Shock reflection problems: existence, stability and regularity of global solutions

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Shock reflection by a wedge: Regular reflection
Shock reflection by a wedge: Mach reflection
Shock reflection by a wedge: Irregular Mach reflection.

Self-similar flow: \((\vec{u}, p, \rho)(x, t) = (\vec{u}, p, \rho)(\frac{x}{t})\).
Shock reflection

First described by E. Mach 1878. Reflection patterns: Regular reflection, Mach reflection.

J. von Neumann, 1940s: on transition between patterns


Analysis: Special models (Transonic small disturbance eq., pressure-gradient system, nonlinear wave eq.): Gamba, Rosales, Tabak, Canic, Keyfitz, Kim, Lieberman, Y. Zheng, G.-Q. Chen-W. Xiang.

Local existence results: S.-X. Chen.
More recent results for potential flow:

Existence of global shock reflection solutions for potential flow: G.-Q.Chen-F., Elling

The complete up-to-date results on existence of regular reflection solutions and their proofs are presented in the monograph ”The Mathematics of Shock Reflection-Diffraction and von Neumann conjectures” by G.-Q.Chen-F, 2018.

Other self-similar shock reflection problems:

Prandtl Reflection: Elling-Liu, Bae-G.-Q.Chen-F

Shock interactions/reflection for Chaplygin gas: D. Serre


Global existence of shock reflection solutions in the framework of compressible Euler system is an open problem.
Shock reflection as a Riemann problem in domain with boundary, with slip boundary conditions

Initial data: Constant (uniform) states (0) and (1):

State (0): velocity $\vec{u}_0 = (0, 0)$, density $\rho_0$, pressure $p_0$.

State (1): velocity $\vec{u}_1 = (u_1, 0)$, density $\rho_1$, pressure $p_1$.

$t > 0$: Self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\xi)$, where

$\xi = \frac{x}{t}$. 
Uniqueness/nonuniqueness for 2-D Riemann problems in whole space

Riemann problem in whole space for Euler system:

Chiodaroli-DeLellis-Kreml(2015): 2D isentropic Euler system

1) Entropy solutions of Riemann problem are non-unique in the class of entropy solutions isentropic Euler system. Specifically, for certain initial data in the form of two constant states separated by flat shock, there exist: (a) an explicit solution in the form of moving flat shock separating the constant states given above; and (b) multiple ”wild” solutions.

2) Self-similar solutions of 1D structure in 2D with flat shock are unique (reduced to 1D system of conservation laws).

Non-uniqueness results for 2D full Euler system: S. Markfelder and C. Klingenberg (2017), Al Baba, Klingenberg, Kreml, Macha, Markfelder (2019)
Shock Reflection as 2D Riemann problem in domain with boundary

1. Time-dependent solutions for compressible Euler system: non-uniqueness for normal reflection (wedge angle $\frac{\pi}{2}$, i.e. reflection from flat wall), using technique of Chiodaroli-DeLellis-Kreml; other cases are open,

2. Potential flow: uniqueness/nonuniqueness of general time dependent or general self-similar solutions is open,

3. Potential flow, self similar solutions of regular reflection structure: We show:
   (a) Existence of ”admissible” regular reflection solutions (G.-Q. Chen - F.); regularity (M. Bae - G.-Q. Chen - F.);
   (b) convexity of shocks for ”admissible solutions” (G.-Q. Chen - F.-W. Xiang).
   (c) uniqueness of regular reflection solutions with convex shocks: (G.-Q. Chen - F.-W. Xiang).
Equations of gas dynamics

Isentropic Compressible Euler system:

\[ \partial_t \rho + \text{div}(\rho \vec{u}) = 0, \]
\[ \partial_t (\rho \vec{u}) + \text{div}(\rho \vec{u} \otimes \vec{u}) + \nabla p = 0, \]

where:
\( \vec{u} = (u_1, u_2) \) – velocity
\( \rho \) – density
\( p = \rho^\gamma \) – pressure
\( \gamma > 1 \) – adiabatic exponent (it is a given constant)

Potential flow: Conservation of mass, Bernoulli’s law

\[ \rho_t + \text{div}(\rho \nabla \Phi) = 0, \]
\[ \Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} = \text{const} \]

where \( \Phi \) – velocity potential: \( \vec{u} = \nabla_x \Phi \).
Regular reflection in self-similar coordinates $\xi = \frac{x}{t}$

Given:
State (0): velocity $\vec{u}_0 = (0, 0)$, density $\rho_0$, pressure $p_0$.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density $\rho_1$, pressure $p_1$.

