BV Solutions to a Hydrodynamic Limit of Flocking Type

Cleopatra Christoforou

University of Cyprus

Collaborator: Debora Amadori (L'Aquila, Italy)

Advances and Challenges in Hyperbolic Conservation Laws

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Outline

- Self-Organized Systems - Models
- Euler-Type Flocking system -Set Up

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  1. Global Existence and Structure
     - New Ideas - Front Tracking
     - Difficulties - Vacuum

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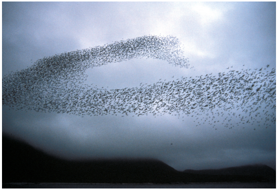
  2. Time-Asymptotic Flocking
     - ⋆ New Functionals
     - ⋆ Special condition

Self-Organized Systems

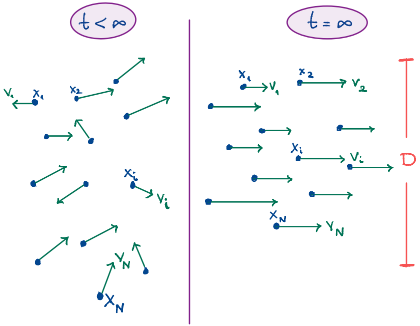



Biology: flocking of birds, swarming of insects, fish schools
Traffic Dynamics, crowds, cosmology...
Social science: social networks, economics, linguistics, learning, gossiping...

Emergent behavior

Models of self-organized systems describe dynamics of objects:

$\mathbf{x}_i \in \Omega \subset \mathbb{R}^n, \quad i = 1, \dots, N, \qquad \mathbf{v}_i = \dot{\mathbf{x}}_i$



Long-Time dynamics: • alignment: $\lim_{t\to\infty} \max_i |\mathbf{v}_i - \bar{\mathbf{v}}| = 0$

• flocking: $\sup_{i,j} |\mathbf{x}_i - \mathbf{x}_j| \le D < \infty$

Mathematical Models

- Particle: Cucker-Smale (2007)

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \frac{\lambda}{N} \sum_{j=1}^{N} \phi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{v}_j - \mathbf{v}_i) \end{cases}$$

- Kinetic

$$f_t^\epsilon + \omega \cdot \nabla_x f^\epsilon + \mathrm{div}_\omega (f^\epsilon L[f^\epsilon]) = \frac{1}{\epsilon} \Delta_\omega f^\epsilon + \frac{1}{\epsilon} \mathrm{div}_\omega (f^\epsilon (\omega - \mathrm{v}^\epsilon))$$

$$L[f] \doteq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y) f(y,w)(w-\omega) dw\, dy$$

- Hydrodynamic Limits: $\epsilon \to 0+$

Carrillo–Fornasier–Toscani–Vecil; Shvydkoy; Ha–Tadmor; Ha–Liu; Motsch–Tadmor; Karper–Mellet–Trivisa; Ha–Huang–Wang; Ha–Kang–Kwon.

Euler-type Flocking system

$$\partial_t \rho + \partial_x(\rho \mathrm{v}) = 0,$$

$$\partial_t(\rho \mathrm{v}) + \partial_x \left(\rho \mathrm{v}^2 + p(\rho)\right) = \int_{\mathbb{R}} \rho(x,t)\rho(x',t)\left(\mathrm{v}(x',t) - \mathrm{v}(x,t)\right)\, dx'$$

Pressure $\qquad p(\rho) = \alpha^2 \rho\,, \qquad \alpha > 0\,.$

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$$(\rho, \mathrm{v})(x,0) = (\rho_0(x), \mathrm{v}_0(x)) \qquad x \in \mathbb{R}\,.$$

Initial Data

$$\begin{cases} \mathrm{supp}\{\rho_0\} = I_0 \doteq [a_0, b_0] \\ \operatorname{ess\,inf}_{I_0} \rho_0 > 0\,, \\ (\rho_0(x), \mathrm{v}_0(x)) = 0 \qquad \forall\, x \notin I_0\,. \end{cases}$$

