Shock Waves and Entropy

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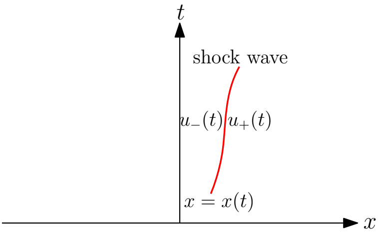
System of Hyperbolic Conservation Laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0,\ \boldsymbol{u} \in \mathbb{R}^n$$

$$\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) = \lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}),\ \lambda_1(\boldsymbol{u}) < \cdots < \lambda_n(\boldsymbol{u}).$$

Shock wave $(\boldsymbol{u}_+, \boldsymbol{u}_-)$ along $x = x(t)$ with speed $\sigma = x'(t)$ and states $\boldsymbol{u}_\pm = \boldsymbol{u}_\pm(t) = \boldsymbol{u}(x(t) \pm 0, t)$.



Rankine-Hugoniot condition

$$\sigma\big(\boldsymbol{u}_+ - \boldsymbol{u}_-\big) = \big(\boldsymbol{f}(\boldsymbol{u}_+) - \boldsymbol{f}(\boldsymbol{u}_-)\big),\ \sigma = \sigma(\boldsymbol{u}_+, \boldsymbol{u}_-).$$

- Shock wave represents an irreversible process in the gas flow.
- From second law of thermodynamics, the entropy increases during an irreversible process.
- Bethe-Weyl: For ideal gases, the increase of the entropy as the gas flows across a shock guarantees the mathematical stability and physical admissibility of the shock.
- This is not so for gases with general equation of state. The increase of the entropy is a necessary, but not sufficient condition for the admissibility of the shock.
- Goal: To interpret the admissibility of a shock from the view point of differential equations in terms of the production of the entropy in thermodynamics.

Euler equations for compressible media (in Lagrangian coordinates):

$$\begin{pmatrix} \tau \\ v \\ E \end{pmatrix}_t + \begin{pmatrix} -v \\ p \\ pv \end{pmatrix}_x = \mathbf{0}, \quad \begin{pmatrix} \text{mass} \\ \text{momentum} \\ \text{energy} \end{pmatrix}.$$

ρ density, $\tau = 1/\rho$ specific volume, v velocity, p pressure, e internal energy, $E = e + v^2/2$ total energy.

$$p = p(\tau, e), \; p = \tilde{p}(\tau, s), \; \text{Constitutive Law.}$$

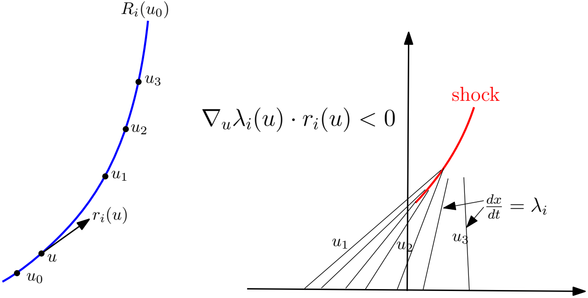
$$de = \theta ds - pd\tau, \quad \text{Thermodynamics Relation.}$$

θ temperature, s entropy.
Acoustic waves are isentropic pressure waves:

$$c = \sqrt{-\tilde{p}_\tau}, \; \text{acoustic speed}, \; \tilde{p}_\tau = p_\tau - pp_e.$$

Poisson 1808, Stokes 1848 *i*-simple waves and shock formation for Euler equations. For general system $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{0}$:

$$\begin{cases} \boldsymbol{u}(x,t) \in \boldsymbol{R}_i(\boldsymbol{u}_0), \text{ integral curve of } \boldsymbol{r}_i(\boldsymbol{u}) \text{ through } \boldsymbol{u}_0 \\ \boldsymbol{u}(x,t) = \boldsymbol{u}(x - \lambda_i t, 0),\ \lambda_i = \lambda_i(\boldsymbol{u}(x,t)) = \lambda_i(\boldsymbol{u}(x - \lambda_i t, 0)),\ t \le 0. \end{cases}$$



Formation of shock through compression of simple wave

Two considerations for Euler equations

1. Formation of shocks through compression: $\nabla_{\boldsymbol{u}}\lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) \neq 0$. For the acoustic modes with speed $\tilde{p}_\tau$ of the Euler equations, the compressibility condition becomes $\tilde{p}_{\tau\tau} \neq 0$.
2. Shock is irreversible, entropy s increases as the gas flow across a shock.

