Can the flow behind weak Mach reflections be shock-free?

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Dedicated to Constantine Dafermos on the occasion of his 80th birthday.

Based on earlier work: Eun A Chong (PhD thesis, UC Davis, 2014)

Transonic flows

Consider steady transonic flow of inviscid, compressible fluid in two space dimensions. Basic problem for multi-d hyperbolic conservation laws.

Suppose fluid velocity $\mathbf{u}$ is perturbation of uniform sonic flow $(-c_0, 0)$ in $-x$ direction:

$$\mathbf{u}(x, y) = (-c_0, 0) + (u(x, y), v(x, y))$$

where u, v are the x, y velocity perturbations.

Look for asymptotic solutions of steady compressible Euler equations with $v \ll u \ll c_0$ and $\partial_y \ll \partial_x$.

TSD equation

After rescaling, find that (u, v) satisfy the (steady) transonic small disturbance equation (TSD) equation (Guderley, 1945; von Kármán 1947). Pressure and density perturbations are proportional to u.

$$\left(\frac{1}{2}u^2\right)_x + v_y = 0, \qquad u_y - v_x = 0$$

Mixed type conservation law: hyperbolic in $u < 0$ (supersonic flow); elliptic in $u > 0$ (subsonic flow). Changes type across sonic line $u = 0$, or across transonic shocks.

Despite simple appearance, captures fundamental phenomena in transonic flows in which shocks are weak and vorticity and entropy effects are small.

Easier model to use than steady potential flow equations.

Regular shock reflection

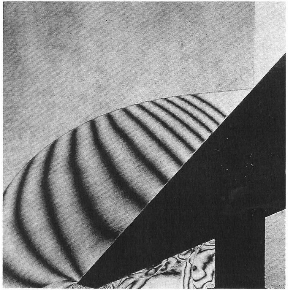


Figure 2.3 Holographic interferogram of a regular reflection - RR with M_s = 1.17 and $\theta_w = 50^o$ in air at T_0 = 287.6K and $p_0 \approx$ 760 torr (courtesy of Professor K. Takayama, Shock Wave Research Center, Institute of Fluid Science, Tohoku University, Sendai, Japan).

Unsteady self-similar solutions analyzed by Čanić, Keyfitz, and Kim (2002, 2006) for TSD models; Chen and Feldman (2010,...,2018) for potential flows.

No transonic coupling between hyperbolic and elliptic regions (non-generic Keldysh-type tangency of characterisics at sonic line). Leads to (difficult!) elliptic problem.

Weak shock Mach reflection

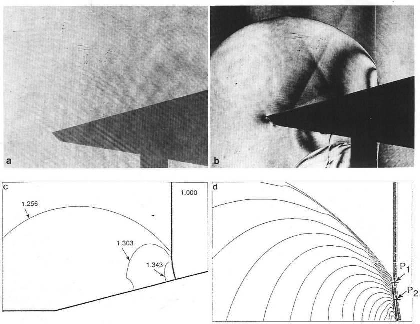


Fig. 1a–d. Examples of NMR, $M_s = 1.15$, $\theta_w = 15°$, **a** Unreconstructed hologram, **b** Infinite fringe interferogram, **c** Isopycnics obtained by numerical simulation; the labeled densities are those normalized by the upstream density, **d** Enlargement of **c** in shock intersection region

See triple point where incident, reflected and Mach shocks meet.

For weak shocks, it's inconsistent with jump conditions to have only three shocks (and a contact discontinuity) meeting at a point (von Neumann triple point paradox, 1943).

Steady Mach reflection for TSD equation

Following BVP for the TSD equation in a rectangle $x_L < x < x_R$, $0 < y < 1$ describes a slightly supersonic jet that enters a channel from the right and hits a thin wedge of slope $\tilde{a}$. Shock generated from corner of wedge reflects off rigid top wall of the channel.

