On the singular local limit of conservation laws with nonlocal fluxes

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Joint works with Maria Colombo, Gianluca Crippa, Elio Marconi

Nonlocal conservation law

Equation:

$$\partial_t u + \operatorname{div}\big[uV(u * \eta)\big] = 0$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $u(t, x) \in \mathbb{R}$, $V : \mathbb{R} \to \mathbb{R}^d$ smooth

$$\eta \in C^1_c(\mathbb{R}^d), \quad \eta \geq 0, \quad \int_{\mathbb{R}^d} \eta(x) dx = 1$$

Convolution:

$$[u * \eta](t, x) = \int_{\mathbb{R}^d} u(t, x - y)\eta(y) dy$$

Existence & uniqueness results for Cauchy problem: Blandin, Bressan, R Colombo, Crippa, Garavello, Goatin, Keimer, Mercier-Leroux, Pflug, Shen..

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Applications to traffic models ($d = 1$)

- Classical models (LWR)

$$\partial_t u + \partial_x \left[u V(u) \right] = 0$$

u: density of cars moving at speed V
$V' < 0$, e.g. $V(u) = 1 - u$

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Drivers look around!

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Singular limit

Original nonlocal equation

$$\partial_t u + \partial_x \big[u V(u * \eta) \big] = 0$$

Rescale: fix $\varepsilon > 0$

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon} \eta \left(\frac{x}{\varepsilon} \right) \quad \Longrightarrow \quad \eta_\varepsilon \overset{*}{\rightharpoonup} \delta_{x=0} \text{ as } \varepsilon \to 0^+$$

$$\begin{cases} \partial_t u_\varepsilon + \partial_x \big[u_\varepsilon V(u_\varepsilon * \eta_\varepsilon) \big] = 0 \\ u_\varepsilon(0, x) = u_0(x) \end{cases} \longrightarrow \begin{cases} \partial_t u + \partial_x \big[u V(u) \big] = 0 \\ u(0, x) = u_0(x) \end{cases}$$

Question (Amorim, R Colombo, Teixeira)

Does u^ε converge to the entropy admissible solution u?

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Answer (Zumbrun)

YES, provided $V(u) = u$, u is regular, η_ε is even

Answer (Amorim, R Colombo, Teixeira)

YES, based on numerical evidence

Answer (M Colombo, Crippa, S.)

NO: exhibit three counter-examples. In general u^ε does NOT converge to u, not even weakly or up to subsequences

The counter-examples are explicit. We single out a property that is i) satisfied by u_ε ; ii) **stable** under *weak* or *strong* convergence and iii) **not** satisfied by u

One ounterexample rules out convergence to **distributional** solutions (Elio Marconi)

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Anisotropic traffic models

Nonlocal LWR:

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where

1. $0 \leq u_0 \leq 1$
2. $V' < 0$ e.g. $V(u) = 1 - u$
3. drivers only look forward, not backward (Goatin et al)

$$u * \eta(x) = \int_x^{+\infty} \tilde{\eta}(x-y) u(y) dy \qquad \eta = \mathbb{1}_{]-\infty,0]} \tilde{\eta}$$

Then

- well-posedness Cauchy problem, maximum principle
- counterexamples **do not** apply

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Singular limit for anisotropic traffic models

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Answer (Blandin & Goatin, Keimer & Pflug)

YES, under suitable conditions on V, if u_0 is monotone

If u_0 monotone then $u_\varepsilon(t,\cdot)$ monotone

Answer (Bressan & Shen)

YES, under mild conditions on V, if $u_0 \in BV(\mathbb{R})$, $u_0 \geq a > 0$ and $\eta(x) = e^x \mathbb{1}_{]-\infty,0]}$

Change of variables $\longrightarrow$ system with relaxation

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Singular limit for anisotropic traffic models (cont'd)

Answer (M Colombo, Crippa, Marconi, S.)

YES, under mild assumptions on η *and* V, *if* $u_0 \geq a > 0$ *and*

$$Lip^- u_0 = -\inf_{x<y} \frac{u_0(y) - u_0(x)}{y - x} \leq L < +\infty$$

Oleĭnik type estimate:

$$Lip^- u_\varepsilon(t, \cdot) \leq \frac{1}{ct,} \quad c > 0$$

Key point in all the above results:

$$(*) \quad \mathrm{TotVar}\, u_\varepsilon(t, \cdot) < M \text{ (does not depend on } \varepsilon)$$

Weak* compactness does not suffice!

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Under 1, 2, 3, can we always hope for $(*)$?

Answer (Numerical evidence)

YES

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Total Variation increase

Material derivative: $\partial_t u + \partial_x [ub] = 0 \Longrightarrow \dot{u} = -u \partial_x b$

Take $V(u) = 1 - u$, $\eta_\varepsilon(x) = \mathbb{1}_{]-\varepsilon, 0]} \Longrightarrow u_\varepsilon * \eta_\varepsilon(t, x) = \fint_x^{x+\varepsilon} u_\varepsilon(t, y) dy$,

$$b(t, x) = 1 - \fint_x^{x+\varepsilon} u_\varepsilon(t, y) dy \Rightarrow -\partial_x b = \frac{u_\varepsilon(\cdot + \varepsilon) - u_\varepsilon}{\varepsilon}$$

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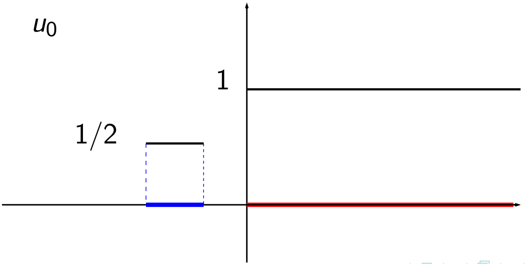
$$b(t,x) = 1 - \fint_x^{x+\varepsilon} u_\varepsilon(t,y)dy \Rightarrow -\partial_x b = \frac{u_\varepsilon(\cdot+\varepsilon) - u_\varepsilon}{\varepsilon} \Rightarrow \dot{u}_\varepsilon(0) = \begin{cases} u_0 \dfrac{1-1}{\varepsilon} = 0 \\ u_0 \dfrac{1-u_0}{\varepsilon} > 0 \end{cases}$$



However...

Question

Under 1, 2,3, does u_ε converge to the entropy admissible solution u?

Answer (Coclite, Coron, De Nitti, Keimer, Pflug)

YES, under mild conditions on V, if $u_0 \in BV(\mathbb{R})$ and $\eta(x) = e^x \mathbb{1}_{]-\infty,0]}$

Note: $(*)$ may fail, **but**

$$\mathrm{TotVar}\,(u_\varepsilon * \eta_\varepsilon(t,\cdot)) < M \text{ (does not depend on } \varepsilon)$$

Thank you!
Happy Birthday!!