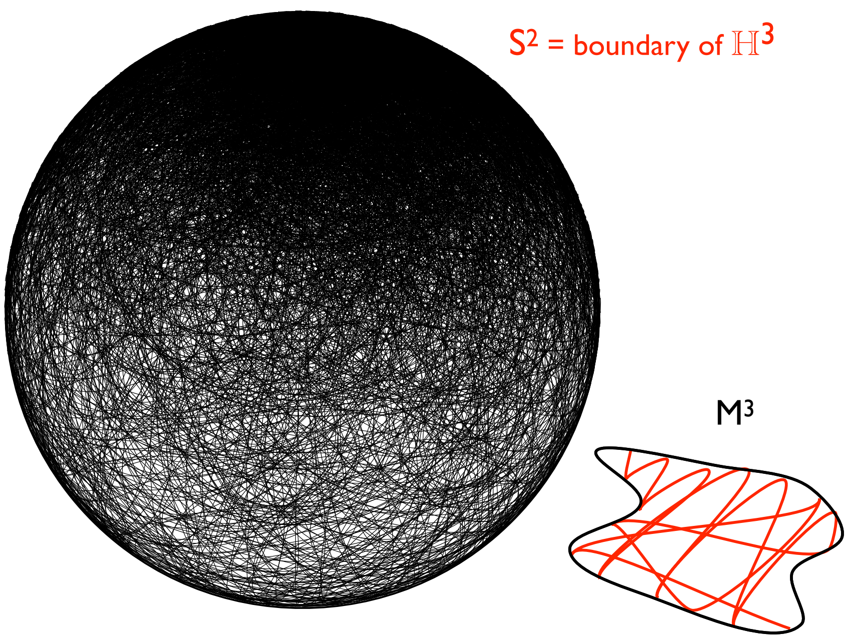
Orbit of a circle, $\Gamma \cdot C$



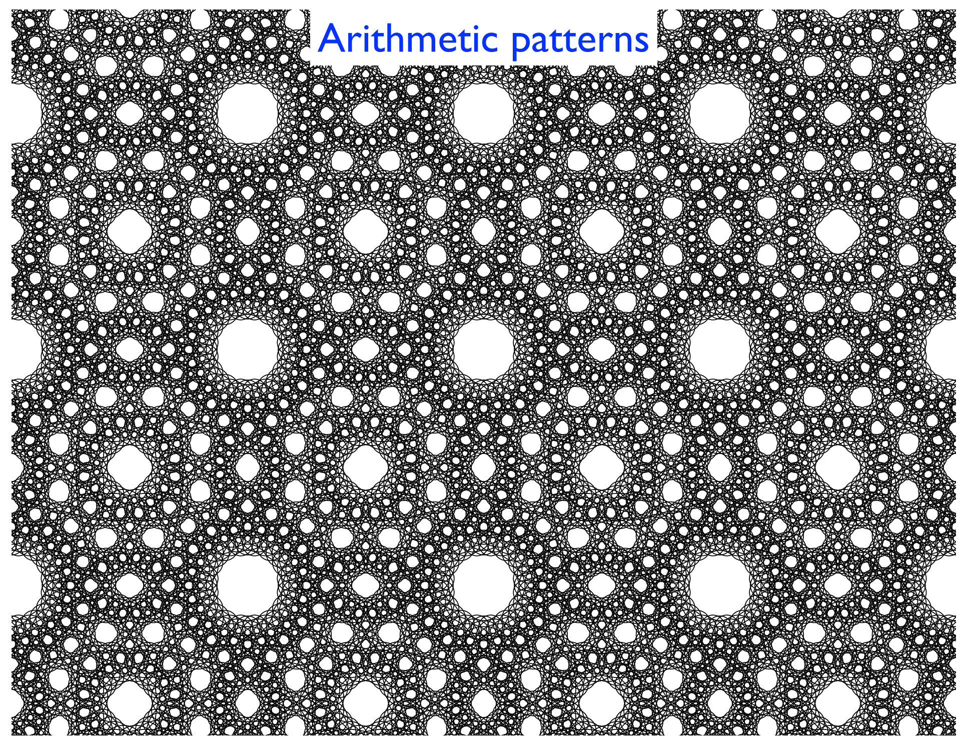
Closed, totally geodesic surface in M



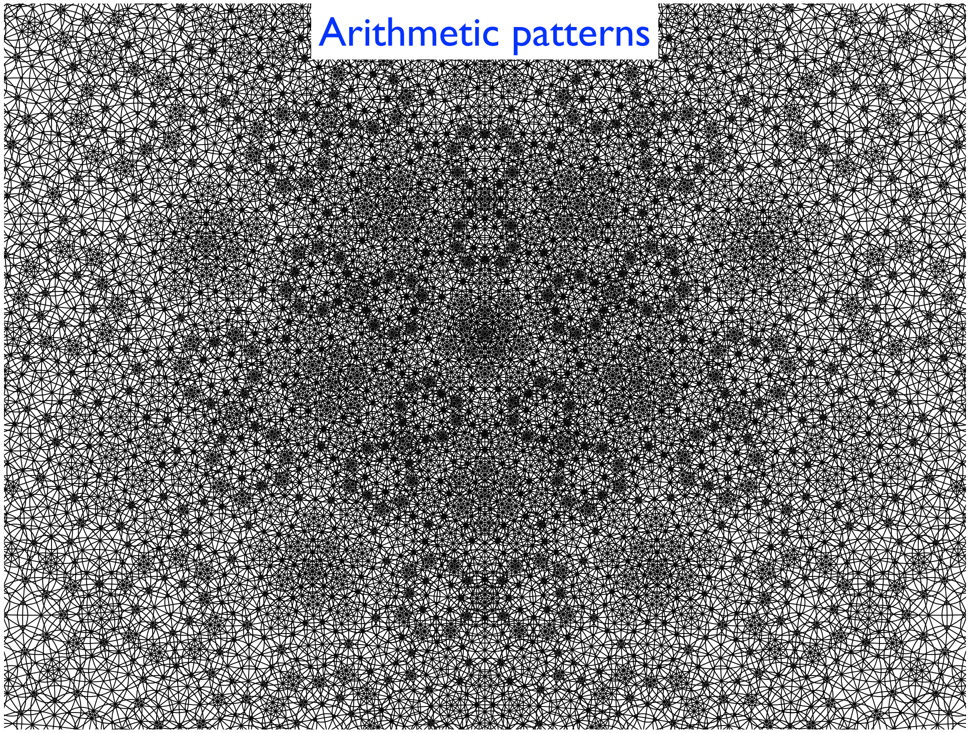
Dense plane in M



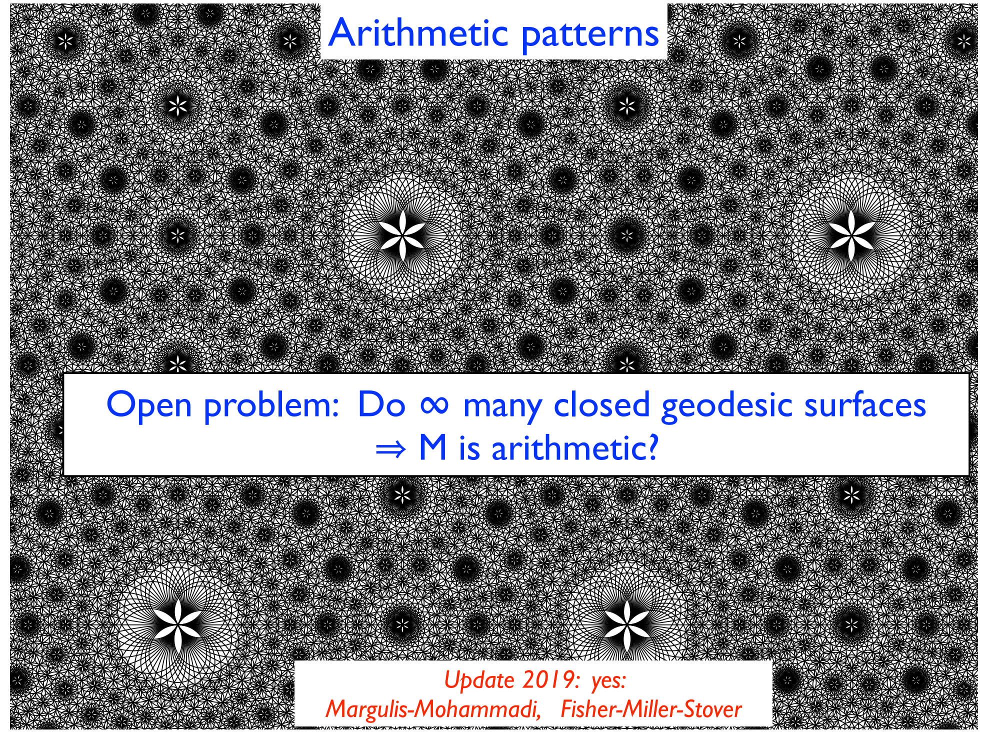
Arithmetic patterns



