Structure Preserving Numerical Methods for Hyperbolic Systems of Conservation and Balance Laws

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joint work with
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Systems of Balance Laws

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U})$$

Examples:

- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:

- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

Examples:

- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:

- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows

Systems of Balance Laws

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U}) \quad \text{or} \quad \boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Challenges: certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, assymptotic regimes, etc.) are essential in many applications;

- Goal: to design numerical methods that are not only consistent with the given PDEs, but
  - preserve the structural properties at the discrete level – **well-balanced numerical methods**
  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[P. LeFloch; 2014]

Asymptotic Preserving (AP) Methods

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Solutions of many hyperbolic systemes reveal a multiscale character and thus their numerical resolution presence some major difficulties;
- Such problems are typically characterized by the occurence of a small parameter by $0 < \varepsilon \ll 1$;
- The solutions show a nonuniform behavior as $\varepsilon \to 0$;
- the type of the limiting solution is different in nature from that of the solutions for finite values of $\varepsilon > 0$.

Goal:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limitting model as $\varepsilon \to 0$.

Shallow Water System with Coriolis Force

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_x = -ghB_y - fhu \end{cases}$$

- h: water height
- u, v: fluid velocity
- g: gravitational constant
- $B \equiv 0$ – bottom topography
- $f = 1/\varepsilon$ – Coriolis parameter

Dimensional Analysis

Introduce

$$\widehat{x} := \frac{x}{\ell_0}, \quad \widehat{y} := \frac{y}{\ell_0}, \quad \widehat{h} := \frac{h}{h_0}, \quad \widehat{u} := \frac{u}{w_0}, \quad \widehat{v} := \frac{v}{w_0}.$$

Substituting them into the SWE and dropping the hats in the notations, we obtain the dimensionless form:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_x + (huv)_y = \dfrac{1}{\varepsilon}hv, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_y = -\dfrac{1}{\varepsilon}hu, \end{cases}$$

in which

$$\mathrm{Fr} := \frac{w_0}{\sqrt{gh_0}} = \varepsilon$$

is the reference Froude number

Explicit Discretization

Eigenvalues of the flux Jacobian:

$$\left\{ u \pm \frac{1}{\varepsilon}\sqrt{h},\, u \right\} \quad \text{and} \quad \left\{ v \pm \frac{1}{\varepsilon}\sqrt{h},\, v \right\}$$

This leads to the CFL condition

$$\Delta t_{\text{expl}} \le \nu \cdot \min \left(\frac{\Delta x}{\max\limits_{u,h} \left\{ |u| + \frac{1}{\varepsilon}\sqrt{h} \right\}}, \frac{\Delta y}{\max\limits_{v,h} \left\{ |v| + \frac{1}{\varepsilon}\sqrt{h} \right\}} \right) = \mathcal{O}(\varepsilon \Delta_{\min}).$$

where $\Delta_{\min} := \min(\Delta x, \Delta y)$

- $0 < \nu \le 1$ is the CFL number
- Numerical diffusion: $\mathcal{O}(\lambda_{max} \Delta x) = \mathcal{O}(\varepsilon^{-1} \Delta x)$.
- We must choose $\Delta x \approx \varepsilon$ to control numerical diffusion and the stability condition becomes

$$\Delta t = \mathcal{O}(\varepsilon^2)$$

Low Froude Number Flows

Low Froude number regime $(0 < \varepsilon \ll 1) \Longrightarrow$ very large propagation speeds

Explicit methods:

- very restrictive time and space dicretization steps, typically proportional to ε due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:

- uniformly stable for $0 < \varepsilon < 1$;
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of ε

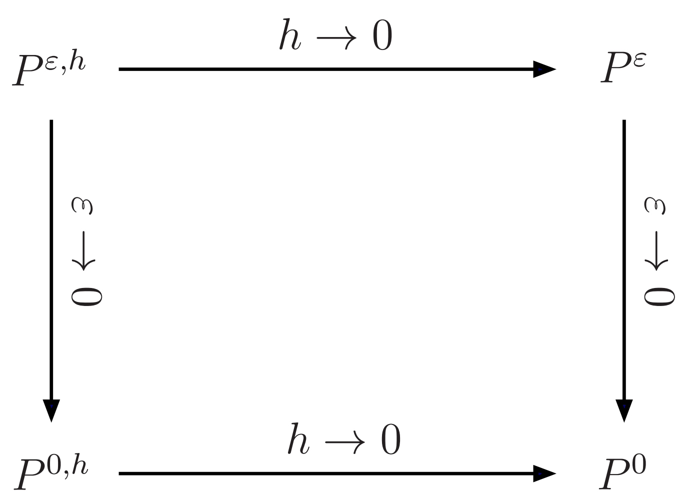
Asymptotic-Preserving (AP) Methods

Introduced in [Klar; 1998, Jin; 1999], see also [Jin, Levermore; 1991],

[Golse, Jin, Levermore; 1999].

Idea:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limitting model as $\varepsilon \to 0$.



Analysis for the Low Froude Number Limit

We plug the formal asymptotic expansions

$$h = h^{(0)} + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \cdots$$
$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots$$
$$v = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots$$

into the SW system:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_x + (huv)_y = \dfrac{1}{\varepsilon} fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_y = -\dfrac{1}{\varepsilon} fhu \end{cases}$$

and then collect the like powers of ε ...