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$\rho_0(x), \mathrm{v}_0(x) \in BV(\mathbb{R})$

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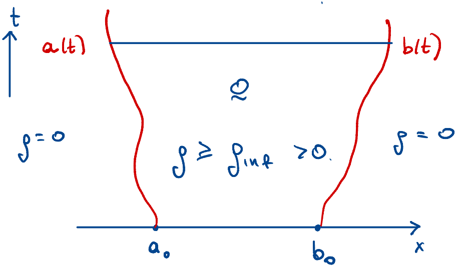
$$\rho_0(x), \mathrm{v}_0(x) \in BV(\mathbb{R}) \qquad q := \frac{1}{2}\mathrm{TV}\left\{\ln(\rho_0)\right\} + \frac{1}{2\alpha}\mathrm{TV}\left\{\mathrm{v}_0\right\} > 0$$

Aim

For entropy weak solutions,

global existence + time-asymptotic flocking



$\Omega = \{(x,t) : x \in I(t), \ t \geq 0\}$

$I(t) = [a(t), b(t)]$.

Global Existence and Structure

Theorem (Amadori-Chr., preprint (2021))

- *The Cauchy problem admits an entropy weak solution* $(\rho, \rho \mathrm{v})$ *on* $\mathbb{R} \times [0, +\infty)$.
- *There exist two locally Lipschitz curves* $t \mapsto a(t)$*,* $b(t)$*,* $t \in [0, +\infty)$ *and a value* $\rho_{inf} > 0$ *s.t.:*

(i) $a(0) = a_0\,, \quad b(0) = b_0\,; \qquad a(t) < b(t) \qquad$ *for all* $t > 0\,;$

(ii)

$$\begin{cases} \mathrm{supp}\{\rho(\cdot,t)\} = I(t) \doteq [a(t), b(t)]\,, \\ \mathrm{ess\,inf}_{I(t)}\, \rho(\cdot,t) \geq \rho_{inf} > 0\,, \\ (\rho, \mathrm{v})(x,t) = 0 & \forall\, x \notin I(t)\,. \end{cases}$$

- *Conservation of mass and momentum.*

Strategy of the Proof STEP 1:

$$M \doteq \int_{\mathbb{R}} \rho_0(x)\,dx \;\; > 0\,, \qquad M_1 \doteq \int_{\mathbb{R}} \rho_0(x) \mathtt{v}_0(x)\,dx\,.$$

$$\begin{aligned} RHS &= \rho(x,t)\left\{ \int_{\mathbb{R}} \rho(x',t)\mathtt{v}(x',t)\,dx' - \mathtt{v}(x,t)\int_{\mathbb{R}} \rho(x',t)\,dx' \right\} \\ &= \rho(x,t)\left(M_1 - \mathtt{v}(x,t)M\right) \\ &= M\rho(x,t)\left(\bar{\mathtt{v}} - \mathtt{v}(x,t)\right), \qquad \text{with } \bar{\mathtt{v}} \doteq \frac{M_1}{M} \end{aligned}$$

System equivalent to

$$\begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0, \\ \partial_t(\rho \mathtt{v}) + \partial_x\left(\rho \mathtt{v}^2 + p(\rho)\right) = -M\rho\left(\mathtt{v} - \bar{\mathtt{v}}\right). \end{cases} \tag{1}$$

$$p(\rho) = \alpha^2 \rho\,, \qquad \alpha > 0\,.$$

Translate to $\bar{\mathrm{v}} = 0$:

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Balance Laws:

Dafermos-Hsiao ... *dissipative source* (strongly or weakly)

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Balance Laws:
Dafermos-Hsiao ... *dissipative source* (strongly or weakly)
Main Difficulty:
loss of strict hyperbolicity - vacuum present in $\mathbb{R} \setminus [a_0, b_0]$
Physical vacuum