Theorem (Hans Bethe 1942, Hermann Weyl 1949)

Consider the Euler equations and suppose that

$$\tilde{p}_{\tau\tau} \neq 0, \text{ convexity condition for shock formation.}$$

Then a shock is compressible if and only if the entropy increases as the gas flows across the shock.

Generalization to general system of hyperbolic conservation laws $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \mathbf{0}$:

- Lax 1957:

$$\begin{aligned} &\nabla_{\boldsymbol{u}}\lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) \neq 0, \ \text{genuine nonlinearity.} \\ &\lambda_i(\boldsymbol{u}_-) > \sigma > \lambda_i(\boldsymbol{u}_+), \ \text{Lax entropy condition.} \end{aligned}$$

- Lax 1971: Entropy pair $(\eta(\boldsymbol{u}), q(\boldsymbol{u}))$ for general system $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \mathbf{0}$ if $\eta''(\boldsymbol{u}) > 0$ and $\eta(\boldsymbol{u})_t + q(\boldsymbol{u})_x = 0$ for smooth solution $\boldsymbol{u}$, or $\eta'(\boldsymbol{u})\boldsymbol{f}'(\boldsymbol{u}) = q'(\boldsymbol{u})$.

Theorem (Lax 1971)

For the genuinely nonlinear field, $\nabla_{\boldsymbol{u}}\lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) \neq 0$, the entropy inequality $\eta(\boldsymbol{u})_t + q(\boldsymbol{u})_x < 0$ across a shock holds if and only if the Lax entropy condition holds.

- For scalar laws, $u \in \mathbb{R}$, the convexity condition becomes $f''(u) \neq 0$. There is a theory for the general case when $f''(u)$ changes sign.
- any convex function $\eta(u)$ is an entropy with entropy flux given by $q(u) = \int^u \eta'(v) f'(v) dv$.
- Entropy inequality: $\eta(u) + q(u)_x \leq 0$ for all entropy pairs iff

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} \leq \frac{f(u) - f(u_-)}{u - u_-} \text{ for all } u \text{ between } u_- \text{ and } u_+$$

Oleinik entropy condition.

- The Russian School of Oleinik 1959 and Kruzkov 1970 completed the theory for scalar laws making essential use of entropy inequality.

- Entropy pair $(\eta(\boldsymbol{u}), q(\boldsymbol{u}))$ for general system $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{0}$ if $\eta''(\boldsymbol{u}) > 0$ and $\eta'(\boldsymbol{u})\boldsymbol{f}'(\boldsymbol{u}) = q'(\boldsymbol{u})$.
- For Euler equations $(s, 0)$ is essentially the only entropy pair. For a general system, the existence of the entropy pair is exceptional.
- The existence of an entropy pair is a constitutive hypothesis.

Theorem (Godunov 1961)

A system is symmetrizable if and only if an entropy pair exists.

- The scalar theory requiring an abundance of entropy pairs does not works for the general systems.
- How to capture the entropy condition, such as the Oleinik condition with only one entropy pair?
- How about the systems, for which there is essentially only one entropy pair $(\eta(\boldsymbol{u}), q(\boldsymbol{u}))$?

Theorem (Dafermos 1973)

For scalar laws, consider the entropy function $\eta(u) = u^2/2$ and the associated entropy flux $q(u) = \int^u zf'(z)dz$. A resolution of discontinuity yields maximum entropy production if and only if the shock waves in the resolution satisfy the Oleinik entropy condition.

The same holds for the p-system with physical entropy pair

$$\begin{cases} \tau_t - v_x = 0, \\ v_t + p(\tau)_x = 0 \end{cases}, \quad \eta(\tau, v) = \frac{v^2}{2} - \int^{\tau} p(z)dz, \quad q(\tau, v) = pv,$$

and the Wendroff E-condition for 1-shock (or 2-shock) $(\boldsymbol{u}_-, \boldsymbol{u}_+)$:

$$\frac{p(\tau_+) - p(\tau_-)}{\tau_+ - \tau_-} \geq (or \leq) \frac{p(\tau) - p(\tau_-)}{\tau - \tau_-}$$

for all τ between τ_- and τ_+.