$$\left(\frac{1}{2}u^2\right)_x + v_y = 0, \qquad u_y - v_x = 0$$

$$u(x_R, y) = -1, \quad v(x_R, y) = 0 \qquad 0 < y < 1,$$

$$v(x, 1) = 0 \qquad x_L < x < x_R,$$

$$v(x, 0) = \tilde{a} \qquad x_0 < x < x_1,$$

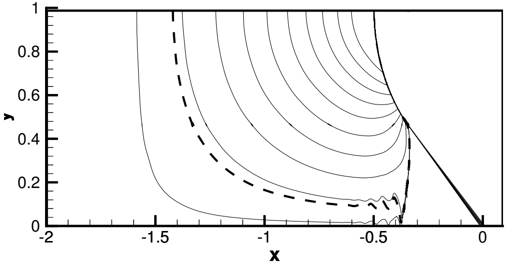
$$u(x, 0) = u_0 \qquad x_L < x < x_0,$$

$$v(x, 0) = 0 \qquad x_1 < x < x_R$$

Numerical solutions show a Mach reflection for parameter values

$$x_L = -2,\ x_0 = -0.4,\ x_1 = 0.0,\ x_R = 0.2,\ \tilde{a} = 0.67,\ u_0 = -0.09$$

Global numerical solution H. and Tesdall (2003)

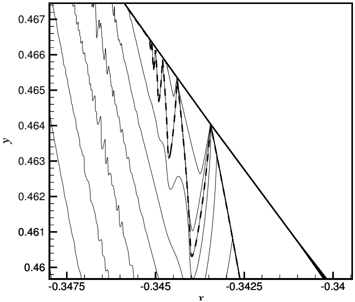


Plot of u-contours. Sonic line $u = 0$ is dashed. Observe triple point where incident, reflected and Mach shocks meet

Immediate resolution of triple point paradox: There is an additional expansion fan at the triple point (Vasilev and Kraiko, 1999).

Get tiny, but complex, transonic structure behind Mach shock with multiple triple points and supersonic patches (Tesdall and H., 2002; Tesdall et. al. 2015).

Solution near triple point

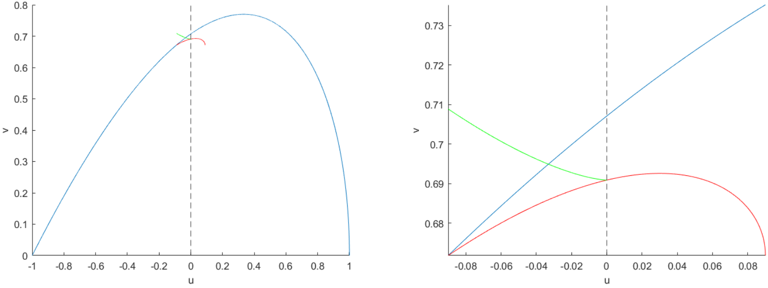


Plot of u contours near triple point, sonic line $u = 0$ dashed.

See sequence of sonic triple points, expansion fans, and supersonic patches in tiny region behind Mach shock.

Basic question: How many triple points and patches can occur (finitely many, countably infinite)? In particular, can there be only one triple point and smooth flow behind Mach shock?

Shock polars



Blue: Shock polar of state ahead of the incident and Mach shocks
Red: Shock polar of state behind the reflected shock
Green: Expansion fan at leading triple point

Guderley Mach Reflection (GMR)

Structure of 'forked shocks' with expansion fan at triple point and single supersonic patch behind it was proposed by Guderley (1947, 1962) so refer to this type of reflection as a Guderley Mach reflection (GMR)

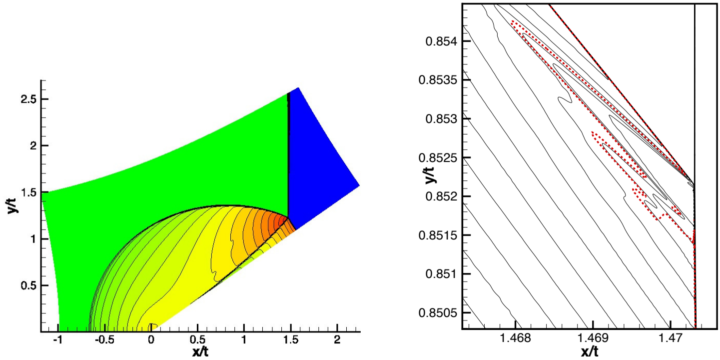
Guderley didn't suggest sequence of triple points or patches, only remarked:

> *Careful analysis (cf. Guderley, 1947) indicates, however, that a singularity results at [the rear point of the supersonic patch] or that also in this case a solution in Tricomi's sense is not possible.*

Numerical solutions indicate any singularity is resolved into a further supersonic patch. Want to revist the question of whether smooth flow behind weak Mach shock is possible.