T.-P. Liu, Liu–Yang, Huang–Pan, Huang–Marcati–Pan, Huang–Pan–Wang, etc

STEP 2:

Recast from Eulerian to Lagrangian variables: $(x,t) \mapsto (y,\tau)$

$$y = \chi(x,t) = \int_{-\infty}^{x} \rho(x',t)\,dx' \in [0,M], \qquad \tau = t$$

$$u \doteq 1/\rho\,, \quad v(y,t) \doteq \mathtt{v}(x,t)\ .$$

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$$\begin{cases} \partial_\tau u - \partial_y v = 0, \\ \partial_\tau v + \partial_y(\alpha^2/u) = -Mv \end{cases} \tag{2}$$

for the unknown $(u(y,t), v(y,t))$ while $y \in [0,M]$ and $t \in [0,\infty)$.

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- I.D.: $(u_0, v_0) \in BV(0,M)\,,\ \operatorname{ess\,inf}_{(0,M)} u_0 > 0\,,\ \int_0^M v_0(y)\,dy = 0\,.$
- Boundary Data: $non-reflecting$ boundary conditions at $y = 0, M\,.$

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Existence: Nishida ($M=0$), Dafermos (1995), Luo–Natalini–Yang (2000), Amadori–Guerra (2001) –but Cauchy problems; Frid - different bdy data

Step 3: Front-tracking approximate solutions (u^ν, v^ν)

- interactions
- time steps $t^n = n\Delta t$
- standby fronts

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Linear Functionals:

$$L(t) = \sum_{\beta \in J(t)} |\varepsilon_\beta|\,, \qquad L_{in}(t) = \sum_{j=1}^{N(t)} |\varepsilon_j|\,.$$

$$0 < y_1 < y_2 < \ldots < y_{N(t)} < M$$

$$L(t) = L_{in}(t) + L_{0,out}(t) + L_{M,out}(t), \quad \forall t\,.$$

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$$L(t) = L_{in}(t) + L_{0,out}(t) + L_{M,out}(t), \quad \forall t\,.$$

- $L_{in}(t)$, $L(t)$ are non-increasing in time
- $L_{0,out}(t)$, $L_{M,out}(t)$ standby fronts at the bdy $y = 0$, $y = M$.
- bounds on u^ν and v^ν

$0 < u^\nu_{inf} \le u^\nu(y,t) \le u^\nu_{sup}, \quad |v^\nu(y,t)| \le \widetilde{C}_0 \quad \forall\, y \in (0,M), t, \nu\,.$

- finite number of interactions

STEP 4: A weighted functional

$$L_\xi(t) = \sum_{j=1,\ \varepsilon_j>0}^{N(t)} |\varepsilon_j| + \xi \sum_{j=1,\ \varepsilon_j<0}^{N(t)} |\varepsilon_j| \qquad \xi \geq 1\,. \tag{3}$$

Amadori–Guerra ('01); Amadori–Corli ('08); Amadori–Baiti–Corli–Dal Santo ('15)

$$L_{in}(t) \leq L_\xi(t) \leq \xi L_{in}(t) \leq \xi L_{in}(0+) \qquad \forall\, t > 0\,.$$

If $1 \leq \xi \leq \dfrac{1}{c(q)}$, then $\Delta L_\xi(t) \leq 0$ for any $t \neq t^n$

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- At interactions (same family), $\Delta L_\xi(t) + (\xi - 1)\,|\varepsilon_{refl}| \leq 0$
- At time steps t^n, $\Delta L_\xi(t^n) \leq \frac{M}{2}\Delta t\,(\xi - 1) L_{in}(t^n)$

Step 5: Vertical Traces:

Given $y \in (0, M)$ and $t > 0$,

$$W_y^\nu(t) = \frac{1}{2}\mathrm{TV}\left\{\ln(u^\nu))(y,\cdot);(0,t)\right\}.$$

and use $L_\xi(t)$ to prove:

$$W_y^\nu(T) \le \widetilde{C}_1 L_{in}(0) + \widetilde{C}_2 M \int_0^T L_{in}(t)\,dt\,.$$

For every $T > 0$, there exist positive constants C and L independent on $\nu \in \mathbb{N}$ such that for all $y \in [0, M]$

$$\mathrm{TV}\left\{u^\nu(y,\cdot);[0,T)]\right\} \le C\,, \quad \mathrm{TV}\left\{v^\nu(y,\cdot);[0,T)]\right\} \le C\,,$$

and

$$\int_0^T \left|v^\nu(y_1,t) - v^\nu(y_2,t)\right|\,dt \le L\,|y_2 - y_1| \qquad \forall\, y_1\,,\, y_2 \in [0, M]\,.$$

Step 6: Convergence in Lagrangian!

- a subsequence of $(u^\nu, v^\nu) \to (u,v) \in L^1_{loc}((0,M)\times[0,+\infty))$
- the map $t \mapsto (u,v)(\cdot,t) \in L^1(0,M)$ is Lipschitz cts in the L^1–norm
- $0 < u_{inf} \le u(y,t) \le u_{sup}\,, \quad |v(y,t)| \le \widetilde{C}_0$
- as $\nu \to \infty$

$$\int_0^y u^\nu(y',t)\,dy' \;\to\; \int_0^y u(y',t)\,dy' \qquad \forall\, y \in (0,M)\,,\; t \ge 0\,.$$

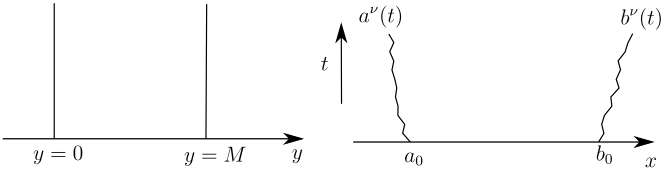
- for every $T>0$, the map $y \mapsto (u,v)(y,\cdot) \in L^1(0,T)$ $v(0+,t) \in L^1_{loc}([0,+\infty)) \quad \int_0^t v^\nu(0+,s)\,ds \;\to\; \int_0^t v(0+,s)\,ds$
- Entropy weak solution (u,v) to

$$\begin{cases} \partial_\tau u - \partial_y v = 0, \\ \partial_\tau v + \partial_y(\alpha^2/u) = -Mv \end{cases}$$

Step 7: From Lagrangian to Eulerian

For each $\nu \in \mathbb{N}$, define the approximate boundaries:

$$a^\nu(t) \doteq a_0 + \int_0^t v^\nu(0+,s)ds\,, \quad b^\nu(t) \doteq a^\nu(t) + \int_0^M u^\nu(y,t)\,dy\,.$$



Lagrangian   Eulerian

$$I^\nu(t) = (a^\nu(t), b^\nu(t))\,, \quad \Omega^\nu = \{(x,t);\ t \geq 0\,,\ x \in I^\nu(t)\}$$

Set $\rho^\nu(x,t) = \dfrac{1}{u^\nu(\chi^\nu(x,t),t)}\,, \qquad \mathrm{v}^\nu(x,t) = v^\nu(\chi^\nu(x,t),t) \quad x \in I^\nu(t)$

$$\rho^\nu(x,t) = 0 = \rho^\nu(x,t)\mathrm{v}^\nu(x,t) \qquad x \notin I^\nu(t)\,.$$

STEP 8: Conclusion

- As $\nu \to \infty$,

$$a^{\nu}(\cdot) \to a(\cdot)\,, \qquad b^{\nu}(\cdot) \to b(\cdot)$$

uniformly on compact subsets of $[0, +\infty)$.