Arithmetic patterns



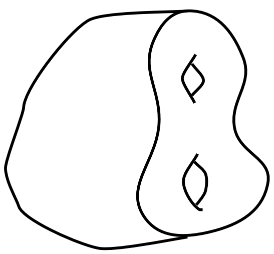
Arithmetic patterns

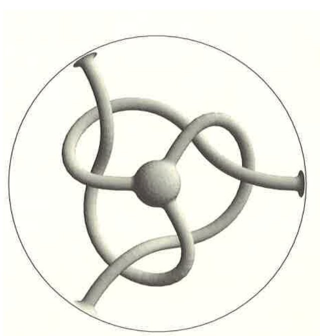


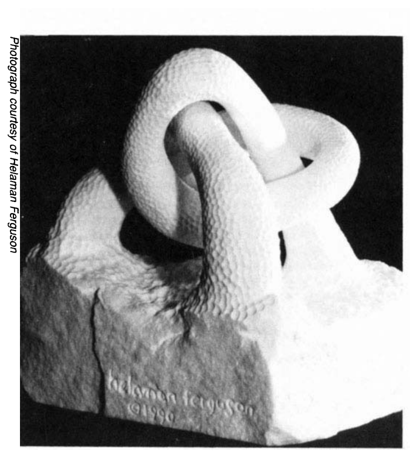
Open problem: Do ∞ many closed geodesic surfaces $\Rightarrow$ M is arithmetic?

Update 2019: yes: Margulis-Mohammadi, Fisher-Miller-Stover

