Non-arithmetic lattices

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Complex hyperbolic space

Complex hyperbolic plane $H^n_{\mathbb{C}}$

- Unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ with Bergman metric
- In homogeneous coordinates ($\mathbb{C}^n$=affine chart of $\mathbb{P}^n$), set of negative lines for $\langle z, w\rangle = -z_0\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, with distance function

$$\cosh(\frac{1}{2}d([z],[w])) = \frac{|\langle v, w\rangle|}{\sqrt{\langle v, v\rangle\langle w, w\rangle}}$$

- Holomorphic isometry group: $PU(n,1)$.

Lattices

- No known classification of lattices (discrete subgroups of $PU(n,1)$ with finite covolume)
- There are non-arithmetic lattices in $PU(n,1)$ for $n \leq 3$ (Mostow 1980, Deligne-Mostow 1986, D.-Parker-Paupert 2016, 2021, D. 2020)
- It is widely beleived that they should exist also for every $n \geq 1$ (cf. $PO(n,1)$)

Known constructions of lattices

- Arithmetic constructions (come from $\mathbb{G}_{\mathbb{Z}}$ in $\mathbb{G}_{\mathbb{R}}$ for an algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$)
- Explicit generating set/fundamental domains
- Uniformization (period mappings, Aubin-Yau)

Fundamental domains

- No totally geodesic real hypersurfaces
- Bisectors have quadratic equations
- A key step is to determine the combinatorics/topology of a semi-algebraic set
- Then apply the Poincaré polyhedron theorem

Triangle groups

(Thompson, Parker-Paupert) Can parametrize $(p, q, r; n)$ triangle groups, i.e groups generated by three complex reflections R_1, R_2, R_3 of the same order k, such that

$$(R_1 R_2)^{p/2} = (R_2 R_1)^{p/2},$$
$$(R_2 R_3)^{q/2} = (R_3 R_2)^{q/2},$$
$$(R_3 R_1)^{r/2} = (R_1 R_3)^{r/2},$$
$$(R_1 \cdot R_3^{-1} R_2 R_3)^{n/2} = (R_3^{-1} R_2 R_3 \cdot R_1)^{n/2}$$

D.-Parker-Paupert 2016, 2021

- Many $(p, q, r; n)$ triangle groups are lattices, some non-arithmetic
- The original proof uses heavy computation (Spocheck)
- Can determine the commensurability classes (trace fields, Margulis commensurator theorem + volume estimates).

Recover all previously known non-arithmetic lattices in PU(2,1)

- Mostow 1980
- Deligne-Mostow 1986
- Couwenberg-Heckman-Looijenga 2005

Produce more examples: 22 commensurability classes in $PU(2, 1)$.

List of examples

	$\mathbb{P}^2_{\mathbb{C}}$	$\mathbb{E}^2_{\mathbb{C}}$	$\mathbb{H}^2_{\mathbb{C}}$	
Type $p, q, r; n$	k	k	k	Alternative description
2,3,3; 3	2,3	4	5,6,7,8,9,10,12,18	DM/Livné
2,3,4; 4	2	3	4,5,6,8,12	DM
2,3,5; 5	2		3,4,5,10	σ_{10}, CHL(H_3)
2,3,6; 6		2	3,4,6	
3,3,3; 2	2,3	4	5,6,7,8,9,10,12,18	DM
3,3,3; 3	2		4,5,6,7,8,9,10,12,18	DM
3,3,3; 4	2		3,4,5,6,8,12	DM
3,3,3; 5	2		3,4,5,10	DM
3,3,3; 6	2		3,4,6	DM
3,3,3; 7	2		3,−7	DM
3,3,3; 8	2		3,4	DM
3,3,3; 9	2		3	DM
3,3,3; 10	2		3	DM
3,3,3; 12	2		3	DM
3,3,4; 4	2		3,4,5,6,8,12	$\mathbf{S_1} \simeq \overline{\sigma_4}$, CHL($G_{24}$)
3,3,4; 5	2		3,4,5	$\mathbf{S_2}$, CHL(G_{27})
3,3,4; 6		2	3,4,5	$\mathbf{E_1} \simeq \sigma_1$
3,3,4; 7			2,−7	$\mathbf{H_1}$
3,3,5; 5			2,3,±5,10	$\mathbf{H_2}$
3,4,4; 4		2	3,4,6,12	$\mathbf{E_2}$
4,4,4; 4		2	3,4,5,6,8,3,12	DM (finite index)
4,4,4; 5			2,3,4	σ_5
5,5,5; 5			2,3,4,5,10	DM (finite index)

