Necklace Theory and Maximal Cusps of Hyperbolic 3-Manifolds

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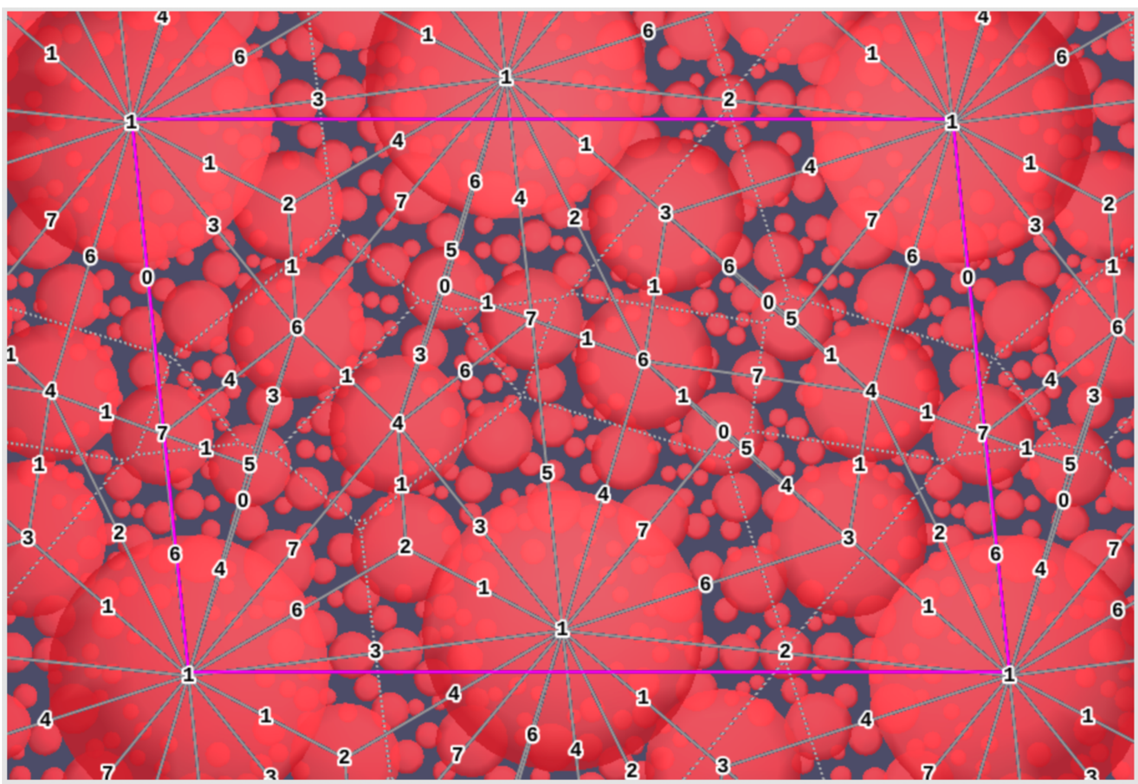

ICERM: Computational Aspects of Discrete Subgroups of Lie Groups

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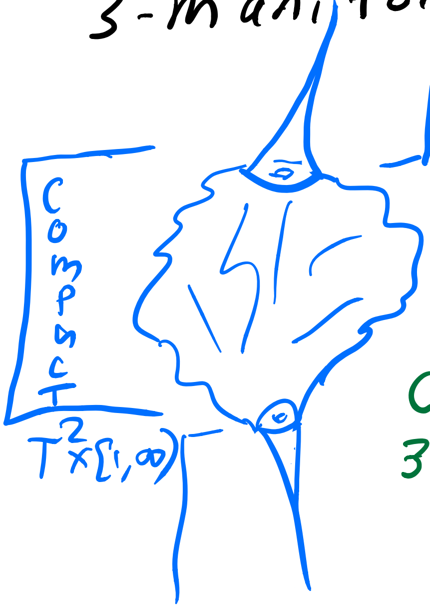
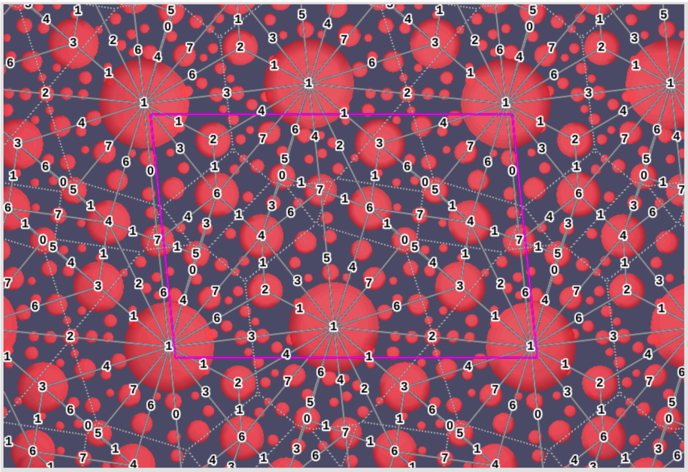
Theorem (G - R. Haraway - R. Meyerhoff - N. Thurston - A. Yarmola)