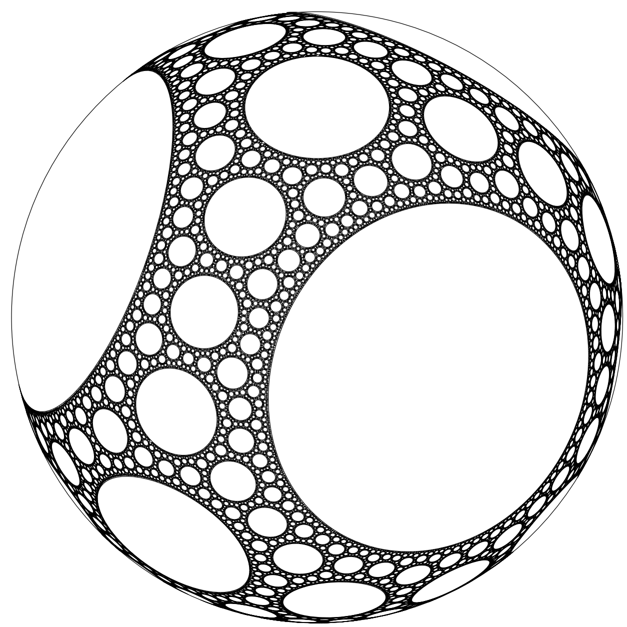
3-manifolds with geodesic boundary



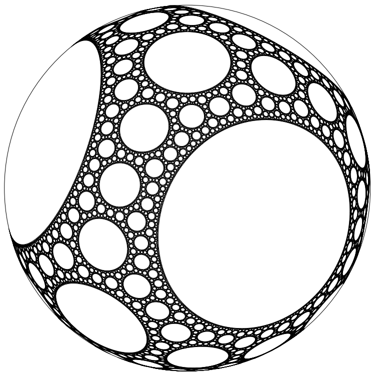




3-manifolds with geodesic boundary



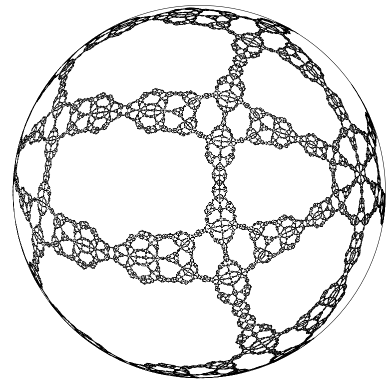
Rigidity holds for planes in 3-manifolds with geodesic boundary….



