Variations on the Theme of Scalar Conservation Law

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Including joint works with A. Bressan, G. Guerra

Overview

Scalar conservation law

$$u_t + f(u)_x = 0$$

Two variations:

1. The flux function $f(t, x, u)$ is discontinuous in t, x

$$u_t + f(t, x, u)_x = 0$$

2. The flux f is nonlocal in (t, x), for example including an integral term

Variation 1: discontinuous flux in (t, x)

Example in traffic flow

$$\rho_t + [\rho\ v(\rho)]_x = 0, \qquad f(\rho) = \rho v(\rho).$$

With rough road condition

$$\rho_t + f(a(x), \rho)_x = 0, \qquad f(a(x), \rho) = a(x)\rho v(\rho)$$

where $a(x)$ is speed limit function, discontinuous in space.

Consider the simple case

$$u_t + [a(x)f(u)]_x = 0, \qquad a(x) \text{ is discontinuous}$$

View this as a 2×2 system of conservation laws

$$u_t + [a\, f(u)]_x = 0, \qquad a_t = 0$$

Characteristic speeds:

$$\lambda_1 = af'(u), \qquad \lambda_2 = 0.$$

Observation

If $f'(u^*) = 0$ for some u^*, then the system is no longer strictly hyperbolic.
$\Longrightarrow$ nonlinear resonance, blowup of total variation. (Temple 1986).

The first few papers:
Elasticity theory (Keyfitz & Kranzer 1980); Oil recovery, water flooding, polymer flooding (Temple & Isaacson 1982; Risebro et al 1990);

Double limits

$$u_t + f(a(x), u)_x = 0$$

Two combined approximations:

(i) Approximate $a(\cdot)$ by a sequence of smooth functions $a_n(x)$, such that $\lim_{n\to\infty} a_n = a$.

(ii) Add a viscosity term εu_{xx} on the right hand side of the PDE.

Double limits $n \to \infty, \varepsilon \to 0$ for: $\quad u_t + f(a_n(x), u)_x = \varepsilon u_{xx}$

In general, the double limits $\varepsilon \to 0$ and $n \to \infty$ do not commute. Infinitely many limits exist. Only in special cases they admit the same limit.
(WS [NoDEA 2017], analysis through traveling waves)

Example: triangular systems

$$\begin{cases} \alpha_t + g(\alpha)_x = 0 \\ u_t + f(\alpha, u)_x = 0 \end{cases}$$

Solution steps:

- Solve for α: $\alpha_t + g(\alpha)_x = 0$

  $\rightarrow$ α can contain shocks and complicated wave interactions

- Solve for u: $u_t + f(\alpha(t,x), u)_x = 0$

  $\rightarrow$ a scalar conservation laws with discontinuous flux

The system is NOT hyperbolic when

$$\lambda_1 = g'(\alpha) = f_u(\alpha, u) = \lambda_2$$

Vanishing Viscosity Solutions

Main objective

The convergence of the viscous approximations, as $\varepsilon \to 0$,

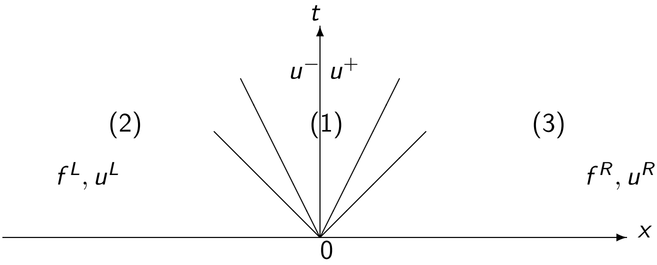
$$u_t + f(\alpha(t,x), u)_x \;=\; \varepsilon\, u_{xx}, \qquad u(0,x) \;=\; \bar{u}(x)$$

to a unique weak solution to the conservation law

$$u_t + f(\alpha(t,x), u)_x = 0, \qquad u(0,x) \;=\; \bar{u}(x)$$

Riemann Problem

$$f(x,u) = \begin{cases} f^L(u), & x < 0 \\ f^R(u), & x > 0 \end{cases}, \qquad u(0,x) = \begin{cases} u^L, & x < 0 \\ u^R, & x > 0 \end{cases}$$



(1) At $x = 0$: Rankine-Hugoniot jump condition $f^L(u^-) = f^R(u^+)$;
(2) On $x < 0$: waves with negative speeds;
(3) On $x > 0$: waves with positive speeds.

But these conditions could still yield infinitely many solutions.

Risebro & Gimse (90)

minimum jump condition is equivalent to vanishing viscosity

There are several other entropy conditions being proposes. In general, these are not vanishing viscosity solutions.