Hugoniot curves $\boldsymbol{H}_i(\boldsymbol{u}_0)$, $i = 1, \cdots, n$, through a state $\boldsymbol{u}_0$:

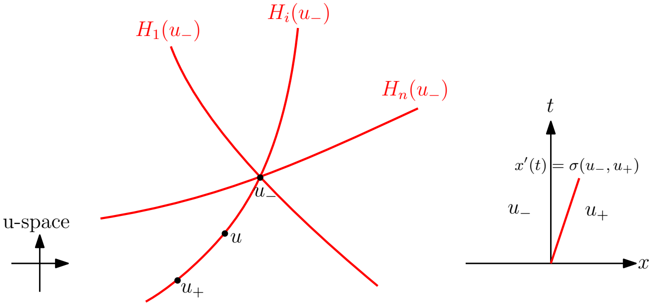
$$\boldsymbol{H}(\boldsymbol{u}_0) = \{\boldsymbol{u} : \ \sigma(\boldsymbol{u} - \boldsymbol{u}_0) = \boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{u}_0) \text{ for some scalar } \sigma = \sigma(\boldsymbol{u}_0, \boldsymbol{u})\},$$

$$\boldsymbol{H}(\boldsymbol{u}_0) = \boldsymbol{H}_1(\boldsymbol{u}_0) \cup \cdots \cup \boldsymbol{H}_n(\boldsymbol{u}_0) \text{ Hugoniot curves},$$

$$\sigma = \sigma(\boldsymbol{u}_0, \boldsymbol{u}) \to \lambda_i(\boldsymbol{u}_0) \text{ as } \boldsymbol{u} \to \boldsymbol{u}_0 \text{ along } \boldsymbol{H}_i(\boldsymbol{u}_0).$$

Liu entropy condition:

$$\sigma(\boldsymbol{u}_-, \boldsymbol{u}_+) \le \sigma(\boldsymbol{u}_-, \boldsymbol{u}) \text{ for } \boldsymbol{u} \in \boldsymbol{H}_i(\boldsymbol{u}_-) \text{ between } \boldsymbol{u}_- \text{ and } \boldsymbol{u}_+.$$



Liu entropy condition:

$$\sigma(\boldsymbol{u}_-, \boldsymbol{u}_+) \leq \sigma(\boldsymbol{u}_-, \boldsymbol{u}) \text{ for } \boldsymbol{u} \in \boldsymbol{H}_i(\boldsymbol{u}) \text{ between } \boldsymbol{u}_- \text{ and } \boldsymbol{u}_+.$$

1. Similar to the Lax theory for genuinely nonlinear case, the Liu condition allows for the unique resolution of the Riemann problem $\boldsymbol{u}(x,0) = \begin{cases} \boldsymbol{u}_l \text{ for } x < 0, \\ \boldsymbol{u}_r \text{ for } x > 0 \end{cases}$ for general systems when $|\boldsymbol{u}_r - \boldsymbol{u}_l|$ is sufficiently small.
2. To obtain global theory, one should aim at specific systems, such as the Euler equations.
3. Liu condition is much more restrictive than the entropy inequality $\eta(\boldsymbol{u})_t + q(\boldsymbol{u})_x \leq 0$.
4. **Goal**: To relate the Liu condition to something about the entropy pair given by physics.

Euler equations

$$\begin{pmatrix} \tau \\ v \\ E \end{pmatrix}_t + \begin{pmatrix} -v \\ p \\ pv \end{pmatrix}_x = \mathbf{0}, \quad \begin{pmatrix} \text{mass} \\ \text{momentum} \\ \text{energy} \end{pmatrix}.$$

Basic hypothesis: $p_e > 0,\ p_\tau < 0.$

Goal:

1. To obtain global theory for the Riemann problem.
2. To relate the Liu condition to the so-called entropy procedure for the formation of shocks.

Analysis:

1. Establish the global monotonicity of Hugoniot curves.
2. Establish the global relation between the entropy production rate and the compressibility of the shock.

