
Alina Chertock

North Carolina State University
chertock@math.ncsu.edu

joint work with
S. Cui, M. Herty, A. Kurganov, X. Liu,
S.N. Özcan and E. Tadmor
Systems of Balance Laws

\[ U_t + f(U)_x + g(U)_y = S(U) \]

Examples:
- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:
- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

\[ U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

Examples:
- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:
- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows
Systems of Balance Laws

\[ U_t + f(U)_x + g(U)_y = S(U) \quad \text{or} \quad U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

- **Challenges:** certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, asymptotic regimes, etc.) are essential in many applications;

- **Goal:** to design numerical methods that are not only consistent with the given PDEs, but

  - preserve the structural properties at the discrete level – **well-balanced numerical methods**

  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[P. LeFloch; 2014]
Asymptotic Preserving (AP) Methods

\[ U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U) \]

- Solutions of many hyperbolic systems reveal a multiscale character and thus their numerical resolution presents some major difficulties;
- Such problems are typically characterized by the occurrence of a small parameter by \( 0 < \varepsilon \ll 1 \);
- The solutions show a nonuniform behavior as \( \varepsilon \to 0 \);
- the type of the limiting solution is different in nature from that of the solutions for finite values of \( \varepsilon > 0 \).

Goal:
- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limiting model as \( \varepsilon \to 0 \).
Shallow Water System with Coriolis Force

\[
\begin{align*}
    h_t + (hu)_x + (hv)_y &= 0 \\
    (hu)_t + \left( hu^2 + \frac{g}{2}h^2 \right)_x + (huv)_y &= -ghB_x + fhv \\
    (hv)_t + (huv)_x + \left( hv^2 + \frac{g}{2}h^2 \right)_x &= -ghB_y - fhu
\end{align*}
\]

- \( h \): water height
- \( u, v \): fluid velocity
- \( g \): gravitational constant
- \( B \equiv 0 \) – bottom topography
- \( f = 1/\varepsilon \) – Coriolis parameter
**Dimensional Analysis**

Introduce

\[ \hat{x} := \frac{x}{\ell_0}, \quad \hat{y} := \frac{y}{\ell_0}, \quad \hat{h} := \frac{h}{h_0}, \quad \hat{u} := \frac{u}{w_0}, \quad \hat{v} := \frac{v}{w_0}. \]

Substituting them into the SWE and dropping the hats in the notations, we obtain the dimensionless form:

\[
\begin{align*}
    h_t + (hu)_x + (hv)_y &= 0, \\
    (hu)_t + \left( hu^2 + \frac{1}{\varepsilon^2} \frac{h^2}{2} \right)_x + (huv)_y &= \frac{1}{\varepsilon} hv, \\
    (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{\varepsilon^2} \frac{h^2}{2} \right)_y &= -\frac{1}{\varepsilon} hu,
\end{align*}
\]

in which

\[ Fr := \frac{w_0}{\sqrt{gh_0}} = \varepsilon \]

is the reference Froude number.
Explicit Discretization

Eigenvalues of the flux Jacobian:

\[
\left\{ u \pm \frac{1}{\varepsilon} \sqrt{h}, u \right\} \quad \text{and} \quad \left\{ v \pm \frac{1}{\varepsilon} \sqrt{h}, v \right\}
\]

This leads to the CFL condition

\[
\Delta t_{\text{expl}} \leq \nu \cdot \min \left( \frac{\Delta x}{\max_{u,h} \left\{ |u| + \frac{1}{\varepsilon} \sqrt{h} \right\}}, \frac{\Delta y}{\max_{v,h} \left\{ |v| + \frac{1}{\varepsilon} \sqrt{h} \right\}} \right) = \mathcal{O}(\varepsilon \Delta_{\text{min}}).
\]

where \( \Delta_{\text{min}} := \min(\Delta x, \Delta y) \)

- \( 0 < \nu \leq 1 \) is the CFL number
- Numerical diffusion: \( \mathcal{O}(\lambda_{\text{max}} \Delta x) = \mathcal{O}(\varepsilon^{-1} \Delta x) \).
- We must choose \( \Delta x \approx \varepsilon \) to control numerical diffusion and the stability condition becomes

\[
\Delta t = \mathcal{O}(\varepsilon^2)
\]
Low Froude Number Flows

Low Froude number regime \((0 < \varepsilon \ll 1) \implies \text{very large propagation speeds}\)

Explicit methods:

- very restrictive time and space discretization steps, typically proportional to \(\varepsilon\) due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:

- uniformly stable for \(0 < \varepsilon < 1\);
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of \(\varepsilon\)
Asymptotic-Preserving (AP) Methods

Introduced in [Klar; 1998, Jin; 1999], see also [Jin, Levermore; 1991], [Golse, Jin, Levermore; 1999].