Analysis for the Low Froude Number Limit

$$\mathcal{O}(\varepsilon^{-2}): \quad h^{(0)} h_x^{(0)} = 0$$

$$h^{(0)} h_y^{(0)} = 0$$

$$\mathcal{O}(\varepsilon^{-1}): \quad h^{(0)} h_x^{(1)} + h^{(1)} h_x^{(0)} = h^{(0)} v^{(0)}$$

$$h^{(0)} h_y^{(1)} + h^{(1)} h_y^{(0)} = -h^{(0)} u^{(0)}$$

$$\mathcal{O}(1) \quad : \quad h_t^{(0)} + (h^{(0)} u^{(0)})_x + (h^{(0)} v^{(0)})_y = 0$$

$$(h^{(0)} u^{(0)})_t + \left[h^{(0)} (u^{(0)})^2\right]_x + (h^{(0)} u^{(0)} v^{(0)})_y + h^{(0)} h_x^{(2)} + h^{(1)} h_x^{(1)}$$

$$(h^{(0)} v^{(0)})_t + \left[h^{(0)} (v^{(0)})^2\right]_y + (h^{(0)} u^{(0)} v^{(0)})_x + h^{(0)} h_y^{(2)} + h^{(1)} h_y^{(1)}$$

$$+ h^{(2)} h_y^{(0)} = -h^{(0)} u^{(1)} - h^{(1)} u^{(0)}$$

$$\dots$$

Analysis for the Low Froude Number Limit

The equations for $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ terms imply that

$$h_x^{(0)} = 0, \;\; h_y^{(0)} = 0 \; (\Rightarrow h^{(0)} \equiv \mathrm{Const}), \qquad h_x^{(1)} = v^{(0)}, \;\; h_y^{(1)} = -u^{(0)},$$

which can be substituted into equations of $\mathcal{O}(1)$ terms to obtain the limit equations:

$$\begin{cases} h_t^{(0)} + (h^{(0)}u^{(0)})_x + (h^{(0)}v^{(0)})_y = 0 \quad \Longrightarrow \quad u_x^{(0)} + v_y^{(0)} = 0 \\ (h^{(0)}u^{(0)})_t + \left[h^{(0)}(u^{(0)})^2\right]_x + (h^{(0)}u^{(0)}v^{(0)})_y + h^{(0)}h_x^{(2)} = h^{(0)}v^{(1)} \\ (h^{(0)}v^{(0)})_t + \left[h^{(0)}(v^{(0)})^2\right]_y + (h^{(0)}u^{(0)}v^{(0)})_x + h^{(0)}h_y^{(2)} = -h^{(0)}u^{(1)} \\ h_t^{(1)} - h^{(0)}\left(h_{xx}^{(1)} + h_{yy}^{(1)}\right)_t = \cdots \\ h_t^{(2)} - \left(h^{(0)}v^{(1)} + h^{(1)}v^{(0)}\right)_{xt} + \left(h^{(0)}u^{(1)} + h^{(1)}u^{(0)}\right)_{yt} = \cdots \end{cases}$$

Goal: To develop an AP numerical methods for the SW system, which yield a consistent approximation of the above limiting equations as $\varepsilon \to 0$

Hyperbolic Splitting

Key idea: Split the stiff pressure term [Haack, Jin, Liu; 2012]

- We first split the stiff pressure gradient term into two parts, i.e.

$$\frac{1}{\varepsilon^2}\frac{h^2}{2} = \underbrace{\frac{1}{\varepsilon^2}\frac{h^2}{2} - \frac{a(t)h}{\varepsilon^2}}_{non-stiff} + \underbrace{\frac{a(t)h}{\varepsilon^2}}_{stiff}$$

- We then split the flux terms in the continuity equation by introducing a weight parameter α so that we can construct the slow dynamic system as a hyperbolic system:

$$hu = \alpha\, hu + (1-\alpha)hu, \qquad hv = \alpha\, hv + (1-\alpha)hv$$

Hyperbolic Flux Splitting

Key idea: Split the stiff pressure term [Haack, Jin, Liu; 2012]

$$\begin{cases} h_t+\alpha(hu)_x+\alpha(hv)_y+(1-\alpha)(hu)_x+(1-\alpha)(hv)_y=0,\\ (hu)_t+\left(hu^2+\dfrac{\frac{1}{2}h^2-a(t)h}{\varepsilon^2}\right)_x+(huv)_y+\dfrac{a(t)}{\varepsilon^2}h_x=\dfrac{1}{\varepsilon}hv,\\ (hv)_t+(huv)_x+\left(hv^2+\dfrac{\frac{1}{2}h^2-a(t)h}{\varepsilon^2}\right)_y+\dfrac{a(t)}{\varepsilon^2}h_y=-\dfrac{1}{\varepsilon}hu. \end{cases}$$

This system can be written in the following vector form:

$$\boldsymbol{U}_t+\underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x+\widetilde{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{non-stiff terms}}+\underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x+\widehat{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{stiff terms}}=\underbrace{\boldsymbol{S}(\boldsymbol{U})}_{\text{source terms}}$$

How to choose parameters α and $a(t)$?

Hyperbolic Flux Splitting

$$\boldsymbol{U}_t + \underbrace{\underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{non-stiff terms}}}_{\text{nonlinear part}} + \underbrace{\underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{stiff terms}} = \underbrace{\boldsymbol{S}(\boldsymbol{U})}_{\text{source terms}}}_{\text{linear part}}$$

Need to ensure: $\boldsymbol{U}_t + \widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y = \boldsymbol{0}$ is both nonstiff and hyperbolic

Eigenvalues of the Jacobians $\partial\widetilde{\boldsymbol{F}}/\partial\boldsymbol{U}$ and $\partial\widetilde{\boldsymbol{G}}/\partial\boldsymbol{U}$:

$$\left\{u \pm \sqrt{(1-\alpha)u^2 + \alpha\frac{h-a(t)}{\varepsilon^2}},\, u\right\}, \quad \left\{v \pm \sqrt{(1-\alpha)v^2 + \alpha\frac{h-a(t)}{\varepsilon^2}},\, v\right\}$$

We then take: $\alpha = \varepsilon^s \quad \text{and} \quad a(t) = \min_{(x,y)\in\Omega} h(x,y,t), \quad s \geq 1$

Remark. It is safe to take $\alpha = \varepsilon^2$ there as in this case the stability time-step restriction will be clearly independent of ε. The results obtained with $\alpha = \varepsilon$ are almost identical and no instabilities have been observed as expected.