Any plane in core(M) is either closed or dense.

M - Mohammadi - Oh 2015

… and without geodesic boundary.



(purely topological assumption: acylindrical)

M - Mohammadi - Oh 2017

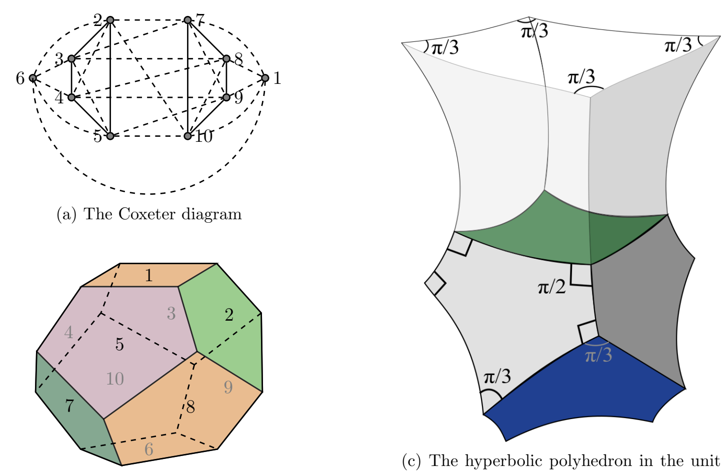
Theorem (Zhang, 2020)

There exists an acylindrical M and a plane P in M that is neither closed nor dense.

Cor

Ratner's theorem fails to generalize *(without some qualification).*

M



(a) The Coxeter diagram

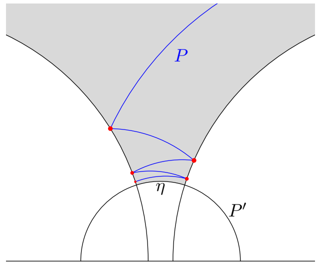
(b) The corresponding polyhedron

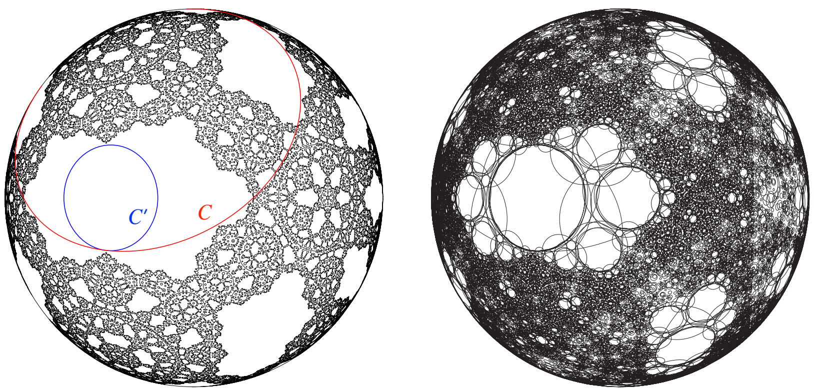
(c) The hyperbolic polyhedron in the unit ball model