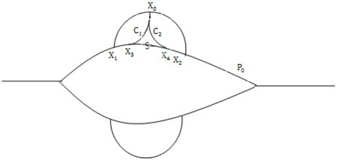
Unsteady GMR for compressible Euler equations

Numerical solution from Tesdall, Sanders, and Popivanov (2015):



Left: Density contours for global solution
Right: Detail of pressure contours near triple point; dashed red line is sonic line

Analog with transonic airfoils



Supersonic patch behind Mach shock analogous to supersonic patch on transonic airfoil. Smooth transonic flows occur for special airfoils.

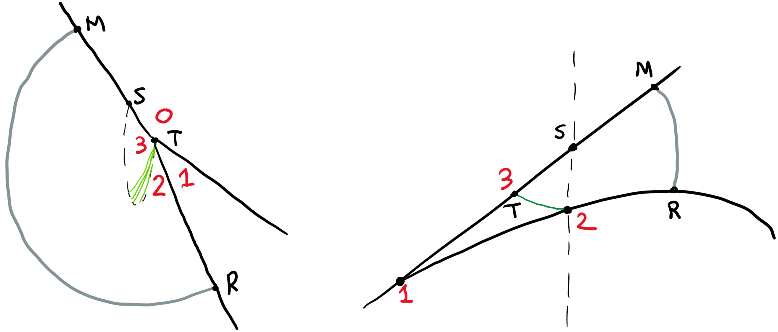
Closed BVPs for hyperbolic PDEs overdetermined: smooth solutions not stable under perturbations in shape of supersonic part of airfoil (Guderley, 1953; Morawetz, 1956).

Supersonic patch is typically terminated by a shock. For airfoils, can terminate single supersonic patch by a normal shock.

For GMR, this reintroduces the von Neumann paradox, resolved by a second expansion wave and second supersonic patch, and so on...

Hypothetical structure for shock-free GMR

Assume expansion fan at triple point originates from sonic point 2.

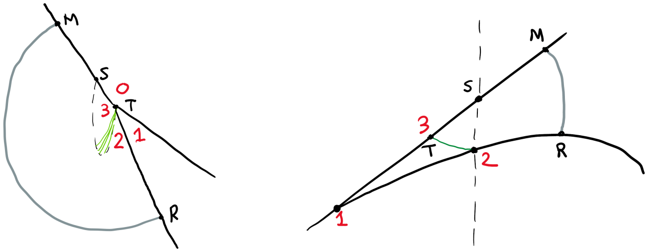


Left: Physical (x, y)-plane.
Expansion fan: green.

Right: Hodograph (u, v)-plane.
Sonic line: dashed.

Shock-free GMR (called ‘Guderley reflection’ in Skews et. al., 2009), appears to occur in some numerical and experimental solutions.
Want to claim: typically due to under-resolution or effects of viscosity.

Simplifications for analysis



(1) Introduce artificial cutoff MR of local triple point solution from global elliptic region.
(2) Assume hodograph transformation is invertible in region behind triple point T (though not at T itself).

Local behavior near triple point influences solution on cutoff MR, but reasonable to suppose effect is weak. Invertibility of hodograph transformation for local solution is consistent with numerics.

Hodograph transform

If change of variable $(x, y) \mapsto (u, v)$ is smoothly invertible in some region, exchange roles of dependent and independent variables, and write

$$x = x(u, v), \qquad y = y(u, v).$$

Then TSD equation

$$uu_x + v_y = 0, \qquad u_y - v_x = 0$$

implies that

$$uy_v + x_u = 0, \qquad x_v - y_u = 0$$

and $y(u, v)$ satisfies linear Tricomi equation

$$y_{uu} + uy_{vv} = 0$$

Shock polars

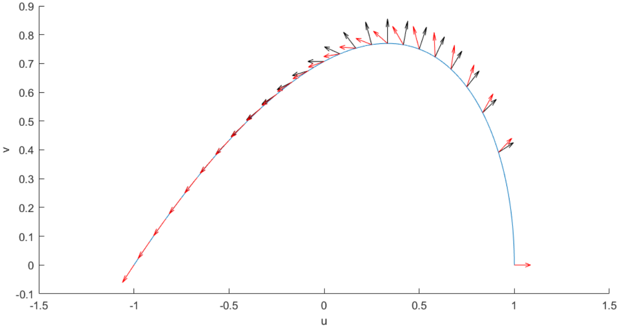
Shock polar for left states (u, v) behind a shock with right state (u_a, v_a) ahead of shock (with $u_a < 0$ and $u_a < u < -u_a$) is

$$v = v_a + f(u, u_a), \qquad f(u, u_a) = (u - u_a)\sqrt{-\left(\frac{u + u_a}{2}\right)}$$

