Computing with hyperbolic structures in dimension 3

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Expanded version with references:
http://dunfield.info/warwick2017
Orientable $M^3$ with comp. metric of curvature -1; compact or finite-vol.

That is, $M = \Gamma \backslash \mathbb{H}^3$ for $\Gamma$ a lattice in $\text{Isom}^+ \mathbb{H}^3 = \text{SO}_0(3,1) = \text{PSL}_2 \mathbb{C}$.

Ex:

$X = S^3 - \bigcirc$ =

Here $\Gamma = \langle (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}) \rangle$ for $\alpha = e^{\pi i/3} = \frac{1 + \sqrt{3}i}{2}$.

$\text{vol}(X) = 2.029883212819307250...$

systole = 1.08707014499573909...

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[Thurston/Perelman] Closed $M^3$ have geometric decompositions. Hyperbolic geom is the most common and the most interesting/mysterious.

[Mostow] The hyperbolic structure is unique when it exists ($\dim \geq 3$.)

**Cor:** Can solve the homeomorphism problem for compact $M^3$. Also, $\pi_1 M^3$ is residually finite and so has solvable word problem.

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Input: Topological description, such as a triangulation, bundle structure, knot/link construction, Heegaard splitting, ...

Note: Most hyperbolic 3-manifolds are not arithmetic.

Output: Description of hyperbolic structure in terms of glued geometric polyhedra.

Can then get volumes, lengths of geodesics, eigenvalues of $\Delta$, harmonic forms, word problem, and decide when two such are isometric.

Can do at scale: 1,000s of tets and millions of examples.
Quick SnapPy Demo 1.

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Finite-volume $M^3$ have ends like:

$\{ \mathbb{T}^2 \times [0, \infty) \}$

Setting: $N$ cpt with $\partial N$ tori; $M = N \setminus \partial N$ hyperbolic.

An *ideal triangulation* $\mathcal{T}$ is a cell complex built from finitely many tets by gluing faces in pairs with $\mathcal{T} \setminus \mathcal{T}^0 \cong M$. 
Geometric ideal tetrahedra:
\[ H^3 = \{(x,y,t) \in \mathbb{R}^3 \mid t > 0\}, \quad g_{H^3} = \frac{1}{t^2} g_{E^3}. \]

\[ z = \text{shape param associated to the edge joining 0 to } \infty \]
Suppose \( T \) is an ideal triangulation of \( M \) and \( z_i \in \mathbb{C} \) with \( \text{Im}(z_i) > 0 \) satisfy the polynomial edge and cusp eqns and also \( \sum \text{arg}(z_i) = 2\pi \) around each edge. Then these shapes give the complete hyperbolic structure on \( M \). In particular,

\[
\text{vol}(M) = \sum \text{Li}_2(z_i) < 1.02(\#\text{tet})
\]

Note: Effectively, \( \#z_i = \#\text{equations} \).

In practice, can solve numerically via Newton’s method starting with all \( z_i = e^{\pi i/3} \).
Method for certifying solutions to Thurston’s equations using interval arithmetic. Given $\epsilon > 0$, returns $z'_i \in \mathbb{Q}(i)$ and a proof that the actual solution $z_i$ satisfies $|z_i - z'_i| < \epsilon$.

Moral: Proof of the inverse function theorem is effective (Interval Newton’s Method; Krawczyk’s test).

Can then compute volume to guaranteed accuracy, provably decide if two such manifolds are isometric, find symm. groups, solve the word problem, but more to do.

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[Weeks]
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“Delaunay dual to Vornoi decomp based at infinity”


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[D-Hirani] Numerical methods for finding harmonic representatives of elements of $H^1(M; \mathbb{R})$, and so can compute regulators, RS torsion, etc.

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