Time Discretization of the Split System

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n - \underbrace{\Delta t \widetilde{\boldsymbol{F}}(\boldsymbol{U})^n_x - \Delta t \widetilde{\boldsymbol{G}}(\boldsymbol{U})^n_y}_{\text{nonlinear part, explicit}}$$
$$- \underbrace{\Delta t \widehat{\boldsymbol{F}}(\boldsymbol{U})^{n+1}_x - \Delta t \widehat{\boldsymbol{G}}(\boldsymbol{U})^{n+1}_y + \Delta t \boldsymbol{S}(\boldsymbol{U})^{n+1}}_{\text{linear part, implicit}}$$

- Nonstiff nonlinear part is treated using the second-order central-upwind scheme
- Stiff linear part reduces to a linear elliptic equation for h^{n+1} and straigtforward computations of $(hu)^{n+1}$ and $(hv)^{n+1}$

For simplicity of presentation: First-order accurate in time

In practice: We implement a two-stage second-order globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2) (all the proofs will apply)

Fully Discrete AP Schemes

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^{n} - \Delta t \underbrace{\left[\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x^n + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y^n\right]}_{\boldsymbol{R}(\boldsymbol{U})^{n}} - \Delta t \left[\widehat{\boldsymbol{F}}(\boldsymbol{U})_x^{n+1} + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y^{n+1} - \boldsymbol{S}(\boldsymbol{U})^{n+1}\right]$$

- We use the notation $\boldsymbol{R}^n := (R^{h,n}, R^{hu,n}, R^{hv,n})^\top$ and rewrite the system

$$h^{n+1} = h^{n} + \Delta t R^{h,n} - \Delta t(1-\alpha)\left[(hu)_x^{n+1} + (hv)_y^{n+1}\right]$$

$$(hu)^{n+1} = \frac{1}{K}\left[(hu)^{n} + \frac{\Delta t}{\varepsilon}(hv)^{n} + \Delta t\Big(R^{hu,n} + \frac{\Delta t}{\varepsilon}R^{hv,n}\Big) - \frac{a^n \Delta t}{\varepsilon^2}\Big(h_x^{n+1} + \frac{\Delta t}{\varepsilon}h_y^{n+1}\Big)\right]$$

$$(hv)^{n+1} = \frac{1}{K}\left[(hv)^{n} - \frac{\Delta t}{\varepsilon}(hu)^{n} + \Delta t\Big(R^{hv,n} - \frac{\Delta t}{\varepsilon}R^{hu,n}\Big) - \frac{a^n \Delta t}{\varepsilon^2}\Big(h_y^{n+1} - \frac{\Delta t}{\varepsilon}h_x^{n+1}\Big)\right]$$

where

$$K := 1 + (\Delta t/\varepsilon)^2$$

Fully Discrete AP Schemes

- We differentiate equations for $(hu)^{n+1}$ and $(hv)^{n+1}$ with respect to x and y, respectively and substitute them into equation into the first equation and obtain the following elliptic equation for h^{n+1}:

$$h^{n+1} - \frac{a^n(1-\alpha)}{\widetilde{K}}\Delta h^{n+1} = h^n + \Delta t R^{h,n} - \frac{\Delta t(1-\alpha)}{K}\Big[(hu)_x^n + (hv)_y^n$$
$$+\frac{\Delta t}{\varepsilon}\Big((hv)_x^n - (hu)_y^n\Big) + \Delta t\Big(R_x^{hu,n} + R_y^{hv,n}\Big) + \frac{(\Delta t)^2}{\varepsilon}\Big(R_x^{hv,n} - R_y^{hu,n}\Big)\Big]$$

where

$$\widetilde{K} := 1 + (\varepsilon/\Delta t)^2$$

- Solve for h^{n+1} and substitute it into the second and third equation to obtain

$$(hu)^{n+1} = \dots$$

$$(hv)^{n+1} = \dots$$

Stability of the Proposed AP Scheme

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n - \Delta t \underbrace{\left[\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x^n + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y^n\right]}_{\boldsymbol{R}(\boldsymbol{U})^n} - \Delta t \left[\widehat{\boldsymbol{F}}(\boldsymbol{U})_x^{n+1} + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y^{n+1} - \boldsymbol{S}(\boldsymbol{U})^{n+1}\right]$$

The stability of the proposed AP scheme is controlled by the CFL condition:

$$\Delta t_{\mathrm{AP}} \le \nu \cdot \min\left(\frac{\Delta x}{\max\limits_{u,h}\left\{|u| + \sqrt{(1-\alpha)u^2 + \alpha\frac{h-a(t)}{\varepsilon^2}}\right\}}, \frac{\Delta y}{\max\limits_{v,h}\left\{|v| + \sqrt{(1-\alpha)v^2 + \alpha\frac{h-a(t)}{\varepsilon^2}}\right\}}\right).$$

The denominators on the RHS are independent of ε (provided $\alpha \sim \varepsilon^s$). Therefore, the use of large time steps of size $\Delta t_{\mathrm{AP}} = \mathcal{O}(\Delta_{\min})$, is sufficient to enforce the stability of the proposed AP scheme.

Proof of Consistency

Recall that for the $\varepsilon \to 0$ limit: equations for $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ terms imply that

$$h_x^{(0)} = 0,\ \ h_y^{(0)} = 0\ (\Rightarrow h^{(0)} \equiv \mathrm{Const}), \qquad h_x^{(1)} = v^{(0)},\ \ h_y^{(1)} = -u^{(0)},$$

which can be substituted into equations of $\mathcal{O}(1)$ terms to obtain the limit equations:

$$\begin{cases} h_t^{(0)} + (h^{(0)}u^{(0)})_x + (h^{(0)}v^{(0)})_y = 0 \quad \Longrightarrow \quad u_x^{(0)} + v_y^{(0)} = 0 \\ (h^{(0)}u^{(0)})_t + \left[h^{(0)}(u^{(0)})^2\right]_x + (h^{(0)}u^{(0)}v^{(0)})_y + h^{(0)}h_x^{(2)} = fh^{(0)}v^{(1)} \\ (h^{(0)}v^{(0)})_t + \left[h^{(0)}(v^{(0)})^2\right]_y + (h^{(0)}u^{(0)}v^{(0)})_x + h^{(0)}h_y^{(2)} = -fh^{(0)}u^{(1)} \\ h_t^{(1)} - h^{(0)}\left(h_{xx}^{(1)} + h_{yy}^{(1)}\right)_t = \cdots \\ h_t^{(2)} - \left(h^{(0)}v^{(1)} + h^{(1)}v^{(0)}\right)_{xt} + \left(h^{(0)}u^{(1)} + h^{(1)}u^{(0)}\right)_{yt} = \cdots \end{cases}$$