Problem: Find self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\xi)$, where $\xi = \frac{x}{t}$, with asymptotic conditions at infinity determined by states (0) and (1), and satisfying $u \cdot \nu = 0$ on the boundary.
Self-similar potential flow

\[ \Phi(\vec{x}, t) = t\psi(\xi, \eta), \quad \rho(\vec{x}, t) = \rho(\xi, \eta) \text{ with } (\xi, \eta) = \frac{\vec{x}}{t} \in \mathbb{R}^2. \]

Pseudo-potential: \[ \varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2). \]

Equation for \( \varphi \):

\[
\text{div} \left( \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \right) + 2\rho(|\nabla \varphi|^2, \varphi) = 0,
\]

with \[ \rho(|\nabla \varphi|^2, \varphi) = (K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla \varphi|^2))^{\frac{1}{\gamma - 1}}. \]

Equation is of mixed type:

**elliptic** \[ |\nabla \varphi| < c(|\nabla \varphi|^2, \varphi, K), \]

**hyperbolic** \[ |\nabla \varphi| > c(|\nabla \varphi|^2, \varphi, K), \]

where **speed of sound** \( c \) is:

\[ c^2 = \rho^{\gamma - 1} = K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla \varphi|^2). \]
Uniform states

Solutions with constant (physical) velocity \((u, v)\):

\[
\varphi(\xi, \eta) = -\frac{\xi^2 + \eta^2}{2} + u\xi + v\eta + \text{const.}
\]

Any such function is a solution.
Also (from formula) density \(\rho(\nabla \varphi, \varphi) = \text{const}\), thus sonic speed \(c = \rho \frac{\gamma - 1}{2} = \text{const}\). Then ellipticity region

\[
|\nabla \varphi(\xi, \eta)| = |(u, v) - (\xi, \eta)| < c
\]

is circle, centered at \((u, v)\), radius \(c\).
Shocks, RH conditions, Entropy condition

Shocks are discontinuities in the pseudo-velocity $\nabla \varphi$:

if $\Omega^+$ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are nonempty and open, and $S := \partial \Omega^+ \cap \Omega$ is a $C^1$ curve where $\nabla \varphi$ has a jump, then

$\varphi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution in $\Omega$ if and only if $\varphi$ satisfies potential flow equation in $\Omega^\pm$ and the Rankine-Hugoniot (RH) condition on $S$:

$$[\varphi]_S = 0,$$

$$[\rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \cdot \nu]_S = 0,$$

where $[\cdot]_S$ is jump across $S$.

Entropy Condition on $S$: density increases across $S$ in the flow direction.
Shock reflection as a free boundary problem

\[ \text{div} \left( \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \right) + 2\rho(|\nabla \varphi|^2, \varphi) = 0 \text{ in } \Omega, \]

\[ \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \cdot \nu = \rho(|\nabla \varphi_1|^2, \varphi_1) \nabla \varphi_1 \cdot \nu \quad \text{on } P_1P_2 \]

\[ \varphi = \varphi_1 \text{ on } P_1P_4 \quad \text{(and prove } D_\nu \varphi = D_\nu \varphi_2 \text{ on } P_1P_4) \]

\[ \varphi_\nu = 0 \text{ on Wedge } P_3P_4, \text{ Symmetry line } P_2P_3, \]

Solve for: Free boundary \( P_1P_2 \) and function \( \varphi \) in \( \Omega \).

Expect equation elliptic in \( \Omega \).
Regular reflection, state (2)

\[\vec{u}_1\]

\[P_0\]

\[\varphi = \text{pseudo-potential between the reflected shock and the wall}\]

\[\varphi_1 = \text{pseudo-potential of state (1)}\]

Denote \(\nabla \varphi(P_0) = (u_2, v_2)\), where \(\varphi = \varphi + \frac{\xi^2 + \eta^2}{2}\). Since \(\varphi_\nu = 0\) on wedge, then \(v_2 = u_2 \tan \theta_w\). Here \(\theta_w\) is wedge angle.