- Uniform bounds on approximate solutions:

$$\mathrm{TV}\,\{\rho^{\nu}(\cdot, t); \mathbb{R}\}, \qquad \mathrm{TV}\,\{\mathrm{v}^{\nu}(\cdot, t); \mathbb{R}\}$$

- Define

$$x_j^{\nu}(t) = a^{\nu}(t) + \int_0^{y_j^{\nu}(t)} u^{\nu}\left(y', t\right) dy' \qquad j = 1, \ldots, N^{\nu}(t)$$

the Rankine-Hugoniot conditions are approximately satisfied across the piecewise linear curves $x_j^{\nu}(t)$

- Approximate total momentum $\mathfrak{I}^\nu(t) = \int_{a^\nu(t)}^{b^\nu(t)} \rho^\nu(x,t)\mathtt{v}^\nu(x,t)\,dx\,,$

$$|\mathfrak{I}^\nu(t)| \le e^{-Mt} \cdot e^{M\Delta t_\nu} \cdot \frac{M}{\nu} + \frac{\widetilde{C}}{M}\eta_\nu + \widetilde{C}\eta_\nu\Delta t_\nu, \qquad t \ge 0 \quad (4)$$

for a suitable constant $\widetilde{C} > 0$, which is independent on t and ν.

- $(\rho^\nu, \mathtt{v}^\nu) \xrightarrow{\nu\to\infty} (\rho, \mathtt{v})$ in $L^1_{loc}(\Omega)$,

$$\mathtt{m}(x,t) = \begin{cases} \rho(x,t)\mathtt{v}(x,t) & (x,t) \in \Omega \\ 0 & (x,t) \in \mathbb{R}\times[0,+\infty) \setminus \overline{\Omega}\,. \end{cases}$$

- Conservation of mass

$$\int_{a^\nu(t)}^{b^\nu(t)} \rho^\nu(x,t)\,dx \xrightarrow{\nu\to\infty} \int_{a(t)}^{b(t)} \rho(x,t)\,dx \qquad \nu\to\infty\,,$$

- Conservation of momentum

$$\int_{a^\nu(t)}^{b^\nu(t)} \rho^\nu(x,t)\mathtt{v}^\nu(x,t)\,dx \xrightarrow{\nu\to\infty} \int_{a(t)}^{b(t)} \rho(x,t)\mathtt{v}(x,t)\,dx = 0.\ \#$$

Time-Asymptotic Flocking

1. $alignment:$ the support $I(t)$ remains bounded $\forall t$

$$\sup_{0\le t<\infty} \{b(t)-a(t)\} < \infty$$

2. $flocking:$ the oscillation of the velocity satisfies

$$\lim_{t\to\infty} \operatorname{osc}\{\mathrm{v}; I(t)\} = 0$$

where

$$\operatorname{osc}\{\mathrm{v}; I(t)\} = \sup_{x_1,x_2\in I(t)} |\mathrm{v}(x_1,t)-\mathrm{v}(x_2,t)|\,.$$

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*** Immediate

$$b(t)-a(t) \leq \frac{1}{\rho_{inf}} \int_{I(t)} \rho(x,t)\,dx = \frac{M}{\rho_{inf}} \qquad \forall\, t>0$$

Time-Asymptotic Flocking

Theorem (Amadori-Chr., preprint (2021))

Let $(\rho, \rho \mathrm{v})$ *be the entropy weak solution with the initial data* $(\rho_0, \mathrm{v}_0) \in BV(\mathbb{R})$ *satisfying* $q > 0$ *as obtained in Theorem 1. Suppose that*

$$e^{2q} M^2 < \alpha \max\left\{\rho_0(a_0+), \rho_0(b_0-)\right\}, \tag{5}$$

holds true, then the solution $(\rho, \rho \mathrm{v})$ *admits time-asymptotic flocking.*