Idea:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limiting model as $\varepsilon \to 0$. 

![Diagram of AP-schemes]

Figure 7: Properties of AP-schemes
Analysis for the Low Froude Number Limit

We plug the formal asymptotic expansions

\[ h = h^{(0)} + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \cdots \]
\[ u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots \]
\[ v = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots \]

into the SW system:

\[
\begin{cases}
  h_t + (hu)_x + (hv)_y = 0 \\
  (hu)_t + \left( hu^2 + \frac{1}{\varepsilon^2} \frac{h^2}{2} \right)_x + (huv)_y = \frac{1}{\varepsilon} fhv \\
  (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{\varepsilon^2} \frac{h^2}{2} \right)_y = -\frac{1}{\varepsilon} fhu
\end{cases}
\]

and then collect the like powers of \( \varepsilon \) ...

Analysis for the Low Froude Number Limit

\( \mathcal{O}(\varepsilon^{-2}) : \quad h^{(0)}h_x^{(0)} = 0 \)
\( h^{(0)}h_y^{(0)} = 0 \)

\( \mathcal{O}(\varepsilon^{-1}) : \quad h^{(0)}h_x^{(1)} + h^{(1)}h_x^{(0)} = h^{(0)}v^{(0)} \)
\( h^{(0)}h_y^{(1)} + h^{(1)}h_y^{(0)} = -h^{(0)}u^{(0)} \)

\( \mathcal{O}(1) : \quad h_t^{(0)} + (h^{(0)}u^{(0)})_x + (h^{(0)}v^{(0)})_y = 0 \)
\( (h^{(0)}u^{(0)})_t + \left[ h^{(0)}(u^{(0)})^2 \right]_x + (h^{(0)}u^{(0)}v^{(0)})_y + h^{(0)}h_x^{(2)} + h^{(1)}h_x^{(1)} \)
\( (h^{(0)}v^{(0)})_t + \left[ h^{(0)}(v^{(0)})^2 \right]_y + (h^{(0)}u^{(0)}v^{(0)})_x + h^{(0)}h_y^{(2)} + h^{(1)}h_y^{(1)} \)
\( + h^{(2)}h_y^{(0)} = -h^{(0)}u^{(1)} - h^{(1)}u^{(0)} \)
\( \ldots \)
Analysis for the Low Froude Number Limit

The equations for $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ terms imply that

$$ h^{(0)}_x = 0, \quad h^{(0)}_y = 0 \quad (\Rightarrow h^{(0)} \equiv \text{Const}), \quad h^{(1)}_x = v^{(0)}, \quad h^{(1)}_y = -u^{(0)}, $$

which can be substituted into equations of $O(1)$ terms to obtain the limit equations:

$$
\begin{align*}
\begin{cases}
 h^{(0)}_t + (h^{(0)}u^{(0)})_x + (h^{(0)}v^{(0)})_y = 0 & \Rightarrow u^{(0)}_x + v^{(0)}_y = 0 \\
 (h^{(0)}u^{(0)})_t + \left[h^{(0)}(u^{(0)})^2\right]_x + (h^{(0)}u^{(0)}v^{(0)})_y + h^{(0)}h^{(2)}_x &= h^{(0)}v^{(1)} \\
 (h^{(0)}v^{(0)})_t + \left[h^{(0)}(v^{(0)})^2\right]_y + (h^{(0)}u^{(0)}v^{(0)})_x + h^{(0)}h^{(2)}_y &= -h^{(0)}u^{(1)} \\
 h^{(1)}_t - h^{(0)} \left(h^{(1)}_{xx} + h^{(1)}_{yy}\right) &= \cdots \\
 h^{(2)}_t - \left(h^{(0)}v^{(1)} + h^{(1)}v^{(0)}\right)_x + \left(h^{(0)}u^{(1)} + h^{(1)}u^{(0)}\right)_y &= \cdots 
\end{cases}
\end{align*}
$$

Goal: To develop an AP numerical methods for the SW system, which yield a consistent approximation of the above limiting equations as $\varepsilon \to 0$
Hyperbolic Splitting

Key idea: Split the stiff pressure term [Haack, Jin, Liu; 2012]

- We first split the stiff pressure gradient term into two parts, i.e.
  \[
  \frac{1}{\varepsilon^2} \frac{h^2}{2} = \frac{1}{\varepsilon^2} \frac{h^2}{2} - \frac{a(t)h}{\varepsilon^2} + \frac{a(t)h}{\varepsilon^2}
  \]
  non-stiff
  stiff