Goal: To have a consistent approximation of the above limiting equations

Proof of Consistency

We consider the asymptotic expansions for the unknowns

$$\begin{aligned}
\bar{h}^n_{j,k} &= h^{(0),n}_{j,k} + \varepsilon h^{(1),n}_{j,k} + \varepsilon^2 h^{(2),n}_{j,k} + \dots, \\
u^n_{j,k} &= u^{(0),n}_{j,k} + \varepsilon u^{(1),n}_{j,k} + \varepsilon^2 u^{(2),n}_{j,k} + \dots, \\
v^n_{j,k} &= v^{(0),n}_{j,k} + \varepsilon v^{(1),n}_{j,k} + \varepsilon^2 v^{(2),n}_{j,k} + \dots, \\
a^n &= h^{(0),n} + \varepsilon a^{(1),n} + \varepsilon^2 a^{(2),n} + \dots,
\end{aligned}$$

and assume that the discrete analogs of the first four equations are satisfied at time level $t = t^n$:

$$h^{(0),n}_{j,k} = h^{(0),n}, \quad D_x u^{(0),n}_{j,k} + D_y v^{(0),n}_{j,k} = 0,$$

$$v^{(0),n}_{j,k} = D_x h^{(1),n}_{j,k}, \quad u^{(0),n}_{j,k} = -D_y h^{(1),n}_{j,k}, \quad \forall j,k.$$

Proof of Consistency

- From the elliptic equation for $\overline{h}_{j,k}^{\,n+1}$, we have

$$\left[I - \frac{a^n(1-\alpha)}{\widetilde{K}}\Delta\right](\overline{h}_{j,k}^{\,n+1} - h^{(0),n}) = \mathcal{O}(\varepsilon),$$

  where $a^n(1-\alpha)/\widetilde{K} = h^{(0),n} + \mathcal{O}(\varepsilon)$.

  Matrix $I - \frac{a^n(1-\alpha)}{\widetilde{K}}\Delta$ is positive definite and non-singular (with eigenvalues bounded away from zero independently of ε), therefore

$$\overline{h}_{j,k}^{\,n+1} = h^{(0),n} + \mathcal{O}(\varepsilon)$$

- We also have $\overline{(hu)}_{j,k}^{\,n+1} = \mathcal{O}(1)$ and $\overline{(hv)}_{j,k}^{\,n+1} = \mathcal{O}(1)$, which gives

$$u_{j,k}^{n+1} = u_{j,k}^{(0),n+1} + \mathcal{O}(\varepsilon) \quad \text{and} \quad v_{j,k}^{n+1} = v_{j,k}^{(0),n+1} + \mathcal{O}(\varepsilon)$$

Proof of Consistency

We plug the asymptotic expansions

$$h^n = h^{(0),n} + \varepsilon h^{(1),n} + \varepsilon^2 h^{(2),n}, \quad h^{n+1} = h^{(0),n+1} + \varepsilon h^{(1),n+1} + \varepsilon^2 h^{(2),n+1}$$

$$u^n = h^{(0),n} + \varepsilon u^{(1),n} + \varepsilon^2 u^{(2),n}, \quad u^{n+1} = u^{(0),n+1} + \varepsilon u^{(1),n+1} + \varepsilon^2 u^{(2),n+1}$$

$$v^n = v^{(0),n} + \varepsilon v^{(1),n} + \varepsilon^2 v^{(2),n}, \quad v^{n+1} = v^{(0),n+1} + \varepsilon v^{(1),n+1} + \varepsilon^2 v^{(2),n+1}$$

into the implicit-explicit scheme and equate the like powers of ε to obtain the following equations...

$$\mathcal{O}(\varepsilon^{-2}): \quad h^{(0),n+1} h_x^{(0),n+1} = 0$$

$$h^{(0),n+1} h_y^{(0),n+1} = 0$$

$$\mathcal{O}(\varepsilon^{-1}): \quad h^{(0),n+1} h_x^{(1),n+1} + h_x^{(0),n+1} h^{(1),n+1} = h^{(0),n+1} v^{(0),n+1}$$

$$h^{(0),n+1} h_y^{(1),n+1} + h_y^{(0),n+1} h^{(1),n+1} = -h^{(0),n+1} u^{(0),n+1}$$

$$\mathcal{O}(1): \qquad \ldots$$

Proof of Consistency

- The equations of $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ terms imply that

$$D_x h_{j,k}^{(0),n+1} \equiv 0, \quad D_y h_{j,k}^{(0),n+1} \equiv 0 \quad \Longrightarrow \quad h_{j,k}^{(0),n+1} \equiv h^{(0),n+1} = \text{Const}$$

- For the $\mathcal{O}(1)$ terms, we obtain

$$u_{j,k}^{(0),n+1} = -D_y h_{j,k}^{(1),n+1}, \quad v_{j,k}^{(0),n+1} = D_x h_{j,k}^{(1),n+1}, \ \ \forall j,k$$

- Taking central differences of the above equations with respect to y and x, respectively, we obtain

$$D_y v_{j,k}^{(0),n+1} + D_x u_{j,k}^{(0),n+1} = D_x D_y h_{j,k}^{(1),n+1} - D_y D_x h_{j,k}^{(1),n+1} = 0,$$

which implies that the divergence-free condition for the discrete velocity holds at all time levels.

- ...

Summary

Theorem. *The proposed hyperbolic flux splitting method coupled with the described fully discrete scheme is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number* $\varepsilon \to 0$.

Remark. In practice, the fully discrete scheme is both second-order accurate in space and time as we increase a temporal order of accuracy to the second one by implementing a two-stage globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2). The proof holds as well.