Figure 3.1: Combinatorial data and visualization of the polyhedron

The exotic plane P

…accumulates on P′

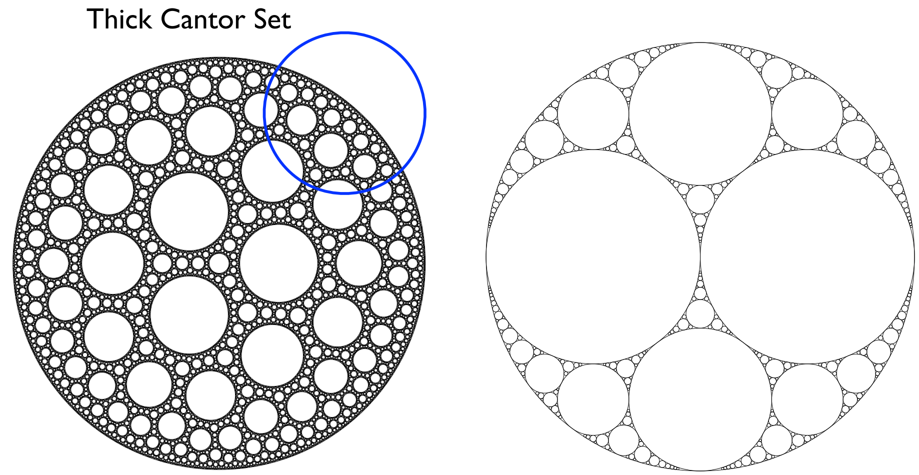




(a) The limit set Λ and an exotic circle C

(b) The orbit $\Gamma \cdot C$

Methods and the frontier

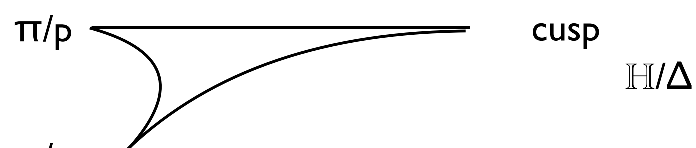


what happens for planes in the Apollonian manifold?

II.
The arithmetic of non-arithmetic groups

Triangle groups

$\Delta(p,q,\infty) \subset SL_2(\mathbb{R})$ *lattice*



invariant trace field

$$K_{pq} = \mathbb{Q}(\,\mathrm{Tr}(g^2) : g \in \Delta(p,q,\infty))$$

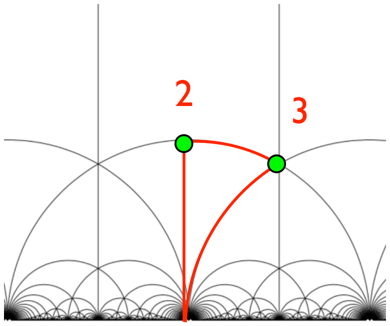
$$= \mathbb{Q}(\cos(2\pi/p),\ \cos(2\pi/q),\ \cos(\pi/p)\cos(\pi/q))$$

$\Delta(p,q,\infty)$ is arithmetic $\Leftrightarrow$ $K_{pq} = \mathbb{Q}$

Arithmetic case

$\Delta(2,3,\infty) = SL_2(\mathbb{Z}) =$

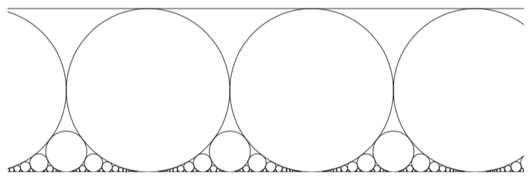
$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$



-1 0 1

$z \to z+1$

$z \to -1/z$



0 1/2 1 2

matrix entries = $\mathbb{Z}$

columns (a,b), gcd=1

cusp = $\mathbb{Q} \cup \{\infty\}$

Non-arithmetic case $\Delta(p,q,\infty)$

is more mysterious!

matrix entries = ?

columns (a,b) ?

cusps = ? $\cup \{\infty\}$

Theorem

The cusps of $\Delta(p,q,\infty)$ coincide with $K_{pq} \cup \{\infty\}$, and satisfy quadratic height bounds, whenever $\deg(K_{pq}/\mathbb{Q}) = 2$.

The golden Hecke group

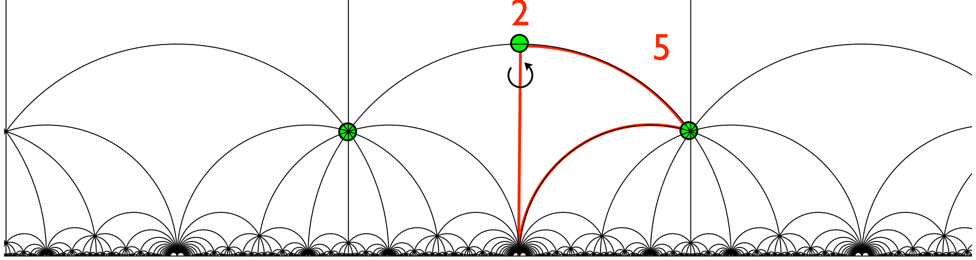
$\gamma = (1+\sqrt{5})/2$

$z \to z+\gamma$

$\Gamma = \Delta(2,5,\infty) =$

$$\left\langle \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$z \to -1/z$