Rankine-Hugoniot condition for Euler equations

$$\begin{cases} \sigma(\tau - \tau_0) = -(v - v_0), \\ \sigma(v - v_0) = p - p_0, \\ \sigma\big(e + \frac{1}{2}v^2 - e_0 - \frac{1}{2}(v_0)^2\big) = pv - p_0 v_0, \text{ or,} \end{cases}$$

$$S_1 = (\tau-\tau_0)(p-p_0)+(v-v_0)^2 = 0,\ S_2 = (\tau-\tau_0)(p+p_0)+2(e-e_0) = 0.$$

Tangent to Hugoniot curve is $\nabla S_1 \times \nabla S_2$. Along $\boldsymbol{H}_1(\boldsymbol{u}_0)$:

$$\begin{pmatrix} \dot{v} \\ \dot{p} \\ \dot{\tau} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 2(\tau - \tau_0)\big(1 - \frac{p_e}{p_\tau}p_0\big) + \frac{2}{p_\tau}(p - p_0) \\ 2(v - v_0)\big(\frac{p_e}{p_\tau}(p + p_0) - 2\big) \\ -\frac{2}{p_\tau}(v - v_0)\big(2 + p_e(\tau - \tau_0)\big) \\ \frac{1}{\sigma\theta p_\tau}(\sigma - \lambda_1)(\sigma - \lambda_3)(v - v_0) \end{pmatrix}$$

Lemma (Monotonicity)

Along $\boldsymbol{H}_1(\boldsymbol{u}_0)$ and $\boldsymbol{R}_1(\boldsymbol{u})$, $\boldsymbol{u} \neq \boldsymbol{u}_0$, v and p are strictly monotone, and $(p - p_0)(\tau - \tau_0) < 0$, $(v - v_0)(p - p_0) < 0$.

Lemma (Shock and characteristic speeds)

Along the Hugoniot curve $\boldsymbol{H}_1(\boldsymbol{u}_0)$, $\dot{\sigma}(\boldsymbol{u}, \boldsymbol{u}_0) > 0$, if and only if $\sigma(\boldsymbol{u}, \boldsymbol{u}_0) < \lambda_1(\boldsymbol{u})$.

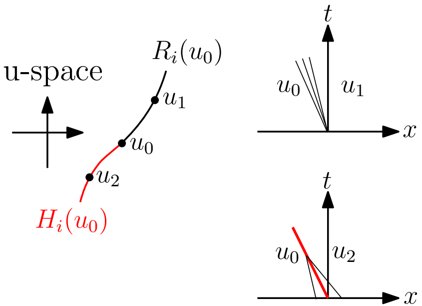
Lemma (Tangency and convexity)

When the shock speed has a critical point, $\dot{\sigma}(\boldsymbol{u}, \boldsymbol{u}_0) = 0$, the Hugoniot curve $\boldsymbol{H}_1(\boldsymbol{u}_0)$ is tangent to the characteristic curve $\boldsymbol{R}_1(\boldsymbol{u})$. At the critical point, σ has maximum, $\ddot{\sigma} < 0$ if and only if $\nabla_{\boldsymbol{u}}\lambda_1(\boldsymbol{u}) \cdot \boldsymbol{r}_1(\boldsymbol{u}) < 0$.

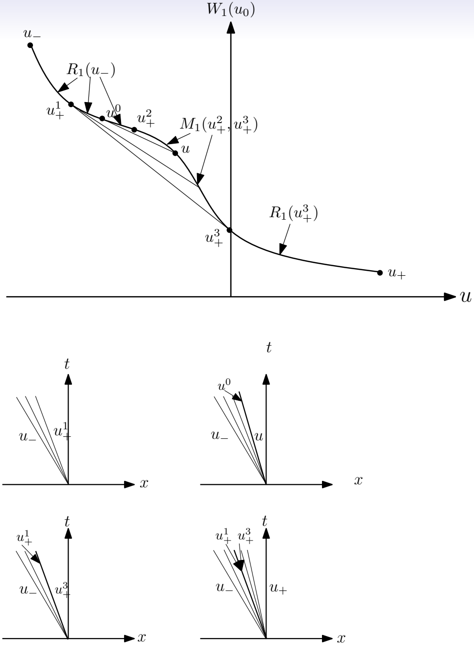
Lemma (Entropy and compressibility)

Along $\boldsymbol{R}_1(\boldsymbol{u}_0)$, the entropy is constant. Along $\boldsymbol{H}_1(\boldsymbol{u}_0)$, the entropy $s = s(\boldsymbol{u})$ increases if and only if $\sigma(\boldsymbol{u}_0, \boldsymbol{u}) > \lambda_1(\boldsymbol{u})$.