Remark. The proposed AP scheme is also asymptotically well-balanced in the sense that it preserves geostrophic equilibria in the zero Froude number limit at the discrete level: implies

$$u = -\frac{1}{\varepsilon} h_y, \quad v = \frac{1}{\varepsilon} h_x$$

Example — 2-D Stationary Vortex

[E. Audusse, R. Klein, D. D. Nguyen, and S. Vater, 2011]

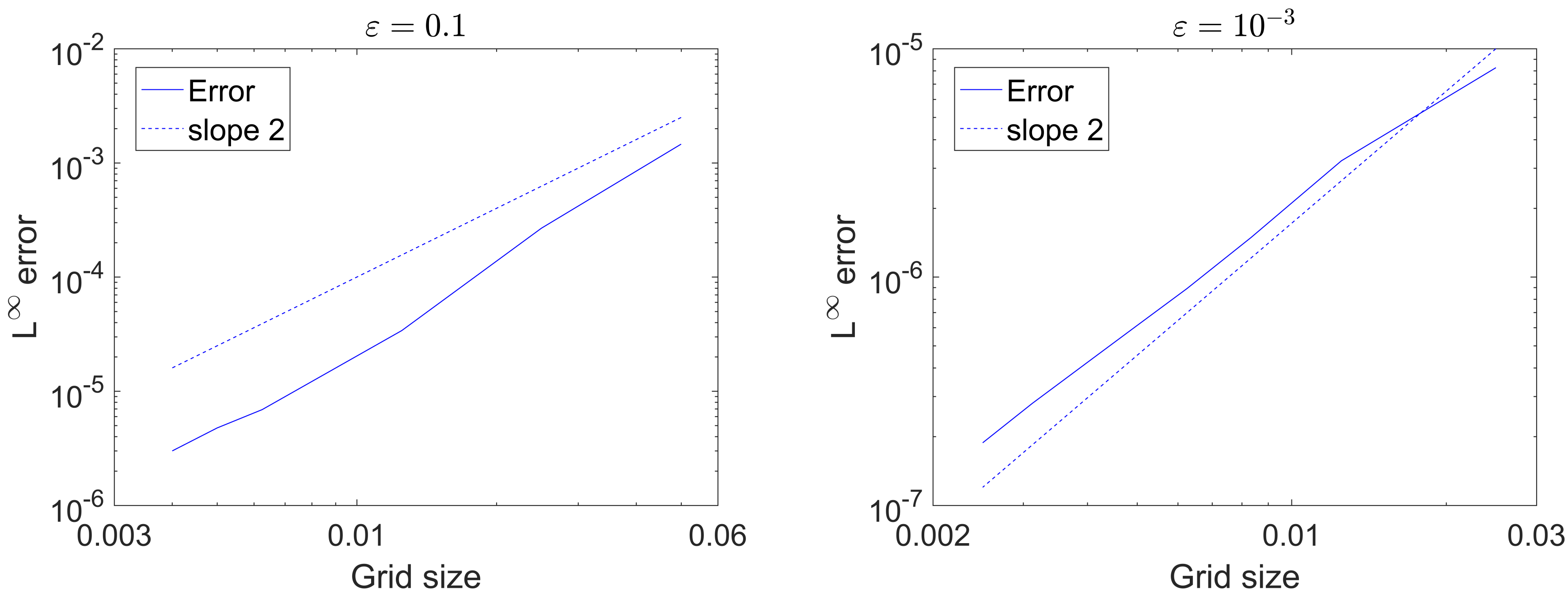
$$h(r,0) = 1+\varepsilon^2 \begin{cases} \frac{5}{2}(1+5\varepsilon^2)r^2 \\ \frac{1}{10}(1+5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2}r^2 + \varepsilon^2(4\ln(5r) + \frac{7}{2} - 20r + \frac{25}{2}r^2) \\ \frac{1}{5}(1 - 10\varepsilon + 4\varepsilon^2 \ln 2), \end{cases}$$

$$u(x,y,0) = -\varepsilon y \Upsilon(r), \quad v(x,y,0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 5, & r < \frac{1}{5} \\ \frac{2}{r} - 5, & \frac{1}{5} \le r < \frac{2}{5} \\ 0, & r \ge \frac{2}{5}, \end{cases}$$

Domain: $[-1,1] \times [-1,1], \quad r := \sqrt{x^2+y^2}$

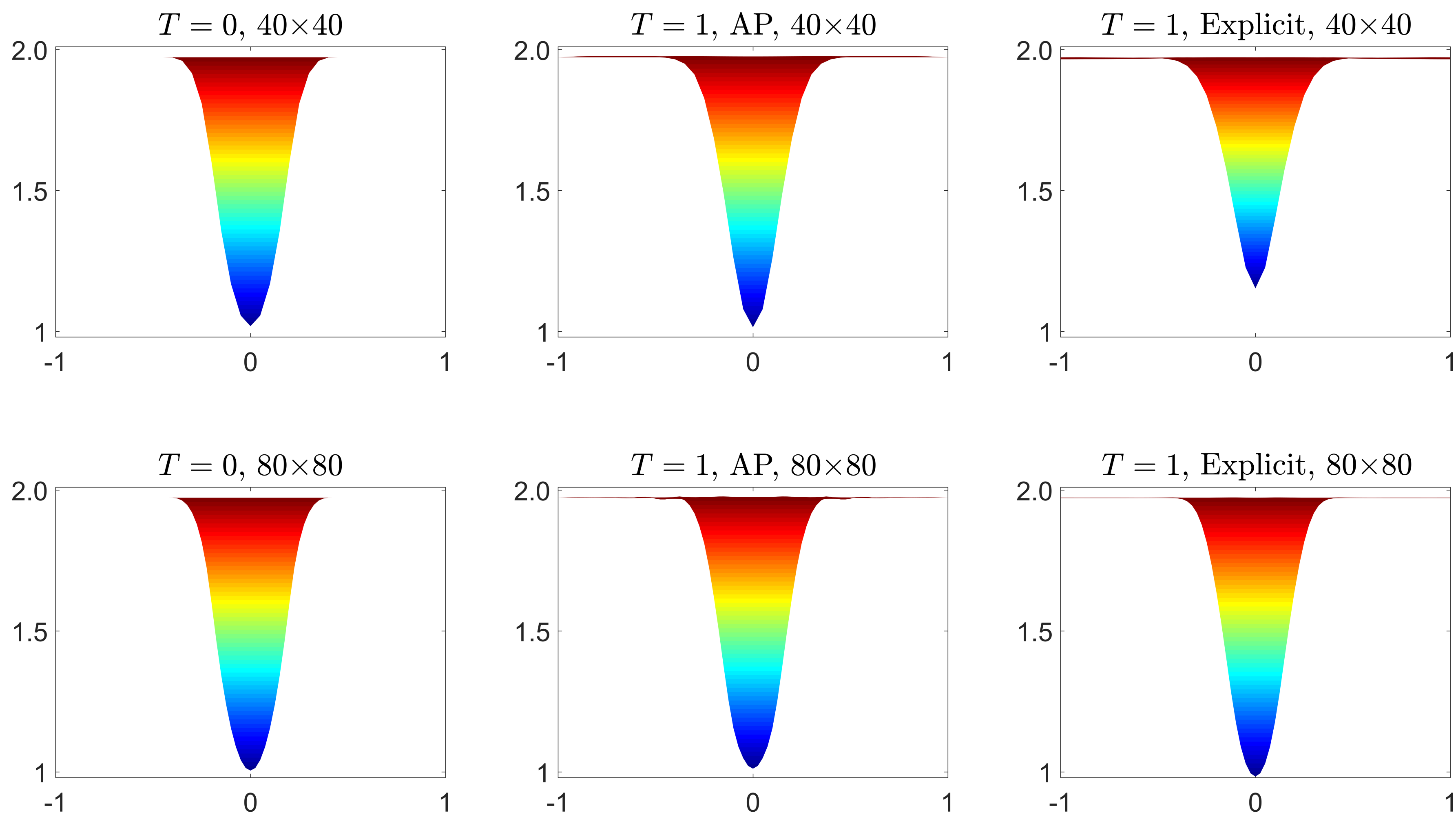
Boundary conditions: a zero-order extrapolation in both x- and y-directions

