Supramaximal representations of planar surface groups

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Abstract

Recently Deroin, Tholozan and Toulisse found connected components of relative character varieties of surface group representations in a Hermitian Lie group $G$ with remarkable properties. For example, although the Lie groups are never compact, these components are compact. In this way they behave more like relative character varieties for compact Lie groups. (A relative character variety comprises equivalence classes of homomorphisms of the fundamental group of a surface $S$, where the holonomy around each boundary component of $S$ is constrained to a fixed conjugacy class in $G$.) The first examples were found by Robert Benedetto and myself in an REU in summer 1992. Here $S$ is the 4-holed sphere and $G = \text{SL}(2,\mathbb{R})$. Although computer visualization played an important role in the discovery of these unexpected compact components, computation was invisible in the final proof, and its subsequent extensions.
A specific example

Fix \((\alpha, \beta, \gamma, \delta) \in \mathbb{R}^3\). Then the equation (in \((\xi, \eta, \zeta) \in \mathbb{R}^3\))

\[
\xi^2 + \eta^2 + \zeta^2 + \xi\eta\zeta = (\alpha\beta + \gamma\delta)\xi + (\beta\gamma + \delta\alpha)\eta + (\alpha\gamma + \beta\delta)\zeta + 4 - (\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \alpha\beta\gamma\delta),
\]

describes a cubic surface in \(\mathbb{R}^3\). For certain \((\alpha, \beta, \gamma, \delta)\) — for example, \((3/2, 3/2, 3/2, -3/2)\) — this surface has one component \(\approx S^2\) and four components \(\approx D^2\). 

Relative Character Varieties

\[ \Sigma \text{ compact oriented surface with boundary } \partial \Sigma = \partial_1 \sqcup \ldots \sqcup \partial_n \]

\[ \pi_1(\Sigma) \text{ with peripheral structure } \pi_1(\partial_i) \hookrightarrow \pi_1, i = 1, \ldots, n. \]

\[ \text{G reductive linear algebraic group over } k \text{ (either } \mathbb{R} \text{ or } \mathbb{C} \text{).} \]

\[ \text{Hom}(\pi, G) \text{ affine algebraic set with algebraic Inn}(G)-action } \]

\[ X(\Sigma, G) := \text{Hom}(\pi, G) \big/ \text{Inn}(G) \text{ categorical quotient.} \]

\[ \text{Restriction to } \pi_1(\partial_i) \text{ defines family } X(\Sigma, G) \rightarrow X(\partial_1, G) \times \ldots \times X(\partial_n, G) \text{ of relative character varieties.} \]

\[ \text{Natural Poisson structure, relative character varieties are symplectic leaves, and the restriction maps are Casimirs.} \]
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  - Natural Poisson structure, relative character varieties are symplectic leaves, and the restriction maps are Casimirs.
Vogt-Fricke theorem and $F_2$

Let $F_2 = \langle X, Y \rangle$ be a two-generator free group. Then $\text{Hom}(F_2, \text{SL}(2)) \cong \text{SL}(2) \times \text{SL}(2)$ and $X(F_2, \text{SL}(2))$ is its quotient under $\text{Inn}(\text{SL}(2))$.

The $\text{Inn}(\text{SL}(2))$-invariant mapping $\text{Hom}(F_2, \text{SL}(2)) \to \mathbb{C}^3, \rho \mapsto \begin{bmatrix} \xi := \text{Tr}(\rho(X)) \\ \eta := \text{Tr}(\rho(Y)) \\ \zeta := \text{Tr}(\rho(XY)) \end{bmatrix}$ defines an isomorphism $X(F_2, \text{SL}(2)) \cong \mathbb{C}^3$.

Characters in $\mathbb{R}^3 \subset \mathbb{C}^3$ (the $\mathbb{R}$-points) $\leftrightarrow$ equivalence classes of representations into $\mathbb{R}$-forms $\text{SU}(2)$ and $\text{SL}(2, \mathbb{R})$ of $\text{SL}(2)$. 
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One-Holed Torus

The fundamental group of is a two-generator free group \( \langle X, Y \rangle \) with redundant geometric presentation

\[
\pi := \langle X, Y, K \mid K = X Y X^{-1} Y^{-1} \rangle
\]

with peripheral generator \( K \).
Boundary trace for the one-holed torus $\Sigma_{1,1}$

Commutator trace function corresponds to the peripheral structure $\partial_1 = K = [X, Y] = XYX^{-1}Y^{-1}$:

$X(F_2, \text{SL}(2)) \sim \kappa \longrightarrow \mathbb{C}(\xi, \eta, \zeta) \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta^{-2} = \text{Tr}[\rho(X), \rho(Y)]$

where $\xi = \text{Tr}(\rho(X)), \eta = \text{Tr}(\rho(Y)), \zeta = \text{Tr}(\rho(XY))$.

Level sets of $\kappa$ are relative character varieties:

- For $\kappa < -2$, level set $\leftrightarrow$ complete hyperbolic structures with ideal boundary: 4 discs (parametrized by spin structures on $\Sigma$) and fixed $\partial$ width.