Rankine-Hugoniot conditions at reflection point \(P_0\), for \(\varphi\) and \(\varphi_1\): algebraic equations for \(u_2, \varphi(P_0)\)
Regular reflection, state (2), detachment angle

If solution exists: Let

$$\varphi_2(\xi, \eta) = -\frac{(\xi^2 + \eta^2)}{2} + u_2\xi + v_2\eta + C,$$

where $C$ determined by $\varphi_2(P_0) = \varphi_1(P_0)$. 

Existence of state (2) is necessary condition for existence of regular reflection

Given $\gamma, \rho_0, \rho_1$, there exists $\theta_{\text{detach}} \in (0, \frac{\pi}{2})$ such that:

state (2) exists for $\theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2})$,
state (2) does not exist for $\theta_w \in (0, \theta_{\text{detach}})$.

If $\varphi_2$ exist, then RH is satisfied along the line $S_1 := \{\varphi_1 = \varphi_2\}$. 
Weak and Strong State (2); Sonic angle

For each $\theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2})$ there exists two possible States (2): weak and strong, with $\rho_{2}^{\text{weak}} < \rho_{2}^{\text{strong}}$. We always choose weak state (2). For strong state (2), existence of global regular reflection solution is not expected, Elling (2011) confirms that.

There exist $\theta_{\text{sonic}} \in (\theta_{\text{detach}}, \frac{\pi}{2})$ such that:
State 2 is supersonic at $P_0$ for $\theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2})$.
State 2 is subsonic at $P_0$ for $\theta_w \in (\theta_{\text{detach}}, \theta_{\text{sonic}})$. 

![Diagram showing the weak and strong states with sonic circles and reflected shocks.](image-url)
Von Neumann’s conjectures on transition between different reflection patterns

Recall: sonic angle \( \theta_{\text{sonic}} \) and detachment angle \( \theta_{\text{detach}} \) satisfy \( 0 < \theta_{\text{detach}} < \theta_{\text{sonic}} < \frac{\pi}{2} \).

Sonic conjecture:
Regular reflection for \( \theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2}) \), Mach reflection for \( \theta_w \in (0, \theta_{\text{sonic}}) \).

Von Neumann’s detachment conjecture:
Regular reflection for \( \theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2}) \), Mach reflection for \( \theta_w \in (0, \theta_{\text{detach}}) \).

G.-Q. Chen - F.(2018): existence of regular reflection for \( \theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2}) \) for potential flow equation if \( \rho_0, \rho_1 \) satisfy \( u_1 \leq c_1 \) (weaker incident shocks), and up to a critical angle otherwise. Given \( \rho_0 > 0 \) there exists \( \rho_1^* > \rho_0 \) such that \( u_1 < c_1 \) for \( \rho_1 \in (\rho_0, \rho_1^*) \) and \( u_1 > c_1 \) for \( \rho_1 > \rho_1^* \).

Structure: supersonic and subsonic regular reflections.
Supersonic regular reflection

State (2) is supersonic at $P_0$.

Structure of solution $\varphi$:

- $\varphi = \varphi_i$ in $\Omega_i$, $i=0,1,2$.
- $\varphi \in C^1(\overline{P_0P_2P_3})$, in particular $C^1$ across sonic arc $P_1P_4$.
- Shock $P_0P_2$ has flat part $P_0P_1$, curved part $P_1P_2$, and is $C^1$ across $P_1$.
- Equation is strictly elliptic in $\overline{\Omega \setminus P_1P_4}$.
Subsonic regular reflection: **State (2) is subsonic at** $P_0$.

**Structure of solution $\varphi$:**

- $\varphi = \varphi_i$ in $\Omega_i$, $i=0,1$.
- $\varphi \in C^1(P_0P_2P_3)$.
- $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at $P_0$.
- Shock $P_0P_2$ is $C^1$.
- Equation is strictly elliptic in $\Omega \setminus \{P_0\}$. 
Existence of regular reflection solutions

**Theorem 1.** (G.-Q. Chen-F.). If \( \rho_1 > \rho_0 > 0, \gamma > 1 \) then a regular reflection solution \( \varphi \) exists for all wedge angles \( \theta_w \in (\theta_{\text{detach}}, \frac{\pi}{2}) \). Here I skip some details related to "attached shocks" with \( P_2 = P_3 \). The type of reflection (supersonic or subsonic) for each \( \theta_w \) is determined by the type of State 2 at the reflection point \( P_0 \) for \( \theta_w \). Moreover, solution satisfies the following additional properties:
Properties of solution: supersonic case

1) Equation is elliptic for $\varphi$ in $\Omega$, ellipticity degenerates near sonic arc $P_1P_4$.