Sides of a family of fundamental domains

Triangle group type: 4,4,4; 3,3,3; 7

Lattice for $p = 3, 4, 5, 6, 8, 12$.

Commensurability invariants:

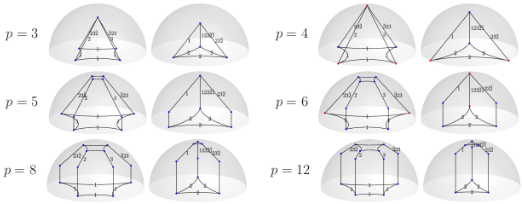
p	χ^{orb}	$\mathbb{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Gamma)$	C?	A?
3	$2/63$	$\mathbb{Q}(\sqrt{21})$	C	A
4	$25/224$	$\mathbb{Q}(\sqrt{7})$	NC	NA(1)
5	$47/280$	$\mathbb{Q}(\sqrt{\frac{5+\sqrt{5}}{14}})$	C	NA(2)
6	$25/126$	$\mathbb{Q}(\sqrt{21})$	NC	NA(1)
8	$99/448$	$\mathbb{Q}(\sqrt{2},\sqrt{7})$	C	NA(2)
12	$221/1008$	$\mathbb{Q}(\sqrt{3},\sqrt{7})$	C	NA(2)

Presentations:

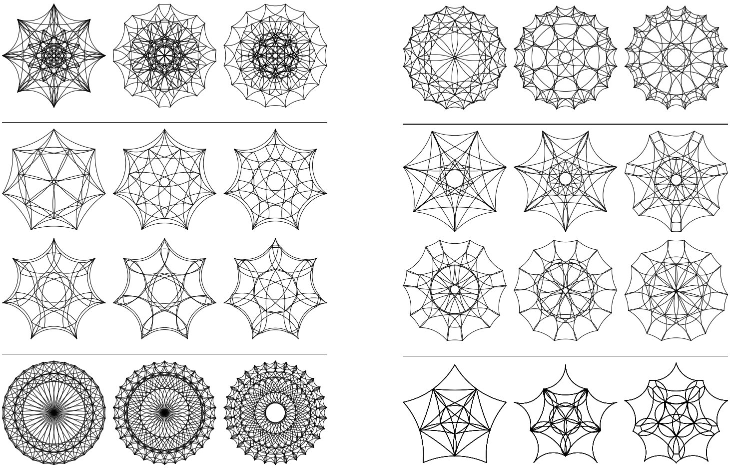
$$\Big\langle R_1, R_2, R_3, J \mid R_1^p,\ J^3,\ (R_1J)^7,\ R_3 = JR_2J^{-1} = J^{-1}R_1J,$$
$$\mathrm{br}_4(R_1,R_2),\ (R_1R_2)^{\frac{4p}{p-4}},\ \mathrm{br}_3(R_1, R_2R_3R_2^{-1}),\ (R_1R_2R_3R_2^{-1})^{\frac{6p}{p-6}} \Big\rangle$$

Rough combinatorics:

Triangle	#(P-orb)	Top trunc.	Top ideal
[4] 1; 2, 3	7	$p = 5, 6, 8, 12$	$p = 4$
[3] 2; 1, $23\bar{2}$	7	$p = 8, 12$	$p = 6$