Let N be a complete finite volume hyperbolic 3-manifold with a maximal cusp of volume ≤ 2.62, then N is obtained by filling either s776 (3-cusps) or one of 15 explicit 2-cusped hyperbolic 3-manifolds.

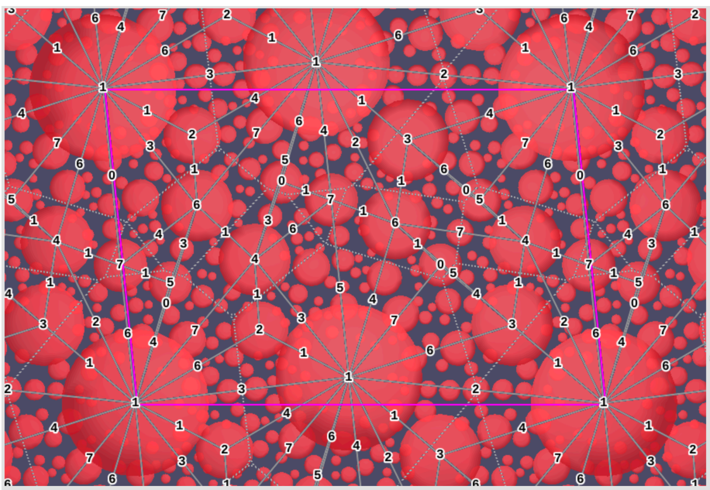


Schematic of a f. volume complete hyperbolic 3-manifold with 2 ends

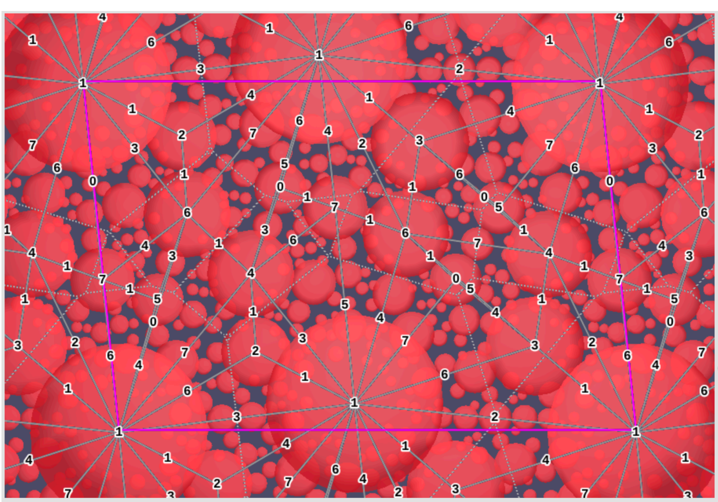
Mostow (1968), Marden, Prasad (≈1971) Rigidity.



Volume 1.52
cusp of
V3276(3,1)
seen from ∞
in upper half
space model



Volume 2.40
cusp of
V3276(3,1)
seen from ∞



Volume 2.855...
cusp of
V3276(3,1)
seen from ∞

"Maximal
cusp"

To what extent does a geometric constraint on N (here: Maximal cusp volume) impose other geometrical, topological, combinatorial, algebraic constraints on N

Hyperbolic Complexity Conjecture 70's 80's (Thurston, Hodgson-Weeks, Matveev-Fomenko)

The complete low volume hyperbolic 3-manifolds can be obtained by filling cusped hyperbolic 3-manifolds of low topological complexity.