To solve the Riemann problem, one constructs the wave curves $\boldsymbol{W}_i(\boldsymbol{u}_0)$ so that for $\boldsymbol{u} \in \boldsymbol{W}_i(\boldsymbol{u}_0)$, $\boldsymbol{u}$ is connected to $\boldsymbol{u}_0$ by an i-wave. For a genuinely nonlinear field, the wave curve is a combination of $\boldsymbol{R}_i(\boldsymbol{u}_0)$ and $\boldsymbol{H}_i(\boldsymbol{u}_0)$.



Waves for a genuinely nonlinear field.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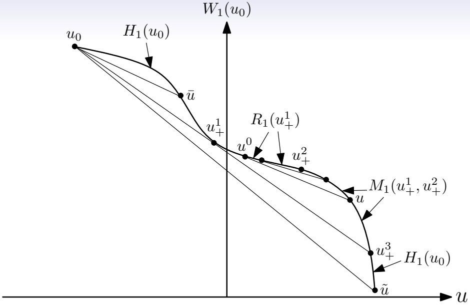
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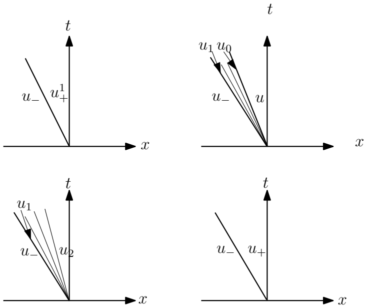
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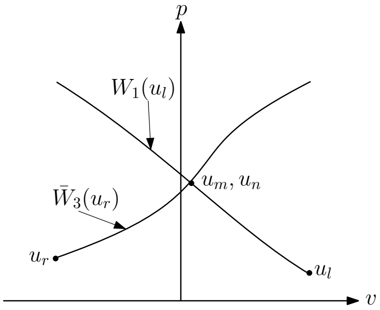
The wave curves $\boldsymbol{W}_1, \ \boldsymbol{W}_3$ are globally monotone in v and in p. To solve the Riemann problem $\boldsymbol{u}_l, \boldsymbol{u}_r$), draw the wave curves $\boldsymbol{W}_1(\boldsymbol{u}_l)$ and $\boldsymbol{W}(\boldsymbol{u}_r)$ to intersect, in the (v, p) plane at $\boldsymbol{u}_n, \boldsymbol{u}_m$ with $v_n = v_m, \ p_n = p_m$. The wave $(\boldsymbol{u}_n, \boldsymbol{u}_m)$ corresponds to the characteristic λ_2 and represents a thermal wave.



Riemann solution

Definition (Liu-Ruggeri 2003)

A 1-shock wave $(\boldsymbol{u}_-, \boldsymbol{u}_+)$ is constructible by the entropy procedure if it can be reached from a wave of zero strength by dynamically increasing its strength through the following procedures:

(1). A shock wave increases its strength by continuously moving its end states $(\boldsymbol{u}_+(\alpha), \boldsymbol{u}_-(\alpha))$ away from each other so that the entropy production increases in the process:

$$\frac{d}{d\alpha}\big(s(\boldsymbol{u}_+(\alpha)) - s(\boldsymbol{u}_-(\alpha))\big) > 0.$$

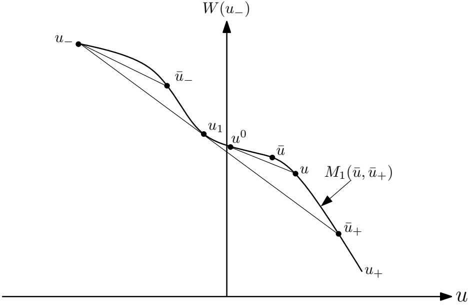
(2). A shock wave increases its strength and entropy production by combining two shock waves of the same speed.

- The entropy procedure is to view the admissibility of a shock through the **dynamic process** of constructing the shock by gradually increasing its strength and entropy production through absorbing the adjacent waves.
- **von Karman: ''Any physical process starts from somewhere and goes to somewhere.''**
  In response to von Neumann about the non-uniqueness of Prandtl construction of shock reflections off a ramp.

Theorem (Journal of Hyperbolic Differential Equations. 2021)

A shock wave of any strength for the Euler equations satisfies Liu entropy condition if and only if it is constructible by the entropy procedure.