2) $\varphi$ is $C^{1,1}$ near and across the sonic arc $P_1P_4$;

3) Reflected shock is $C^{2,\beta}$, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$ between $\vec{e}_\eta = (0, 1)$ and $\vec{e}_{S_1} = P_0P_1$;

4) $\varphi_2 \leq \varphi \leq \varphi_1$ in $\Omega$, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$. 
Properties of solution: subsonic case

1) Equation is elliptic for $\varphi$ in $\Omega$, except for the sonic wedge angle (then ellipticity degenerates at $P_0$).

2) $\varphi$ is $C^{2,\alpha}$ inside $\Omega$, and $C^{1,\alpha}$ near and up to the reflection point $P_0$, and $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at $P_0$;

3) Reflected shock is $C^{2,\alpha}$ away from $P_0$ and $C^{1,\alpha}$ up to $P_0$, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$;

4) $\varphi_2 \leq \varphi \leq \varphi_1$ in $\Omega$, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$. 
Stability of normal reflection as $\theta_w \to \pi/2$

Furthermore, the solutions $\varphi(\theta_w)$ converge in $W^{1,1}_{loc}$ to the solution of the normal reflection as $\theta_w \to \pi/2$. 

Figure: Normal reflection
Proof of Th. 1 is obtained by solving free boundary problem using method of continuity/degree theory in the set of ”admissible solutions”

First consider supersonic reflection case. Free boundary problem with and the solution $\varphi$ in $\Omega$:

$$
\text{div} \left( \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \right) + 2\rho(|\nabla \varphi|^2, \varphi) = 0 \text{ in } \Omega,
$$

$$
\rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \cdot \nu = \rho(|\nabla \varphi_1|^2, \varphi_1) \nabla \varphi_1 \cdot \nu \right \}
$$

on $P_1P_2$

$\varphi = \varphi_1$ on $P_1P_2$

$\varphi = \varphi_2$ on $P_1P_4$ (and prove $D_\nu \varphi = D_\nu \varphi_2$ on $P_1P_4$)

$\varphi_\nu = 0$ on Wedge $P_3P_4$, Symmetry line $P_2P_3$,
Admissible solutions:

(a) Have structure supersonic or subsonic reflections depending on $\theta_w$. Recall: this includes ellipticity in $\Omega$ and regularity of $P_0P_2$ and of $\varphi$ in $P_0P_2P_3$;

(b) $\varphi_2 \leq \varphi \leq \varphi_1$ in $\Omega$;

(c) satisfy nonstrict monotonicity $\partial_e(\varphi_1 - \varphi) \leq 0$ in $\Omega$ for any $e \in Con(e_\eta, e_{S_1})$. 
Solving FBP

- Prove strict monotonicity of $\varphi_1 - \varphi$ for each direction $e \in \text{Cone}(e_\eta, e_{S_1}) \iff \Gamma_{\text{shock}}$ is a graph, $\text{Lip}[\Gamma_{\text{shock}}] \leq C$.
- Derive basic uniform apriori estimates for admissible solutions: $\|\varphi\|_{C^{0,1}(\Omega)} \leq C$, $\text{diam}(\Omega) \leq C$, $0 < \rho_{\text{min}} \leq \rho(\nabla \varphi, \varphi) \leq \rho_{\text{max}}$.
- Prove geometric properties of the free boundary $\Gamma_{\text{shock}}$:
  - Uniform estimates on separation of shock with wedge and the symmetry line, uniform lower bound $\text{dist}(\Gamma_{\text{shock}}, B_{c_1}(O_1)) \geq \frac{1}{C}$.
  - Prove ”ellipticity” $(\xi, \eta) \geq \frac{1}{C} \text{dist}((\xi, \eta), \Gamma_{\text{sonic}})$.
- Derive apriori estimates for $\varphi$ in weighted/scaled $C^{2,\alpha}$ in $\overline{\Omega}$, including for degenerate elliptic region near sonic arc.
- Use method of continuity/degree theory to prove existence of admissible solutions for each wedge angle up to the sonic angle (if $u_1 \leq c_1$, otherwise take into account the possibility of ”attached solutions” with $P_2 = P_3$).
Free boundary problem for subsonic reflection