Experimental order of convergence

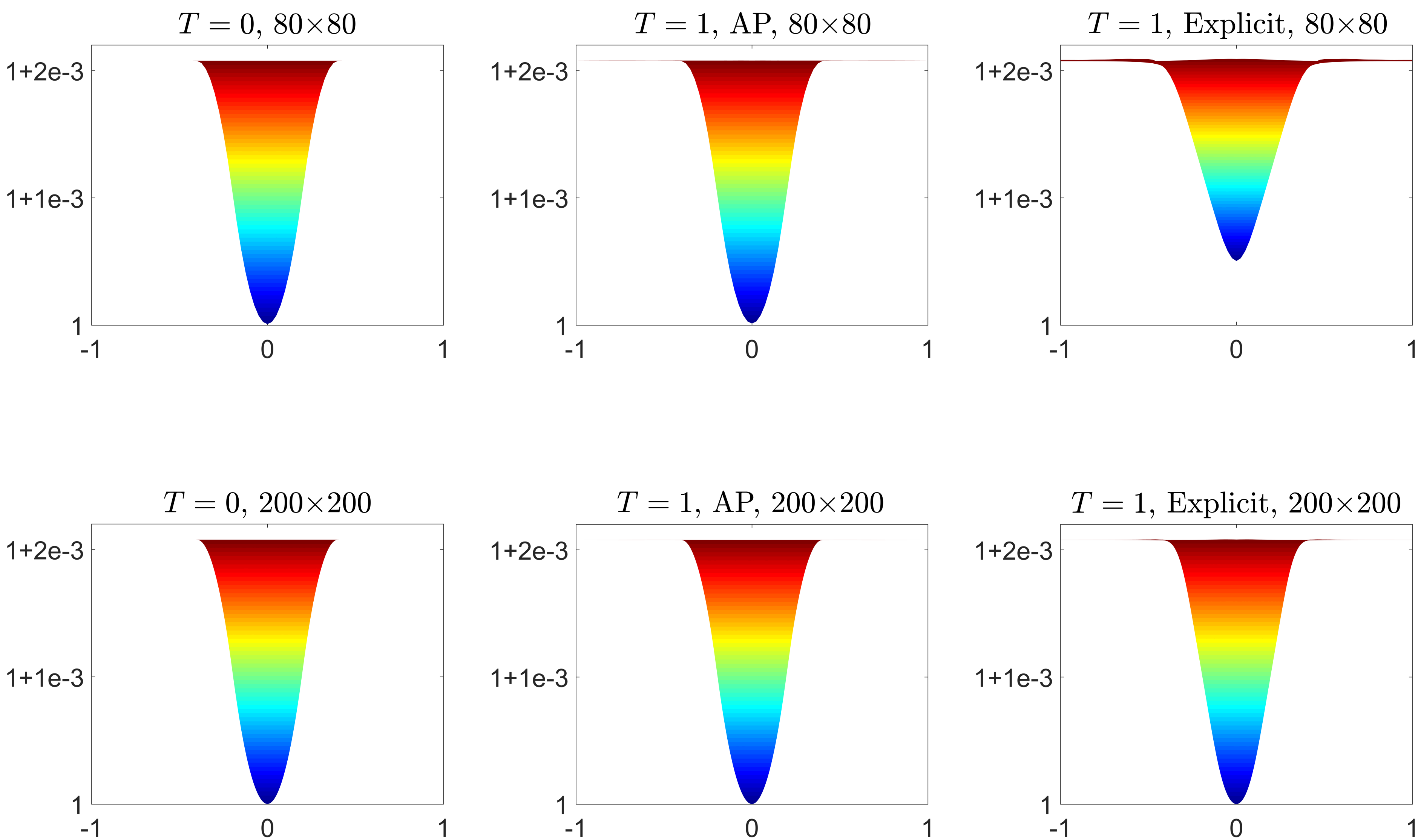


L^∞-errors for h computed using the AP scheme on several different grids for $\varepsilon = 0.1$ (left) and 10^{-3}