Study of hyperbolic 3-manifolds amenable to computer assisted experiments and rigorous proof

- 3 manifolds - defined by Δ'ions
  finite & Combinatorial

- $\mathrm{Isom}(\mathbb{H}^3)$ - $PSL_2(\mathbb{C})$ 2x2 matrices

- (Mostow) Hyperbolic structures determined by $\pi_1(N)$ - "practical group theory"

In addition to experimental and rigorous programs developed to prove the theorem we used

Rigorous Computer Proofs

SnapPy - hyperbolic structures - weeks, Culler-Dunfield

SAGE - verified integral arithmetic for SnapPy - Stein et al

REGINA - Normal Surface theory - Burton-Budney-Petterson

quotpic - group isomorphisms - Holt-Rees

1-Jet arithmetic - For estimating error - [G-Meyerhoff-N. Thurston] Annals 2003

Experiments & sanity checking

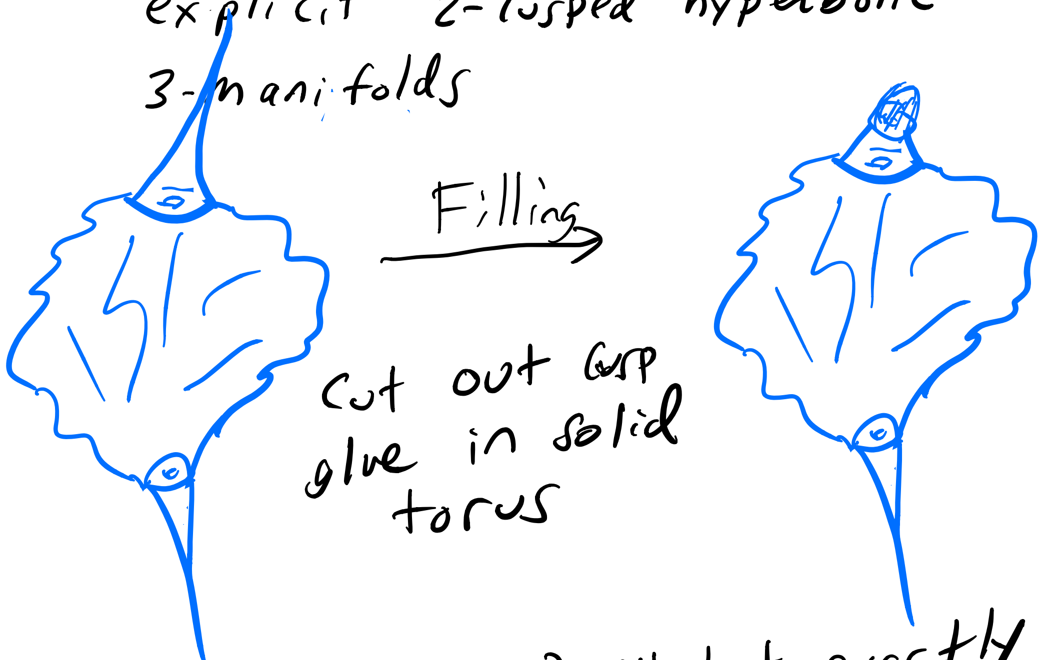
SnapPy

Heegaard - Heegaard splittings from group presentations - Berge

Twister - triangulations from H-splittings - Bell-Hall-Schleimer

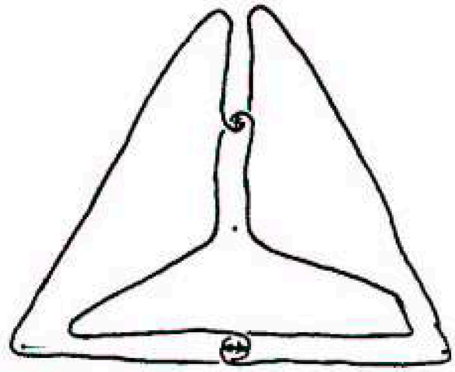
Coover - SnapPy ready Δions from group presentations - Haraway-Yarmola

Let N be a complete finite volume hyperbolic 3-manifold with a maximal cusp of volume ≤ 2.62, then N is obtained by filling either s776 (3-cusps) or one of 15 explicit 2-cusped hyperbolic 3-manifolds



