Computing with hyperbolic structures in dimension 3

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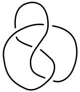
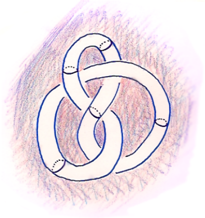
Expanded version with references:
http://dunfield.info/warwick2017

Orientable M^3 with comp. metric of curvature -1; compact or finite-vol.

That is, $M = \Gamma \backslash \mathbb{H}^3$ for Γ a lattice in $\mathrm{Isom}^+ \mathbb{H}^3 = \mathrm{SO}_0(3,1) = \mathrm{PSL}_2\mathbb{C}$.

Ex:

$X = S^3 -$  $=$ 

Here $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \right\rangle$ for $\alpha = e^{\pi i/3} = \frac{1+\sqrt{3}i}{2}$.

$\mathrm{vol}(X) = 2.0298832128193072500\ldots$

systole $= 1.0870701449957390 9\ldots$

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[Thurston/Perelman] Closed M^3 have geometric decompositions. Hyperbolic geom is the most common and the most interesting/mysterious.

[Mostow] The hyperbolic structure is unique when it exists (dim ≥ 3.)

Cor: Can solve the homeomorphism problem for compact M^3. Also, $\pi_1 M^3$ is residually finite and so has solvable word problem.

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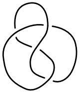
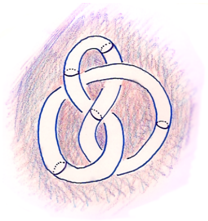
$X = S^3 -$  $=$ 

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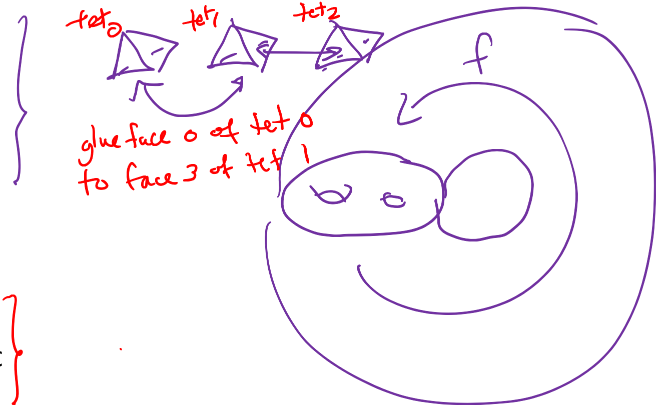
Input: Topological description, such as a triangulation, bundle structure, knot/link construction, Heegaard splitting,...

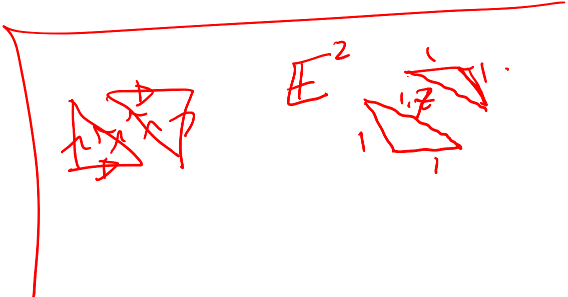
Note: Most hyperbolic 3-manifolds are not arithmetic.

Output: Description of hyperbolic structure in terms of glued geometric polyhedra.

Can then get volumes, lengths of geodesics, eigenvalues of Δ, harmonic forms, word problem, and decide when two such are isometric.

Can do at scale: 1,000s of tets and millions of examples.





Quick SnapPy Demo 1.

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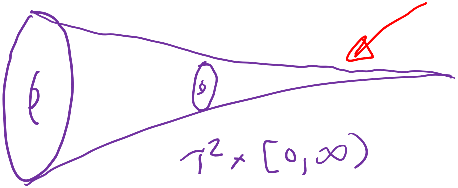
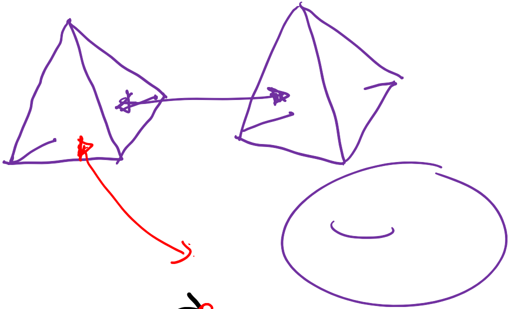
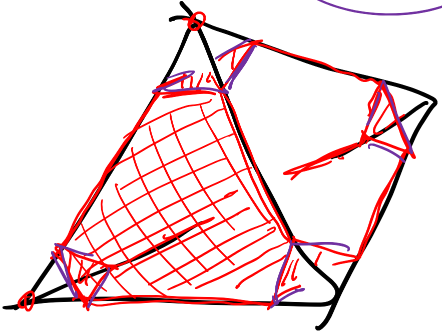
Finite-volume M^3 have ends like:





Setting: N cpt with ∂N tori; $M = N \setminus \partial N$ hyperbolic.

An *ideal triangulation* $\mathcal{T}$ is a cell complex built from finitely many tets by gluing faces in pairs with $\mathcal{T} \setminus \mathcal{T}^0 \cong M$.



Geometric ideal tetrahedra:

$\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid t > 0\}$, $g_{\mathbb{H}^3} = \frac{1}{t^2} g_{\mathbb{E}^3}$.

$z =$ shape param associated to the edge joining 0 to ∞

[Thurston] Suppose $\mathcal{T}$ is an ideal triangulation of M and $z_i \in \mathbb{C}$ with $\mathrm{Im}(z_i) > 0$ satisfy the polynomial edge and cusp eqns and also $\sum \arg(z_i) = 2\pi$ around each edge. Then these shapes give the complete hyperbolic structure on M. In particular

$$\mathrm{vol}(M) = \sum \mathrm{Li}_2(z_i) < 1.02(\#\mathrm{tet})$$

Note: Effectively, $\# z_i = \#$equations.

[Weeks] In practice, can solve numerically via Newton's method starting with all $z_i = e^{\pi i/3}$.

[HIKMOT 2014] Method for certifying solutions to Thurston's equations using interval arithmetic. Given $\epsilon > 0$, returns $z_i' \in \mathbb{Q}(i)$ and a proof that the actual solution z_i satisfies $\left|z_i - z_i'\right| < \epsilon$.

Moral: Proof of the inverse function theorem is effective (Interval Newton's Method; Krawczyk's test).

Can then compute volume to guaranteed accuracy, provably decide if two such manifolds are isometric, find symm. groups, solve the word problem, but more to do.

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