\[
\text{div} \left( \rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \right) + 2 \rho(|\nabla \varphi|^2, \varphi) = 0 \quad \text{in} \quad \Omega,
\]
\[
\rho(|\nabla \varphi|^2, \varphi) \nabla \varphi \cdot \nu = \rho(|\nabla \varphi_1|^2, \varphi_1) \nabla \varphi_1 \cdot \nu \quad \text{on shock} \quad P_0P_2
\]
\[
\varphi = \varphi_1 \quad \text{at} \quad P_0 \quad \text{(and prove} \quad D\varphi = D\varphi_2 \quad \text{at} \quad P_0)\]
\[
\varphi_\nu = 0 \quad \text{on Wedge} \quad P_3P_4, \quad \text{Symmetry line} \quad P_2P_3,
\]

Note: One-point Dirichlet condition at \( P_0 \)
Oblique directions, weak and strong reflections, one-point Dirichlet conditions

\[ \varphi_\nu = 0 \]
on Wedge

Directions at \( P_0 \) of oblique condition \((\text{interior to the domain})\) on shock, from RH cond., using \( \varphi = \varphi_2, \ D\varphi = D\varphi_2 \) at \( P_0 \).
For strong reflection: at \( P_0 \) both directions vectors (on shock and wedge) and domain are on one side of a line through \( P_0 \)
\[ \Rightarrow \] Cannot prescribe One-Point Dirichlet condition at \( P_0 \) (by G. Lieberman’s Harnack estimate).
For weak reflection can prescribe one-point Dirichlet cond. at \( P_0 \)
Regularity in $\Omega$ near sonic arc (supersonic case)

**Theorem 2. (Bae-Chen-F.)**

1) For every $P$ in sonic arc $(P_1P_4]$ (i.e. excluding $P_1$)

$$\varphi \in C^{2,\alpha}(\overline{\Omega \cap B_R(P)}), \text{ for some small } R > 0, \text{ any } \alpha \in (0, 1).$$

2) $D^2\varphi$ has a jump across sonic arc $P_1P_4$:

$$D_{rr}\varphi|_{\Omega} - D_{rr}\varphi_2 = \frac{1}{\gamma+1} \text{ on arc}(P_1P_4].$$

Remark: $(\varphi - \varphi_2)_r = (\varphi - \varphi_2)_r\theta = (\varphi - \varphi_2)_{\theta\theta} = 0$ on $P_1P_4$

3) $D^2\varphi$ in $\Omega$ does not have a limit at $P_1$. 


Convexity of shock, uniqueness

**Theorem 3. (Chen-F.-W. Xiang)** For admissible solutions, shock is strictly convex in its relative interior. Moreover, regular reflection solution satisfying (a)-(b) of the definition of admissible solutions, have cone of monotonicty (c) if and only if the shock is (strictly) convex.

Based on Theorem 2, we prove:

**Theorem 4. (Chen-F.-Xiang)** Admissible solutions are unique (and exist, by Thms. 1, 2).

**Corollary. (Chen-F.-Xiang)** Regular reflections solutions with convex shocks are unique (and exist by Thms. 1, 2).

To put these results in a wider context, compare them with the known results on uniqueness/nonuniqueness for 2-D Riemann problems in whole space discussed above: Similar to that case, we show uniqueness of self-similar solutions of the prescribed structure (regular reflections; convex shocks).
Outline of proof of uniqueness

We prove uniqueness of admissible solutions (thus with convex shock).

Heuristic idea:
By Th. 1, when $\theta_w \to \frac{\pi}{2}$, admissible solutions converge to normal reflection. Also we have uniform estimates for admissible solutions. Then use the method of continuity:

Suppose $\varphi$, $\hat{\varphi}$ are two admissible solutions for some $\theta^*_w \in (\theta^d_w, \frac{\pi}{2})$. Then it is sufficient to:

1. Construct continuous in $C^1$ families $\theta_w \mapsto \varphi(\theta_w)$, $\theta_w \mapsto \hat{\varphi}(\theta_w)$ for $\theta_w \in [\theta^*_w, \frac{\pi}{2})$, with $\varphi(\theta^*_w) = \varphi$, $\hat{\varphi}(\theta^*_w) = \hat{\varphi}$,

2. Show ”local uniqueness”: if two admissible solutions for same $\theta_w$ are close in $C^1$, then they are equal.

Both are achieved if we can linearize FBP at an admissible solution, and linearization is ”good” so that we can construct solutions for close wedge angles by Implicit Function Theorem.
Outline of proof of uniqueness

Rigorously, cannot use linearization for supersonic reflections: elliptic degeneracy near sonic arc requires very detail control of $D^2 \varphi$ on sonic arc $P_1 P_4$ to show well-posedness of linearization. We do not have this control at one point: $P_1$, where shock meets sonic arc.