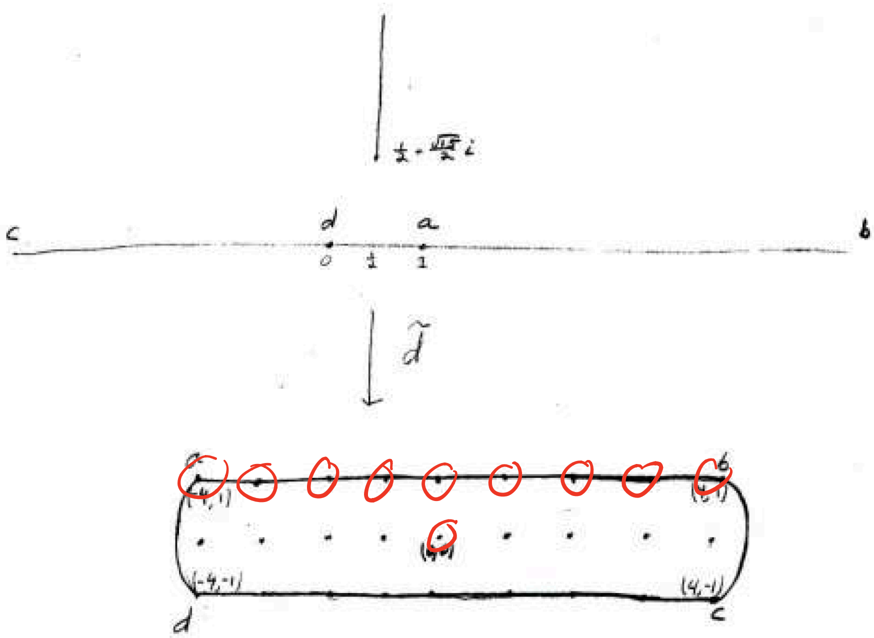
Theorem (Thurston-1978) All but exactly 10 fillings of the figure-8 knot complement are hyperbolic

Theorem (Thurston-1978) A 1-cusped hyperbolic 3-manifold has only finitely many non hyperbolic fillings



Tetrahedron with figure-eight knot, viewed from above

Figures from Thurston's ~1978 Notes



The Non-hyperbolic Fillings

Uniform upper bounds on the number of non hyperbolic fillings

Thurston (using Gromov-Thurston, Perelman, Meyerhoff) 48
2π Theorem

Bleiler-Hodgson (1992) (using above + Adams) 24

Agol, Lackenby (2000) (using 6 theorem; Cao-Meyerhoff) 12

Lackenby-Meyerhoff (2013) (sharp 6 thm + above + Mom technology) 10

Many other results on Non hyperbolic fillings
– See Gordon (1998) survey
Gordon (1998) Conjectured Fig-8 Complement only one with ≥ 9 exceptions

Agol (2010) If N has 1 cusp and $Vol(cusp) > 2\,4/7$ then N has at most 8 non hyperbolic fillings (uses 6 theorem)

<u>Theorem</u> (Crawford – G – Haraway – Meyerhoff – N. Thurston – Yarmola)

The figure-8 knot complement is the unique 1-cusped hyp manifold with ≥ 9 non hyperbolic fillings.

<u>Proof</u> $2\,4/7 = 2.57\cdots < 2.62$ so an exceptional N is obtained by filling one of 16 explicit manifolds.
Martelli-Petronio (2006) For s776 only $N = S^3$- fig-8
True for the other 15 too. $\square$

Idea of Proof of main result

Step 1 If N has a maximal cusp of Vol ≤ 2.62 then "$\pi_1(N)$ has g-exponential length ≤ 7"

Requires Rigorous Computer assistance

Step 2 If $\pi_1(N)$ has g-exponential length ≤ 7 then N is a filling of a compact hyperbolic manifold with handle description of form

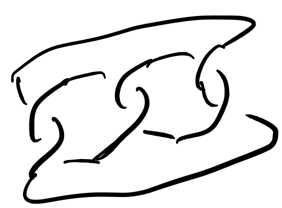
$(T^2 \times I) \cup (1\text{-handle}) \cup (\text{valence} \leq 7 \text{ 2-handle})$

Necklace theory

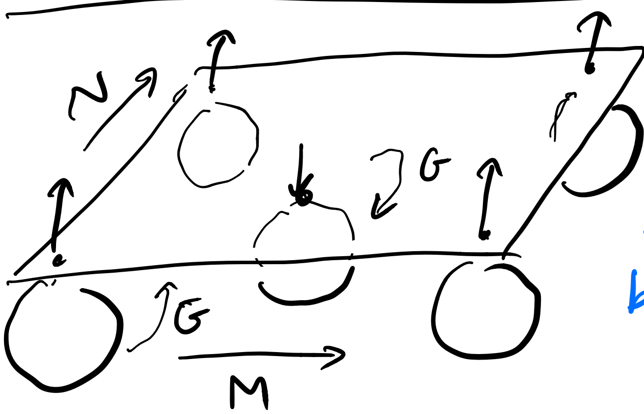
Step 3 There are 28 such hyperbolic manifolds