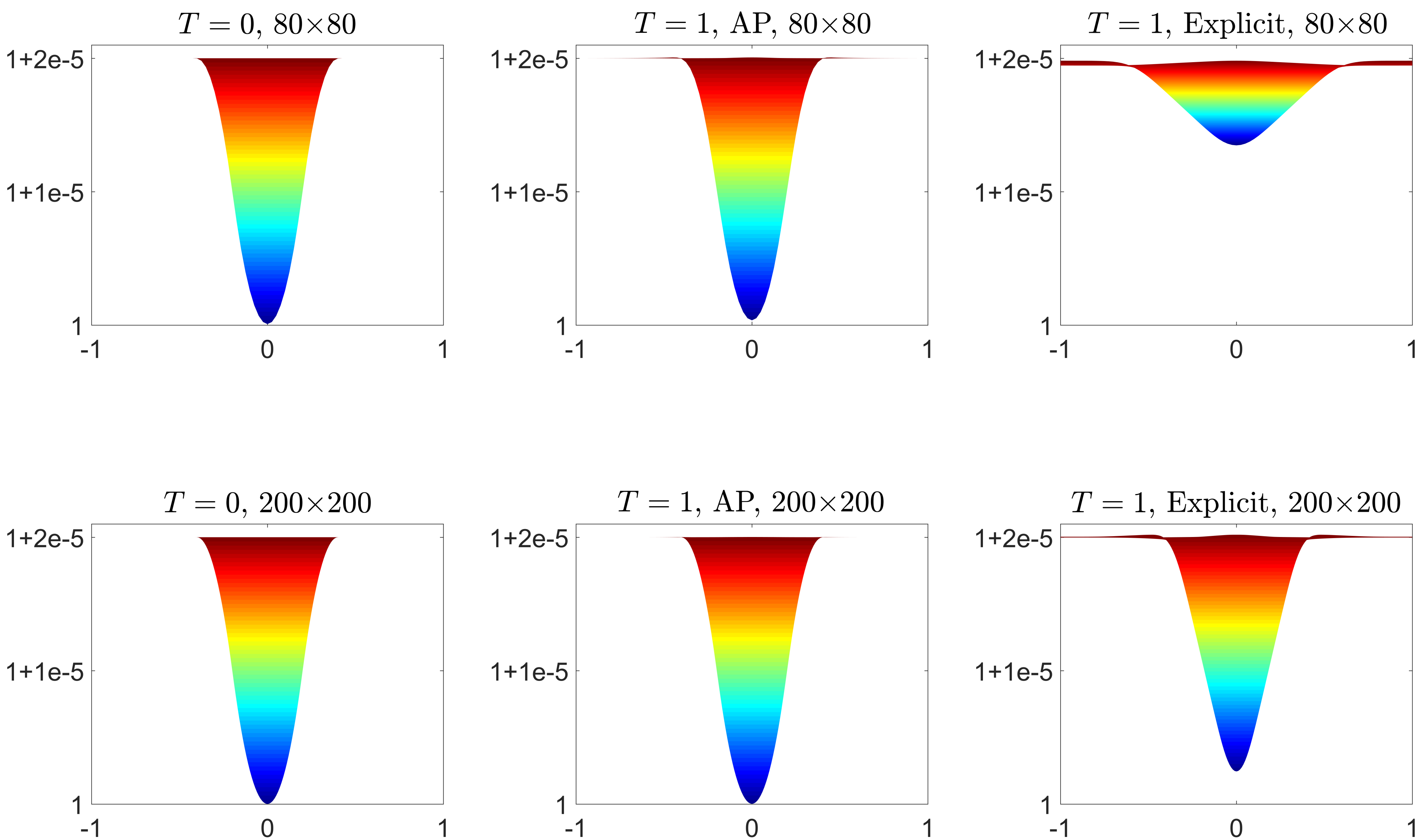
Comparison of non-AP and AP methods, $\varepsilon = 1$



Comparison of non-AP and AP methods, $\varepsilon = 0.1$



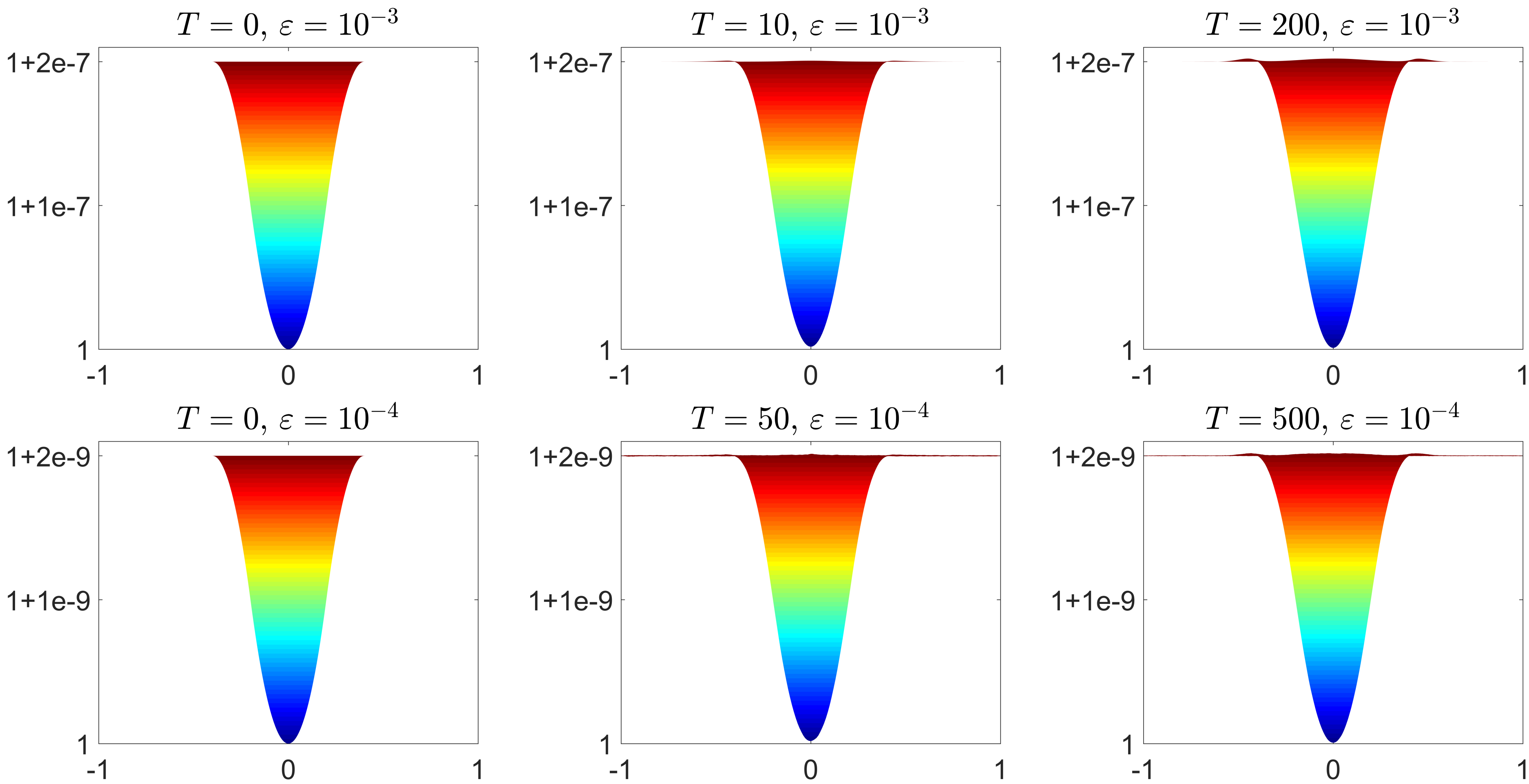
Comparison of non-AP and AP methods, $\varepsilon = 0.01$