- For $\kappa = -2$, level set $\leftrightarrow$ complete finite area hyperbolic structures (Markoff surface), and the origin $(0, 0, 0) \leftrightarrow$ Pauli spin (quaternion) representation in $\text{SU}(2)$.

- For $-2 < \kappa < 2$, the level set has five components, corresponding to hyperbolic structures with a cone point (noncompact) and a compact component corresponding to $\text{SU}(2)$-representations.

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- Quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution
  \[(a, b) \mapsto (a^{-1}, b^{-1}).\]

\[\xi = a + a^{-1}, \quad \eta = b + b^{-1}, \quad \zeta = ab + (ab)^{-1}\]
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    $$\xi = a + a^{-1}, \quad \eta = b + b^{-1}, \quad \zeta = ab + (ab)^{-1}$$
  - For example, $X \xrightarrow{\rho} \begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix}$, $Y \xrightarrow{\rho} \begin{bmatrix} b & * \\ 0 & b^{-1} \end{bmatrix}$. 
R-points: Unitary representations

\[ \begin{align*}
\text{R-points correspond to representations into } \mathbb{R}\text{-forms of } \text{SL}(2) \\
\text{either } \text{SL}(2, \mathbb{R}) \text{ or } \text{SU}(2). \\
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$\mathbb{R}$-points: Unitary representations

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- Markoff surface $\mathbb{R}^3 \cap \kappa^{-1}(-2) \setminus \{O\}$ parametrizes complete finite area hyperbolic structures on $\Sigma_{1,1}$, forming four other components.
Compact components for $\Sigma_{1,1}$

The four noncompact level sets for $-2 < \kappa < 2$ correspond to hyperbolic structures on torus with an isolated singularity of cone angle $\theta$, where $\kappa = -2 \cos(\theta/2)$. A fifth compact component corresponds to $SU(2)$-representations.
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Four-Holed Sphere

The fundamental group of is a three-generator free group $\langle A, B, C \rangle$ given by redundant geometric presentation

$$\pi := \langle X, Y, Z, A, B, C, D \mid X = AB, Y = BC, Z = CA \rangle$$

with peripheral generators $A, B, C, D$. 
Relative character variety for four-holed sphere

The fundamental group is the 3-generator free group $\langle A, B, C, D \mid ABCD = e \rangle$, with boundary traces:

$\alpha := \text{Tr}(\rho(A))$, $\beta := \text{Tr}(\rho(B))$, $\gamma := \text{Tr}(\rho(C))$, $\delta := \text{Tr}(\rho(D))$.

Interior traces are:

$\xi := \text{Tr}(\rho(AB))$, $\eta := \text{Tr}(\rho(BC))$, $\zeta := \text{Tr}(\rho(CA))$.

These seven functions are related by the defining equation

$\xi^2 + \eta^2 + \zeta^2 + \xi\eta\zeta = (\alpha\beta + \gamma\delta)\xi + (\beta\gamma + \delta\alpha)\eta + (\alpha\gamma + \beta\delta)\zeta + 4 - (\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \alpha\beta\gamma\delta)$,

a family of cubic surfaces parametrized by $(\alpha, \beta, \gamma, \delta)$.

(Benedetto-G 1992) For certain $(\alpha, \beta, \gamma, \delta) \in [-2, 2]$ compact components of $\text{SL}(2, \mathbb{R})$-characters exist.
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a family of cubic surfaces parametrized by \((\alpha, \beta, \gamma, \delta)\).

(Benedetto-G 1992) For certain \((\alpha, \beta, \gamma, \delta) \in [-2, 2]\) compact components of \(\text{SL}(2, \mathbb{R})\)-characters exist.
Deroin-Tholozan found compact components of $\text{SL}(2,\mathbb{R})$-characters for $X(\Sigma, n)$ for all $n \geq 3$.

Compact components of relative $\text{SL}(2,\mathbb{R})$-characters only exist for planar surfaces, that is, when $\Sigma$ has genus 0.

Unlike components of Fuchsian characters, $\rho(x)$ is elliptic for every $x \in \pi$ corresponding to a simple closed curve. Otherwise the Hamiltonian flow of the corresponding trace function would be unbounded.

The orbit $\text{Mod}(\Sigma)[\rho]$ is bounded.
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Poisson geometry and holomorphic metrics

For every choice of boundary traces, each compact component of $X(\Sigma_0, n, \text{SL}(2, \mathbb{R}))$ is symplectomorphic to $\mathbb{C}P^n-3$ with its standard Fubini-Study symplectic structure.

Recently Tholozan-Toulisse have found compact components of supramaximal representations from $\text{SL}(2, \mathbb{R})$ in higher rank Hermitian Lie groups: $\text{PU}(p, q)$, $\text{Sp}(2m, \mathbb{R})$, $\text{SO}^*(2m)$.

For every complex structure on $\Sigma_0, n$, $\exists \rho$-equivariant holomorphic map $\tilde{\Sigma}_0, n \rightarrow G/K$.

(analogous to constant map when $G$ is compact)

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