- reduces to 16 since some are fillings of s776

Requires Rigorous Computer assistance

$s776 = S^3 -$ 

Idea of proof of Step 1



Given maximal cusp of N consider bicusped subgroup β of $\pi_1(N)$ generated by $\underbrace{M, N}_{\text{parabolic}}, \underbrace{G}_{\text{Loxodromic}}$

$\beta \subset \pi_1(N)$ is a smallest subgroup s.t. $\mathbb{H}^3/\beta$ has a maximal cusp isometric with that of N.

There is a compact six real parameter space P containing of subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ all such groups. $\mathrm{Vol}(\text{max cusp}) \leq 2.62 \Rightarrow$

Theorem $\beta = \langle M, N, G \mid [M,N]=1, \; w(M,N,G)=1, \ldots \rangle$

and $4 \leq \#$ times g, g^{-1} appears in $w \leq 7$

Proof Chop P into about 1.3B boxes. Either

a) Conclusion holds $\forall$ tor. free discrete parameter in box

b) box fails some boundary condition

c) box dispatched by killer word. $\square$

Biproducts of Step 1

Theorem (GHMTY) Figure-8 complement & sister mfld are the unique complete finite volume hyperbolic manifolds with a maximum cusp of volume $\leq \sqrt{3} = 1.73\ldots$

- Meyerhoff (1986) $\sqrt{3}/4$
- Adams (1987) $\sqrt{3}/2$
- Cao-Meyerhoff (2001) $\frac{3.35}{2} = 1.675$

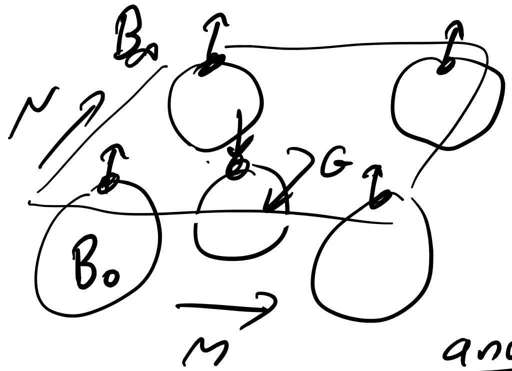
Theorem (GHMTY)

- The maximal cusp volume of one cusp of the whitehead link complement strictly decreases under filling the other cusp.

- There exist 2-cusped manifolds (e.g., m295) such that maximal cusp volume sometimes goes up, sometimes down when filling the other cusp.

whitehead link

Lemma If $\pi_1(N) = \langle M, N, G \mid [M,N]=1, \; w(M,N,G)=1, \dots \rangle$



These horoballs are tangent to B_∞.

and $w(M,N,G)$ has g exponential length $= k$ then

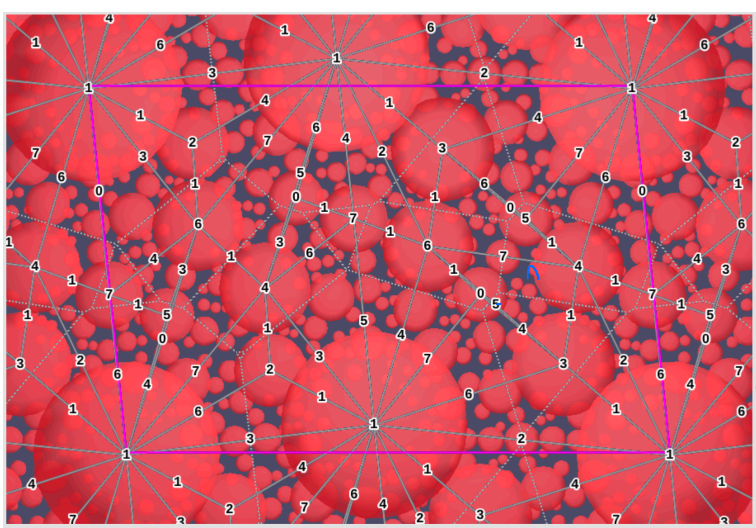
$\mathbb{H}^3 = \tilde{N}$ has a k-Necklace i.e.

a cycle of k-horoballs

$H_0, H_1, \dots, H_{k-1}$ s.t. $\forall i$

$\exists$ a conjugate of g taking

H_i to H_{i+1} modulo k.