Pictures of fundamental domains



Finite UGGR

Some families include finite groups, that must appear in the Shephard-Todd list (Finite UGGR = Unitary Group Generated by complex Reflections).
For each $G \subset U(3)$ finite $UGGR$, the ring of invariants $\mathbb{C}[z_1, z_2, z_3]^G$ is known to be a polynomial algebra generated by explicit homogeneous polynomials of degree d_1, d_2, d_3.
We get an explicit weighted projective space

$$\mathbb{P}^2/G = \mathbb{P}(d_1, d_2, d_3).$$

Can also identify the quotient as an orbifold, the branch locus being given by the union of the mirrors of reflections.

Kobayashi-Nakamura-Sakai

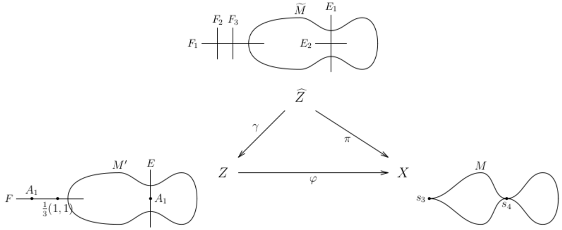
In particular, we get a description of $\mathbb{P}^2/G$ as an orbifold (X, D), where D is an explicit Q-divisor. For groups generated by involutive reflections, $D = \frac{1}{2}C$ where C is an irreducible curve.
Rough idea:

1. Keep X and consider $D = (1 - \frac{1}{k})C$ for positive integers k (or $k = \infty$)
2. Check whether the pair (X, D) has at most log-canonical singularities, and check $K_{(X,D)} := K_X + D$ is ample.
3. Check whether $c_1^2(X, D) = 3c_2(X, D)$.
4. If so, then after removing the components of D with weight ∞ and the non log-terminal locus, we get a ball quotient of finite volume (Kobayashi-Nakamura-Sakai).

Blow-up

The above condition holds very rarely, but it holds many times (but for special values of k) after a suitable blow-up.

1. To get log-canonical singularities, one often needs to blow-up X (blow-up $\mathbb{P}^2$ in a G-equivariant way).
2. The exceptional divisors of the blow-up pick up orbifold weights (determined by the requirement $c_1^2 = 3c_2$).



Ball quotient pairs

For example, when G is the automorphism group of the Klein quartic (simple group of order 168), 7 values of k yield a ball quotient:

$$X^{(3)} = X, D^{(3)} = (1 - \tfrac{1}{3})M$$
$$X^{(4)} = X, D^{(4)} = (1 - \tfrac{1}{4})M$$
$$X^{(5)} = Y, D^{(5)} = (1 - \tfrac{1}{5})M + (1 - \tfrac{1}{10})E$$
$$X^{(6)} = Y, D^{(6)} = (1 - \tfrac{1}{6})M + (1 - \tfrac{1}{6})E$$
$$X^{(8)} = Z, D^{(8)} = (1 - \tfrac{1}{8})M + (1 - \tfrac{1}{4})E + (1 - \tfrac{1}{8})F$$
$$X^{(12)} = Z, D^{(12)} = (1 - \tfrac{1}{12})M + (1 - \tfrac{1}{3})E + (1 - \tfrac{1}{4})F$$
$$X^{(\infty)} = Z, D^{(\infty)} = M + (1 - \tfrac{1}{2})E + (1 - \tfrac{1}{2})F$$

Conclusion

Not obvious how to find explicit matrix generating sets for these lattices (their existence comes from existence of a solution to a Monge-Ampère equation).

- Can prove that the corresponding ball quotients are isometric (compute orbifold fundamental group and search for an isomorphism)
- Using such uniformization techniques we get a new proof of the existence of *most* non-arithmetic lattices constructed by D.-Parker-Pauper (18 of the 22 commensurability classes).