Jump conditions across shock with constant right state (u_a, v_a) lead to oblique derivative boundary condition for $y(u, v)$ at $v = v_a + f(u, u_a)$ behind shock (e.g. see Cole and Cook, 1986)

$$y_u + a(u, u_a)y_v = 0, \qquad a(u, u_a) = \frac{7u + u_a}{5u + 3u_a}\sqrt{-\left(\frac{u + u_a}{2}\right)}$$

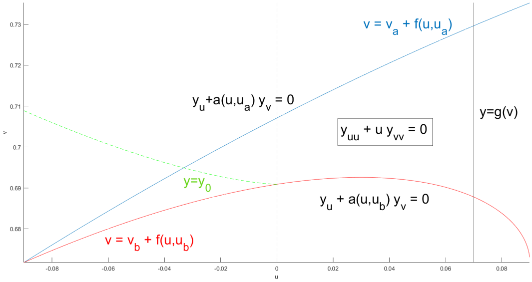
Busemann porcupine



Blue: Shock polar from $(u_a, v_a) = (-1, 0)$
Red: Oblique derivative direction behind shock
Black: Neumann (conormal) derivative for Tricomi equation

Hypothetical shock-free GMR in hodograph plane

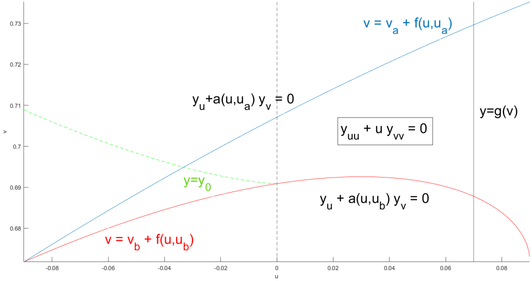
Get BVP for Tricomi equation on Tricomi domain with oblique derivative BCs on shock polars, Dirichlet conditions on cutoff $u = u_c$.



Here (u_a, v_a) and (u_b, v_b) are constant states ahead of incident and reflected shocks, respectively. Blue curve is Mach shock, red curve is reflected shock, and green curve is triple point.

Uniqueness result

Want to use energy estimates to prove (under suitable assumptions) that (admissible) solutions are uniquely determined by oblique derivative BCs on shock polars and cutoff BCs at $u = u_c$.



In general, these solutions won't satisfy $y = y_0$ is constant at triple point. However, less clear how to derive perturbation results here than in airfoil problem, since no flexibility in supersonic region.

ABC-Method of Friedrich and Morawetz (1956)

Multiply PDE $y_{uu} + uy_{vv} = 0$ by

$$By_v + Cy_u$$

($A = 0$) integrate over domain, use Green's theorem, and choose multipliers cfs $B(u, v)$, $C(u, v)$ so that we get positive definite energy estimates.

Make different choices for B, C in hyperbolic ($u < 0$) and elliptic ($u > 0$) regions, with B, C piecewise C^1 and continuous across $u = 0$.

Hyperbolic multipliers. Take $B = B(v)$, $C = 0$ with $B \geq 0$, $B_v > 0$. Get positive definite estimates provided that

$$a^2 - u - 2a\frac{df}{du} \leq 0 \qquad \text{on } v = v_a + f(u, u_a) \text{ with } u \leq 0.$$

This condition is satisfied, in particular, on the shock polars.

Elliptic multipliers

Following Morawetz (1956), define B, C for $u > 0$ by

$$B(u,v) - i\sqrt{u}C(u,v) = e^{f(\lambda)}, \qquad \lambda = \mu + iv, \quad \mu = \frac{2}{3}u^{3/2}$$

where $f(\lambda) = f_1(\mu, v) + if_2(\mu, v)$ is analytic function of λ in the elliptic region.