A 7 Necklace

V3376(3,1)

Theorem (GHMTY) If N has a $k \leq 7$ Necklace then N is a filling of a compact hyperbolic 3-manifold M

$$M = T^2 \times I \cup (1\text{-handle}) \cup (2\text{-handle})$$

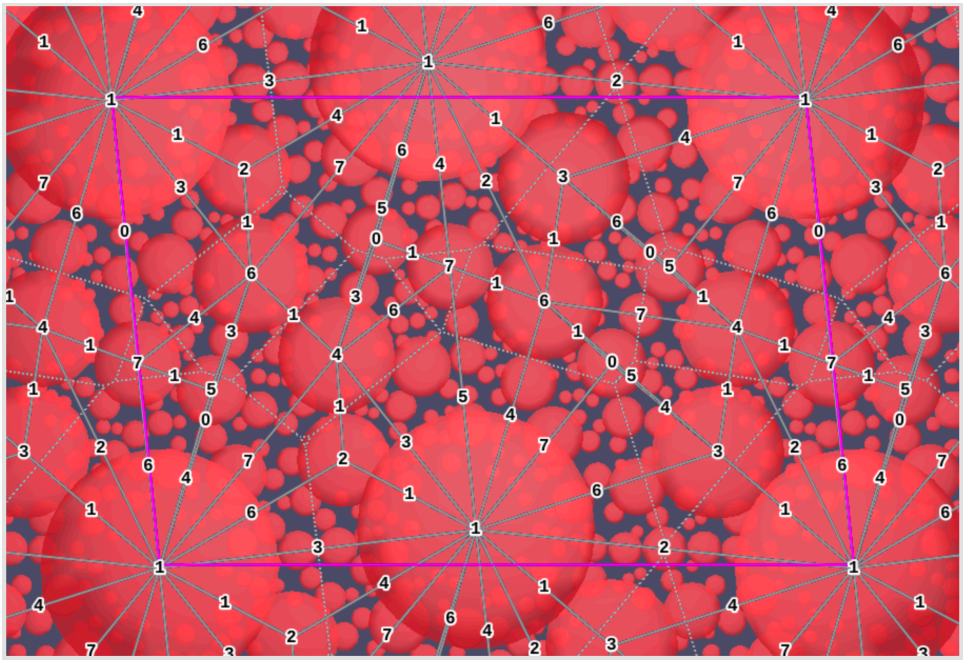
where 2-handle runs over 1-handle $\leq k$ times.

Idea of Proof

Let $T^2 = \partial(\text{Maximal cusp})$ pushed slightly into the cusp.

$T^2 \times I$ = small product nbhd. Fixing $T^2 \times 0$ ← side facing cusp
expand other end into manifold. Tangency with other horoballs creates 1-handle (Morse theory).

The 2-handle corresponds to equivariantly compressing the Necklace. $k \leq 7$ used repeatedly to show such a 2-handle exists.



V3376(3,1)