Then we use a "nonlinear version of linearization": apply degree theory with "small" iteration set, consisting of functions close to the background solution (in appropriate norm). To apply degree theory, we need to show (in particular) that fixed point of iteration map cannot occur on the boundary of the iteration set. This is done using local uniqueness theorem. We use convexity of shock for proof of local uniqueness theorem.
Proof of uniqueness: Role of convexity (heuristic)

When linearize FBP, variations of shock locations introduce an additional zero-order term in the oblique boundary condition derived from RH condition \( \rho D\varphi \cdot \nu = \rho_1 D\varphi_1 \cdot \nu \). This term has the "correct" sign if shock is convex:

Linerization of RH conditions: shock is \( \eta = f(\xi) \) with \( \Omega \subset \{ \eta < f(\xi) \} \) after rotating coordinates. Then RH:

\[
\varphi^\varepsilon(\xi, f^\varepsilon(\xi)) = \varphi_1(\xi, f^\varepsilon(\xi));
\]

\[
\left( (\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon) D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon) \right)(\xi, f^\varepsilon(\xi)) = 0,
\]

where we use that \( \nu = \frac{D\varphi_1 - D\varphi^\varepsilon}{|D\varphi_1 - D\varphi^\varepsilon|} \). Here \( \varphi^\varepsilon = \varphi + \varepsilon \delta \varphi + \ldots \), same for \( f^\varepsilon \). Taking \( \frac{d}{d\varepsilon} \) at \( \varepsilon = 0 \) in 1st condition and using \( \partial_\nu(\varphi_1 - \varphi) > 0 \) and on shock, so \( \partial_\eta(\varphi_1 - \varphi) > 0 \):

\[
\delta f = \frac{1}{\partial_\eta(\varphi_1 - \varphi)} \delta \varphi.
\]
Proof of uniqueness: Role of convexity (heuristic)

Now take $\frac{d}{d\varepsilon}$ at $\varepsilon = 0$ in 2nd RH condition

$$\left( (\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon) \right)(\xi, f^\varepsilon(\xi)) = 0,$$

Get two terms. First, linearization of oblique condition:

$$\frac{d}{d\varepsilon} \left[ \left( (\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon) \right) \right]_{\varepsilon=0}(\xi, f(\xi)) = a\partial_\nu \delta\varphi + b\partial_\tau \delta\varphi + c\delta\varphi,$$

where $a(\xi) \geq \lambda > 0$, $c(\xi) \leq -\lambda < 0$.

Second term comes from the perturbation of shock location:

$$\partial_\eta \left[ \left( (\rho(|D\varphi|^2, \varphi)D\varphi - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi) \right) \right] \delta f$$

$$= A(\varphi_1 - \varphi)_{\tau\tau} \delta f = \frac{A}{(\varphi_1 - \varphi)_\eta} (\varphi_1 - \varphi)_{\tau\tau} \delta\varphi,$$

where $A > 0$. Convexity of shock equivalent to $(\varphi_1 - \varphi)_{\tau\tau} < 0$, and then the coefficient of $\delta\varphi$ has ”correct” sign.
Open problems

1) Prove existence of regular reflection solutions for Euler system. One of difficulties is in vorticity estimates, noticed by D. Serre for isentropic Euler system: vorticity is not in $L^2(\Omega)$. Singularities are expected near the tip of wedge. Thus one has to work in the low regularity framework: velocity is discontinuous (but subsonic) near tip of wedge. On the positive side, this may improve stability of solutions: For potential flow, regular reflection solution does not exist for non-symmetric perturbations of the incoming flow (J. Hu, 2018) because velocity cannot be subsonic and discontinuous in case of potential flow. For Euler system, existence of non-symmetric perturbations can be expected.

2) Uniqueness/nonuniqueness in various classes of solutions. For example, for reflection of oblique shock by a flat wall, in the class of self-similar solutions for Euler system, etc.

3) Mach reflection: develop apriori estimates.