On sonic line, require

$$f_2(0, v) \equiv 0 \pmod{2\pi}$$

to match elliptic multiplier with hyperbolic multiplier where $C = 0$.

Will specify BCs for f_2 in (μ, v)-plane, and construct f_1 as its harmonic conjugate. Then define hyperbolic multiplier by

$$B(v) = e^{f_1(0,v)}.$$

Elliptic multipliers

To get positive definite energy estimates, need to choose harmonic function f_2 to satisfy two conditions on boundary.

(1) $\sin f_2 \leq 0$ on all elliptic boundaries (ensures $C \geq 0$ on boundary)

To describe second condition, introduce two angles:

(i) θ angle of positively-oriented boundary of elliptic region in (μ, v)-plane

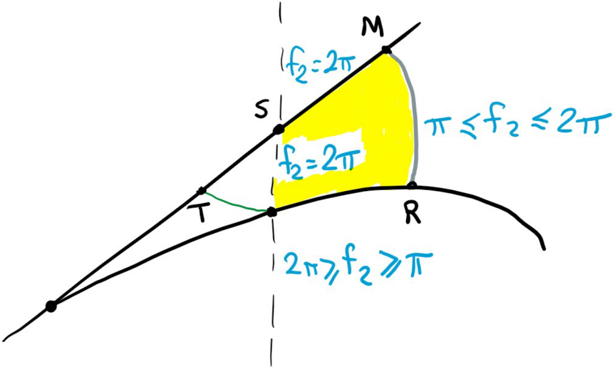
$$\cos\theta = \frac{d\mu}{ds} = \sqrt{u}\frac{du}{ds}, \qquad \sin\theta = \frac{dv}{ds}$$

(ii) α angle associated with oblique derivative BC $y_u + a(u)y_v = 0$

$$\cos\alpha = \frac{a^2 - u}{a^2 + u}, \qquad \sin\alpha = \frac{2a\sqrt{u}}{a^2 + u}$$

(2) $\cos(\alpha + \theta + f_2) \geq 0$ on oblique derivative boundaries (ensures boundary line integrals are nonnegative).

Boundary values for f_2

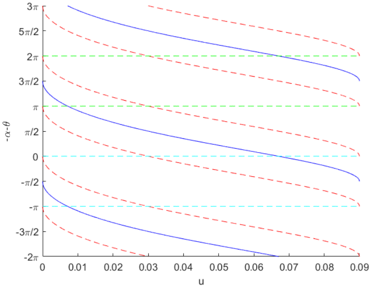


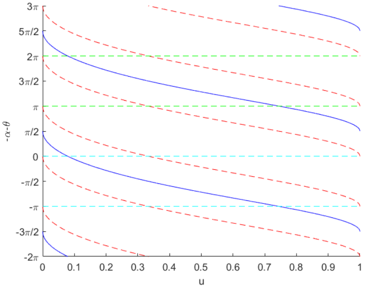
Modulo 2π, need:

$$(1) \quad \pi \leq f_2 \leq 2\pi; \qquad (2) \quad -\frac{\pi}{2} \leq f_2 + \alpha + \theta \leq \frac{\pi}{2}.$$

Choice of f_2

Blue line plots $-(\alpha+\theta)$ on shock polars (for $u_a = -1$, $u_b = -0.09$). Boundary values of f_2 must lie between dotted green lines and dotted red lines. Want f_2 to equal 2π at $u = 0$.





Left: Reflected shock polar. Right: Incident shock polar.

Can construct continuous boundary values for f_2 provided that cutoff value $0 < u_c \leq -u_b/3$ (where $-u_b/3$ corresponds to maximum deflection angle for reflected shock from $u_b < 0$).

Conclusions

Transonic problems with shocks remain extremely challenging, especially in generic case of Tricomi-type, non-tangential characteristics at sonic line. Mathematical progress possible only in special cases.

Partial, but incomplete, results on transonic flows by compensated compactness (Morawetz, 1995; Chen, Slemrod, and Wang, 2008).

Uniqueness results by energy methods for oblique derivative Tricomi problems suggest smooth flows behind weak-shock Mach reflections highly exceptional. (Or don't exist at all?)

Plausible (but totally unproved!) conjecture: typically get infinite sequence of triple points accumulating at rear sonic point in inviscid weak shock Mach reflections.