cusp vol

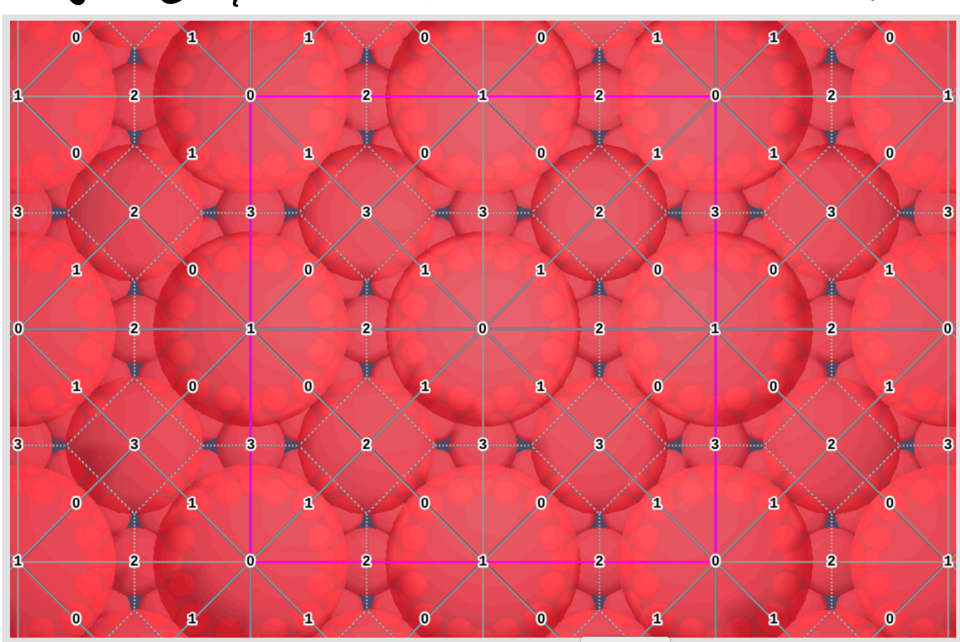
$\approx 2.85\ldots$

Need to show Necklace is a) unknotted

b) unblocked

c) unlinked

This 8-Necklace is blocked



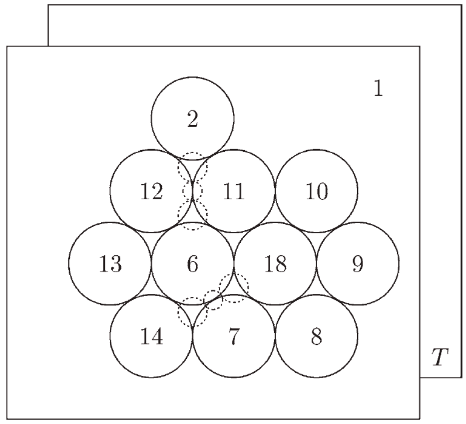
m135

cusp vol =

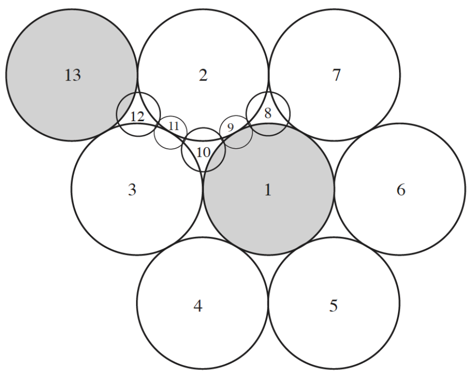
$2\sqrt{2} = 2.82\ldots$

Adams-Knudson (2013) ≤ 6 Necklaces are unblocked

(false for non orientable hyp manifolds)



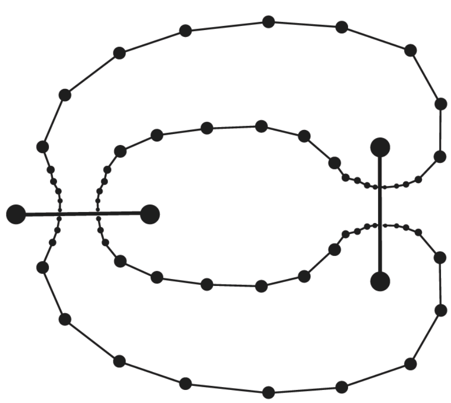
A necklace whose core is a trefoil. Picture from Maehara 2007



Theorem Minimal $K \leq 8$ Necklaces are unknotted.

Conjecture True for $K \leq 13$

Figure 6: A knotted necklace with unknotted core.
The 14th horoball is at infinity.



A Borromean necklace. It is unknotted and unblocked, but is linked. The arcs are ties in a horoball system.

Biproduct of Step 2

Corollary (GHMTY) If N has a maximal cusp of volume ≤ 2.62 then any bicusped subgroup is index-1 in N.

SnapPy (experimentally) For m135, m136 the bicusped groups are index 2.

Conjecture (GHMTY) $\mathrm{Vol}(\text{max cusp}) \leq 2\sqrt{2} \Rightarrow N = m135$ or $m136$ or any bicusped group is index-1 in $\pi_1(N)$.

Theorem (Agol 2010) If N has a maximal cusp of Vol $< \pi$ then any bicusped group to that cusp is finite index in $\pi_1(N)$.

Step 3 N is one of 28 explicit hyperbolic manifolds of which s776 is 3-cusped and the others are 2-cusped.

28 can be reduced to 16 since 12 are fillings of s776.

Conjecture (GHMTY) If N has a maximal cusp of volume $\leq 2\sqrt{2}$ then either $N = m125$ or $m126$ or is a filling of one of 16 explicit hyperbolic 3-manifolds.