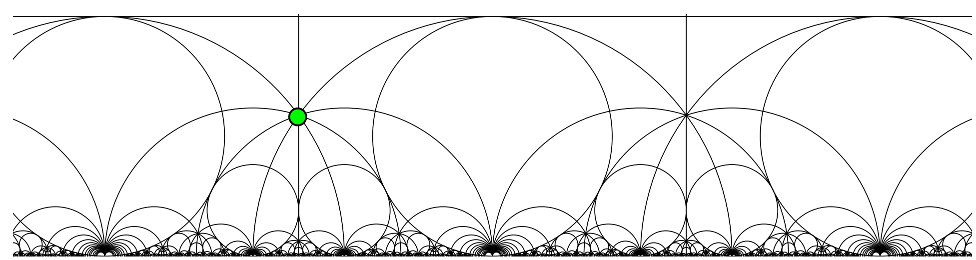
$-\gamma/2$ 0 $\gamma/2$

Cor

The cusps of Γ coincide with $K = \mathbb{Q}(\sqrt{5}) \cup \{\infty\}$.

Leutbecher, 1970s

5 packing



0 $1/\gamma$ 1 γ 2γ

Golden Continued Fractions

Cor

Every x in $\mathbb{Q}(\sqrt{5})$ can be expressed as a *finite* golden continued fraction:

$$x = [a_1, a_2, a_3, \ldots, a_N] = a_1\gamma + \cfrac{1}{a_2\gamma + \cfrac{1}{a_3\gamma + \cdots + \cfrac{1}{a_N\gamma}}},$$

with a_i in $\mathbb{Z}$.

Quadratic height bounds: N, $\max \log |a_i| = O(1+h(x))$.

Golden Fractions

Cor

Every x in $K=\mathbb{Q}(\sqrt{5})$ can be written uniquely as a `golden fraction' $x = a/c$, up to sign.

a,c in $\mathcal{O} = \mathbb{Z}[\gamma] \subset K$ relatively prime

(a,c) column of a matrix in Γ

Quadratic height bounds: $h(a)+h(c) = O(1+h(x)^2)$.

What about matrix entries in $\Delta(2,5,\infty)$?

M = all nonzero matrix entries

$\delta M = \{m'/m : m \text{ is in } M\}$

$R = -\gamma^{-2} \cdot \delta M.$

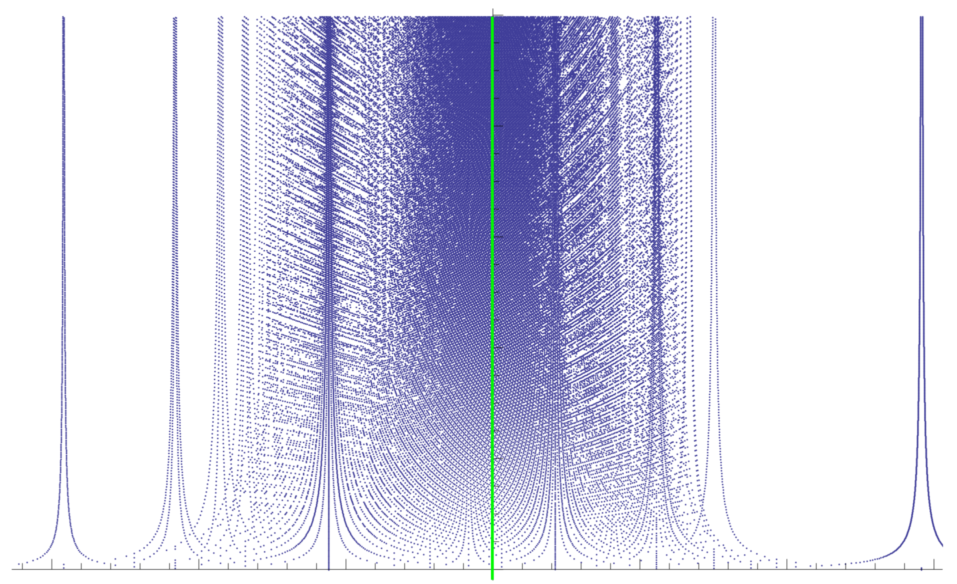
Theorem

The closure of R is a countable semigroup in [-1,1], homeomorphic to $\omega^\omega + 1$.