If. Any shock satisfying Liu condition can be constructed by entropy procedure:

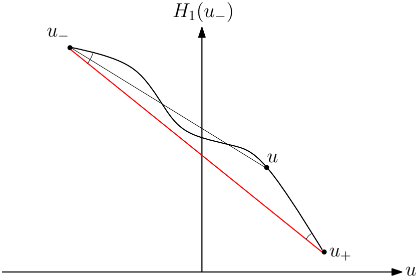


To reduce $(\boldsymbol{u}_-, \boldsymbol{u}_+)$ to zero wave:

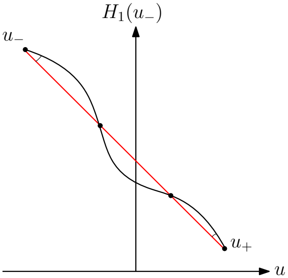
$$(\boldsymbol{u}_-, \boldsymbol{u}_+) \Rightarrow (\boldsymbol{u}_-, \bar{\boldsymbol{u}}_+) = (\boldsymbol{u}_-, \boldsymbol{u}_1) \cup (\boldsymbol{u}_1, \bar{\boldsymbol{u}}_+),$$

$$(\boldsymbol{u}_1, \bar{\boldsymbol{u}}_+) \Rightarrow (\boldsymbol{u}^0, \boldsymbol{u}); \; (\boldsymbol{u}_-, \boldsymbol{u}_1) \Rightarrow (\boldsymbol{u}_-, \bar{\boldsymbol{u}}_-).$$

Only If. Any shock violating Liu condition cannot be constructed by entropy procedure:



(E) implies compressibility



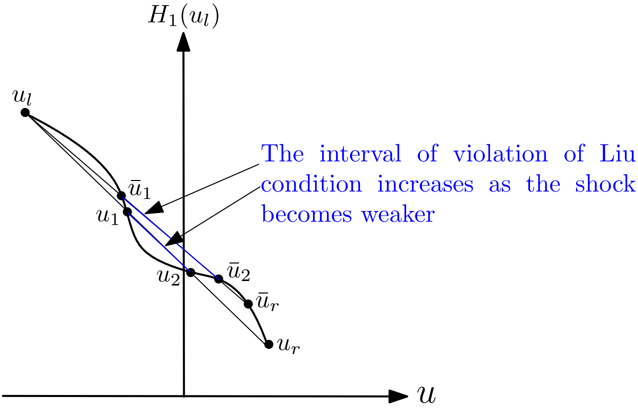
Conpressive, not (E)

Entropy change along the Hugoniot courve:

$$\dot{s} = \frac{1}{\sigma\theta p_\tau}(\sigma - \lambda_1)(\sigma - \lambda_3)|v - v_-| \text{ along } \boldsymbol{H}_1(\boldsymbol{u}_-) \;\Rightarrow$$

Decrease in entropy production with weakening of shock.

Only If. Any shock violating Liu condition cannot be constructed by entropy procedure:



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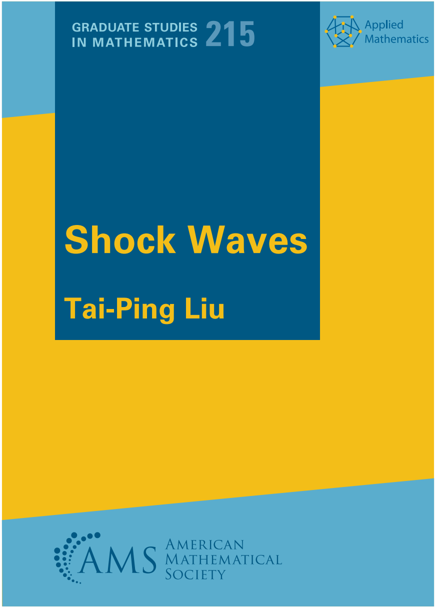

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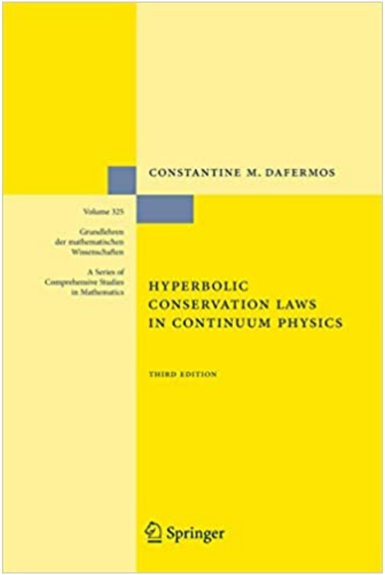
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