Vast literatures! (incomplete list)

Adimurthi, Andreianov, Bürger, Dihel, Garavello, Gimse, Karlsen, Klingenberg, Isaacson, Mishra, Natalini, Panov, Piccoli, Temple, Terracina, Towers, Risebro, Veerappa Gowda,. . .

On the existence and uniqueness of solutions for the Cauchy problems:

- $\alpha = a(x)$ is piecewise smooth with finitely many jumps (Risebro et al, ARMA 2011): GERM
- $\alpha(t,x) = a(x)b(t)$ where $a(\cdot)$ is piecewise smooth with finitely many jumps, $b(\cdot)$ is BV (Risebro et al)
- $\alpha(t,x)$ is a 2D-regulated function (Bressan, Guerra, Shen, JDE 2018) – applicable to certain triangular systems

Regulated functions $\alpha(x)$ with a single variable

Definition

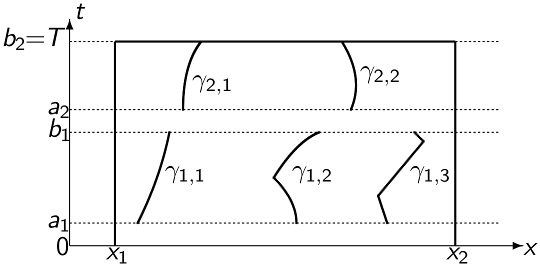
A function of a single variable $\alpha : \mathbb{R} \mapsto \mathbb{R}$ is *regulated* if it admits left and right limits at every point.

Remarks

A function $\alpha : \mathbb{R} \to \mathbb{R}$ is **regulated** if and only if for every interval $[x_1, x_2]$ and every $\epsilon > 0$ there exists a piecewise constant function φ such that

$$\|\varphi - \alpha\|_{\mathbf{L}\infty([x_1,x_2],\mathbb{R})} < \epsilon$$

Extension: regulated functions $\alpha(t,x)$ with two variables



A bounded function $\alpha = \alpha(t,x)$ is **regulated** if, for every intervals $[x_1, x_2]$ and $[0, T]$ and any $\varepsilon > 0$, the following holds.

There exist finitely many disjoint subintervals $[a_i, b_i] \subseteq [0, T]$, Lipschitz continuous curves $\gamma_{i,1}(t) < \gamma_{i,2}(t) < \cdots < \gamma_{i,N(i)}(t)$, $t \in [a_i, b_i]$ and constants $\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,N(i)}$ such that:

[i]. The intervals $[a_i, b_i]$ cover most of $[0, T]$, namely

$$T - \sum_i (b_i - a_i) \leq \varepsilon.$$

[ii]. For every i, k, the time derivative

Variation 2: scalar conservation law with nonlocal flux

Nonlocal traffic flow:

$$
\begin{array}{rll}
\text{Averaging kernel } w: & w(s) \geq 0, & \displaystyle\int w(s)\, ds = 1 \\
\text{Looking "forward":} & w(s) = 0, & s < 0 \\
\text{Decaying importance:} & w'(s) \leq 0, & s > 0
\end{array}
$$

Averaged car density:

$$q(t,x) \;=\; \int_{-\infty}^{+\infty} w(s)\, \rho(t, x+s)\, ds$$

Conservation law

$$\rho_t + (\rho v(q))_x \;=\; 0$$

Global existence and uniqueness

Global existence, uniqueness, continuous dependence of solutions for given averaging kernel w were proved.
(Blandin & Goatin (2016), Keimer & Pflug (2017), Friedrich & Kolb & Göttlich (2018) , Bressan & WS (2020) ... and many more ...)

Approaches: numerical approximations, viscous approximations, Lipschitz solutions, ect.

Nonlocal-local Limit

Rescaled kernels: $w^\varepsilon(s) = \varepsilon^{-1} w(s/\varepsilon)$

$$\rho_t + (\rho v(q))_x \;=\; 0, \qquad q(t,x) \;=\; \int_{-\infty}^{+\infty} w^\varepsilon(s)\,\rho(t,x+s)\,ds \tag{1}$$

Conjecture: As $\varepsilon \to 0$, under suitable assumptions, the solutions to (1) converge to the unique entropy admissible weak solutions to the local conservation law.

Obstacles: Lack of estimates uniform in ε. Typical estimate on the total variation

$$\mathrm{TV}\{\rho(t,\cdot)\} \le e^{K(\varepsilon)t}\,\mathrm{TV}\{\rho(0,\cdot)\}, \qquad K(\varepsilon) \to \infty \;\text{ as }\; \varepsilon \to 0$$

No bounds on TV as $\varepsilon \to 0$!