(Reveals hidden multiplicative structure.)

method: nonabelian modular symbols

Image of M under $(m'/m, H(m))$



Compare to ω^ω in

Pisot numbers,
Weyl spectrum,
3D hyperbolic volumes, …

Thin group perspective

$\Gamma = \Delta(2,5,\infty) \subset SL_2(\mathbb{Z}(\gamma))$

∞

lattice $\cap$ $\qquad$ $\cap$ *lattice*

$SL_2(\mathbb{R}) \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

$V \subset X$

curve $\qquad$ *Hilbert modular surface*

General curves

K = real quadratic field

$$X_K = (\mathbb{H} \times \mathbb{H}) / \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

$$V = \mathbb{H}/\Gamma \looparrowright X_K$$ geodesic curve

Theorem

Either V is a Shimura curve, or the cusps of V coincide with $\mathbb{P}^1(K)$ *and satisfy quadratic height bounds.*

proof by descent, using new height

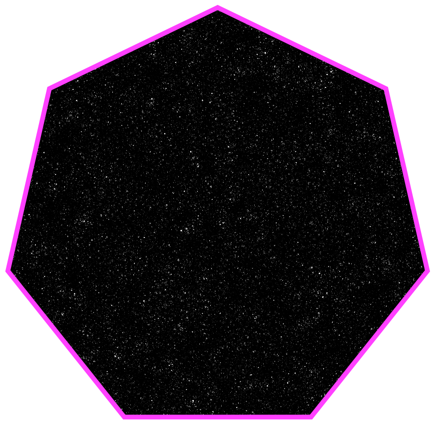
The frontier

What are the cusps of $\Delta(2,7,\infty)$?

$K = \mathbb{Q}(\cos(2\pi/7))$

The frontier

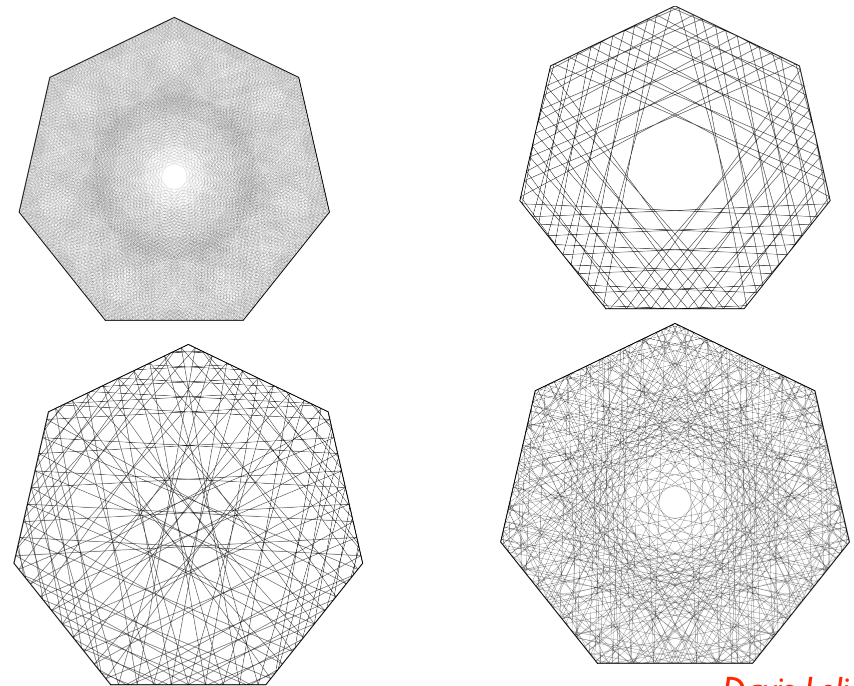
What are the periodic slopes for billiards in a heptagon?



L(s)=7,

L(2 s) = 2190,

…



Davis-Lelievre

References

Geodesic planes in the convex core of an acylindrical 3-manifold (with Mohammadi and Oh)

Billiards, heights, and the arithmetic of non-arithmetic groups

math.harvard.edu/~ctm/papers

Geodesic planes in hyperbolic 3-manifolds (Zhang)

sites.google.com/view/yqzhang/