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Let $(\rho, \rho \mathrm{v})$ be the entropy weak solution with the initial data $(\rho_0, \mathrm{v}_0) \in BV(\mathbb{R})$ satisfying $q > 0$ as obtained in Theorem 1. Suppose that

$$e^{2q} M^2 < \alpha \max\left\{\rho_0(a_0+), \rho_0(b_0-)\right\}, \qquad (5)$$

holds true, then the solution $(\rho, \rho \mathrm{v})$ admits time-asymptotic flocking. More precisely, $\exists\, t_0 > 0$ such that

$$\operatorname{osc}\left\{\mathrm{v}; I(t)\right\} \leq C_2' e^{-C_1' t}, \qquad \forall\, t \geq t_0$$

for some positive constants C_1', C_2'.

Strategy of the Proof STEP 1:

- $k \geq 1$ - generation order

$$F_k(t) \doteq \sum_{\varepsilon>0,\ g_\varepsilon=k} |\varepsilon| + \xi \sum_{\varepsilon<0,\ g_\varepsilon=k} |\varepsilon| \ , \qquad \widetilde{F}_k(t) \doteq \sum_{j\geq k} F_j(t) \quad (6)$$

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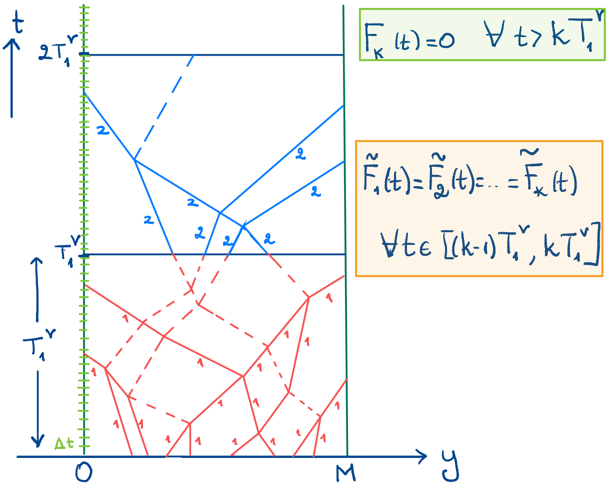
- Decay estimates via the total variation weighted by generation order

$$V(t) = \sum_{k\geq 1} \xi^k F_k(t)$$ NEW

- $$V(t) \leq \left(1 + \frac{(\xi^2-1)}{2} M\Delta t\right)^n V(0+) \ ,$$

while $t \in [t^n, t^{n+1})$ and $\xi \in [1, c(q)^{-1/2}]$.

STEP 2: • the maximal existence time length T_1^ν for the waves of $k=1$ to reach $y=0, M$ is $T_1^\nu = \frac{M}{\alpha} u_{sup}^\nu = \frac{e^{2q}M}{\alpha} \min\{\tilde{u}_0^\nu, \tilde{u}_M^\nu\}$, STEP 2: • the maximal existence time length T_1^ν for the waves of $k=1$ to reach $y=0,M$ is $T_1^\nu = \frac{M}{\alpha} u^\nu_{sup} = \frac{e^{2q}M}{\alpha}\min\{\tilde{u}^\nu_0, \tilde{u}^\nu_M\}$,



- Using that $\xi \geq 1$, we have the relation

$$\tilde{F}_k(t) \leq \frac{1}{\xi^k} \sum_{j \geq k} \xi^j F_j(t) \leq \frac{1}{\xi^k} V(t), \quad \forall t$$

- Prove for some $\bar{\xi} > 1$,

$$\lim_{t \to \infty} \tilde{F}_1(t) = 0, \quad \text{since } MT_1^* < 1 \tag{7}$$

$$L_{in}(t) \leq \tilde{F}_1(t) \leq C_2 e^{-C_1 t} q \ ,$$

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STEP 3: • Time-asymptotic flocking $\#$

THANK YOU

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Χρόνια Πολλά