Counter example: Blowup of total variation (Colombo, Crippa & Spinolo 2019):

Uniform BV Bound: A Relaxation Approach

$$\text{Choose:} \qquad w(s) = e^{-s}, \qquad w^\varepsilon(s) = \varepsilon^{-1} e^{-s/\varepsilon}, \qquad (s \geq 0),$$

$$q(t,x) = \frac{1}{\varepsilon}\int_0^{+\infty} e^{-s/\varepsilon}\rho(t,x+s)\,ds, \qquad q_x = \frac{1}{\varepsilon}(q-\rho)$$

Rewrite as a relaxation system:

$$\begin{cases} \rho_t + (\rho v(q))_x = 0 \\ q_x = \frac{1}{\varepsilon}(q-\rho) \end{cases}$$

Formally, as $\varepsilon \to 0$, we have $q \to \rho$, and the system converges to

$$\rho_t + (\rho v(\rho))_x = 0$$

Coordinate change: $\tau = t - \frac{x}{K}, \quad y = x, \qquad$ where $K > \max_\rho v(\rho)$

$$\Longrightarrow \quad \begin{cases} (K\rho - \rho v(q))_\tau + (K\rho v(q))_y &= 0 \\ q_\tau - Kq_y &= \dfrac{K}{\varepsilon}(\rho - q) \end{cases}$$

Variable change: $u = \ln \rho \quad (\rho > 0) \qquad z = \ln(K - v(q))$

$$\Longrightarrow \quad \begin{cases} u_\tau + K(Ke^{-z} - 1)u_y &= \dfrac{K}{\varepsilon}\Lambda(u, z) \\ z_\tau - Kz_y &= -\dfrac{K}{\varepsilon}\Lambda(u, z) \end{cases}$$

where $\quad \Lambda(u, z) = (\rho(u) - q(z)) \cdot \dfrac{v'(q(z))}{K - v(q(z))}$

$\rightarrow$ nonlinear hyperbolic **transport** equation with relaxation terms

$$\begin{cases} u_{y\tau} + [K(Ke^{-z} - 1)u_y]_y &= \dfrac{K}{\varepsilon}[\Lambda_u u_y + \Lambda_z z_y] \\ z_{y\tau} - K z_{yy} &= -\dfrac{K}{\varepsilon}[\Lambda_u u_y + \Lambda_z z_y] \end{cases}$$

u_y = density of forward moving particles
z_y = density of backward moving particles

Important assumptions: $\Lambda_u \leq 0$, $\Lambda_z \geq 0$

$\Longrightarrow$ Total mass of particles is non-increasing in time

$\Longrightarrow$ TVD estimate for u, z in the coordinate (τ, y)

$\Longrightarrow$ Global uniform BV estimate for ρ, q in (t, x)

$\Longrightarrow$ Existence of the limit solution as $\varepsilon \to 0$, which is a weak solution of the local equation. (Bressan, WS 2020)

Entropy admissibility of the limit

To prove entropy-admissibility, it suffices to check that one strictly convex entropy is dissipated (E.Y.Panov, 1994)

$$\text{entropy:} \quad \eta(\rho) = \frac{\rho^2}{2} \qquad\qquad \text{entropy flux:} \quad \psi(\rho) = \frac{\rho}{2} - \frac{2\rho^3}{3}$$

$$\text{One needs to prove that:} \qquad \eta(\rho)_t + \psi(\rho)_x \;\leq\; 0$$

Bressan & WS 2021

Assume that $v'(\rho) \leq 0$, then the limit solution is entropy admissible.

Another Entropy Condition: Oleinik Decay Estimate

Colombo, Crippa, Marcon, Spinolo, (2021): Under assumptions

$$w'(s) \leq -w(s) \quad (s \geq 0), \qquad v' \leq -\delta < 0,$$

and on the initial data $\bar{\rho}$:

$$\inf \bar{\rho} > 0, \qquad \mathrm{Lip}^{-} \bar{\rho} \leq L, \qquad L > 0,$$

then

$$\mathrm{Lip}^{-} \rho(t, \cdot) \leq \frac{L}{2\delta L t + 1} < \frac{1}{2\delta t} \qquad \text{for every } t \geq 0.$$

Talk by Laura Spinolo on Friday!

Conjecture:

Assume $w'(s) \leq -w(s)$, $(s \geq 0)$, $\quad v' \leq -\delta < 0$, and $\inf \bar{\rho} > 0$. The solution $\rho(t, x)$ satisfies a uniform (in ε) bound on the total variation.