Comparison of non-AP and AP methods, CPU times

	$\varepsilon = 1$		$\varepsilon = 0.1$		$\varepsilon = 0.01$	
Grid	AP	Explicit	AP	Explicit	AP	Explicit
40×40	0.18 s	0.16 s	0.06 s	1.25 s	0.03 s	10.53 s
80×80	1.57 s	1.32 s	0.29 s	4.73 s	0.18 s	47.0 s
200×200	24.11 s	21.36 s	5.36 s	163.36 s	3.37 s	804.15 s

Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$



Smaller times: 200×200, larger times: 500×500

Example — 2-D Traveling Vortex

We take $\varepsilon = 10^{-2}$ and simulate a traveling vortex with the same initial water depth profile as in Example 1 but the initial velocities are now modified by adding a constant velocity vector $(15, 15)^\top$:

$$u(x, y, 0) = 15 - \varepsilon y \Upsilon(r), \quad v(x, y, 0) = 15 + \varepsilon x \Upsilon(r)$$

$$\Upsilon(r) := \begin{cases} 5, & r \le \dfrac{1}{5}, \\ \dfrac{2}{r} - 5, & \dfrac{1}{5} < r \le \dfrac{2}{5}, \\ 0, & r \ge \dfrac{2}{5}, \end{cases}$$

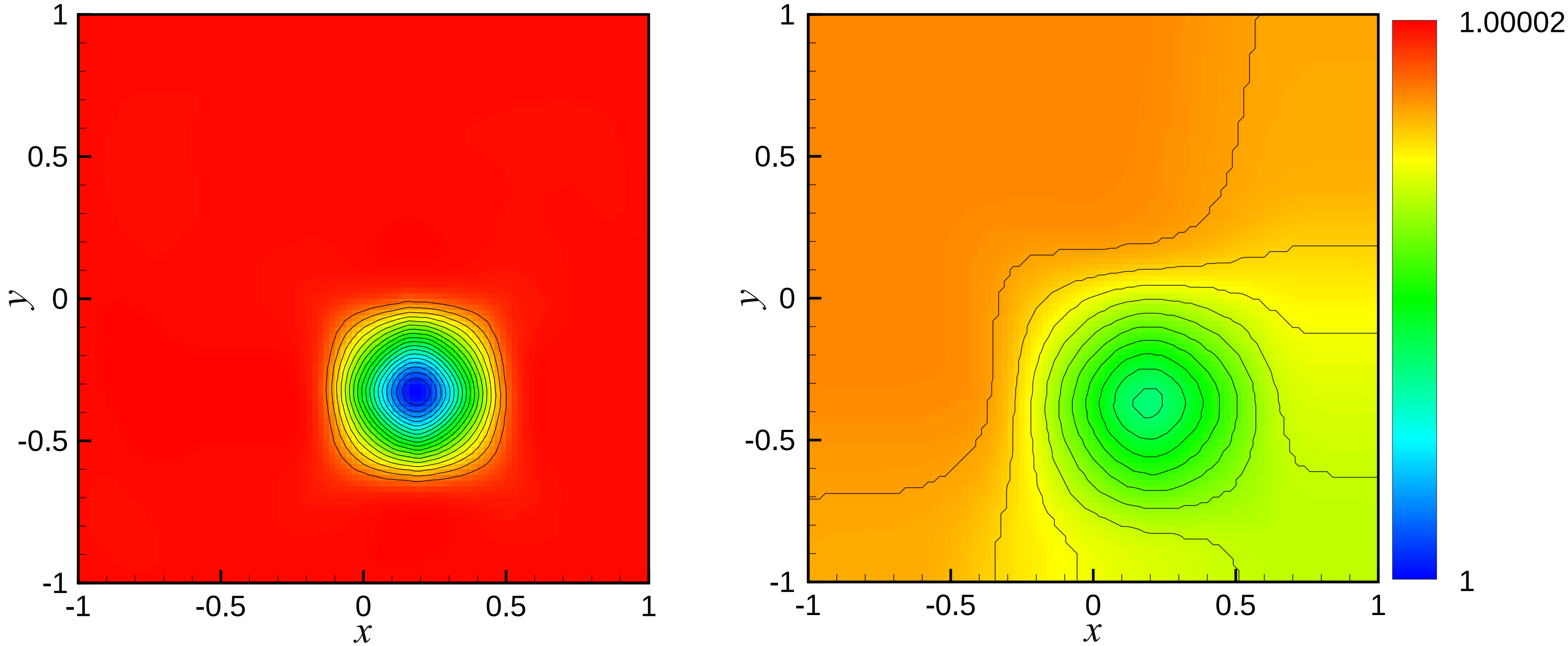
where $r := \sqrt{x^2 + y^2}$.

Domain: $[-1, 1] \times [-1, 1], \quad r := \sqrt{x^2 + y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions

These initial data correspond to a rotating vortex traveling along a circular path

Comparison of non-AP and AP methods, $\varepsilon = 0.01$



$$100 \times 100$$

THANK YOU!