- We then split the flux terms in the continuity equation by introducing a weight parameter \( \alpha \) so that we can construct the slow dynamic system as a hyperbolic system:
  \[
  hu = \alpha hu + (1 - \alpha)hu, \quad hv = \alpha hv + (1 - \alpha)hv
  \]
Hyperbolic Flux Splitting

Key idea: Split the stiff pressure term [Haack, Jin, Liu; 2012]

\[
\begin{align*}
ht + \alpha(hu)_x + \alpha(hv)_y + (1 - \alpha)(hu)_x + (1 - \alpha)(hv)_y &= 0, \\
(hu)_t + \left(hu^2 + \frac{1}{2}h^2 - a(t)h\right)_x + (huv)_y + \frac{a(t)}{\varepsilon^2}hx &= \frac{1}{\varepsilon}hv, \\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}h^2 - a(t)h\right)_y + \frac{a(t)}{\varepsilon^2}hy &= -\frac{1}{\varepsilon}hu.
\end{align*}
\]

This system can be written in the following vector form:

\[
U_t + \tilde{F}(U)_x + \tilde{G}(U)_y + \hat{F}(U)_x + \hat{G}(U)_y = S(U)
\]

non-stiff terms

stiff terms

source terms

How to choose parameters \(\alpha\) and \(a(t)\)?
Hyperbolic Flux Splitting

\[ U_t + \tilde{F}(U)_x + \tilde{G}(U)_y + \hat{F}(U)_x + \hat{G}(U)_y = S(U) \]

- **non-stiff terms**
- **stiff terms**
- **nonlinear part**
- **linear part**
- **source terms**

Need to ensure: \( U_t + \tilde{F}(U)_x + \tilde{G}(U)_y = 0 \) is both nonstiff and hyperbolic

Eigenvalues of the Jacobians \( \partial \tilde{F} / \partial U \) and \( \partial \tilde{G} / \partial U \):

\[
\left\{ u \pm \sqrt{(1 - \alpha)u^2 + \alpha \frac{h - a(t)}{\varepsilon^2}}, u \right\}, \quad \left\{ v \pm \sqrt{(1 - \alpha)v^2 + \alpha \frac{h - a(t)}{\varepsilon^2}}, v \right\}
\]

We then take: \( \alpha = \varepsilon^s \) and \( a(t) = \min_{(x, y) \in \Omega} h(x, y, t), \quad s \geq 1 \)

**Remark.** It is safe to take \( \alpha = \varepsilon^2 \) there as in this case the stability time-step restriction will be clearly independent of \( \varepsilon \). The results obtained with \( \alpha = \varepsilon \) are almost identical and no instabilities have been observed as expected.
Time Discretization of the Split System

\[ U^{n+1} = U^n - \Delta t \tilde{F}(U)^n_x - \Delta t \tilde{G}(U)^n_y \]

nonlinear part, explicit

\[ - \Delta t \hat{F}(U)^{n+1}_x - \Delta t \hat{G}(U)^{n+1}_y + \Delta t S(U)^{n+1} \]

linear part, implicit

- Nonstiff nonlinear part is treated using the second-order central-upwind scheme
- Stiff linear part reduces to a linear elliptic equation for \( h^{n+1} \) and straightforward computations of \((hu)^{n+1}\) and \((hv)^{n+1}\)

For simplicity of presentation: First-order accurate in time

In practice: We implement a two-stage second-order globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2) (all the proofs will apply)
**Fully Discrete AP Schemes**

\[ U^{n+1} = U^n - \Delta t \left[ \frac{\hat{F}(U)^n_x + \hat{G}(U)^n_y}{R(U)^n} \right] - \Delta t \left[ \frac{\hat{F}(U)^{n+1}_x + \hat{G}(U)^{n+1}_y - S(U)^{n+1}}{R(U)^n} \right] \]

- We use the notation \( R^n := (R^{h,n}, R^{hu,n}, R^{hv,n})^\top \) and rewrite the system

\[
\begin{align*}
  h^{n+1} &= h^n + \Delta t R^{h,n} - \Delta t (1 - \alpha) \left[ (hu)^{n+1}_x + (hv)^{n+1}_y \right] \\
  (hu)^{n+1} &= \frac{1}{K} \left[ (hu)^n + \frac{\Delta t}{\varepsilon} (hv)^n + \Delta t \left( R^{hu,n} + \frac{\Delta t}{\varepsilon} R^{hv,n} \right) - \frac{a^n \Delta t}{\varepsilon^2} \left( h^{n+1}_x + \frac{\Delta t}{\varepsilon} h^{n+1}_y \right) \right] \\
  (hv)^{n+1} &= \frac{1}{K} \left[ (hv)^n - \frac{\Delta t}{\varepsilon} (hu)^n + \Delta t \left( R^{hv,n} - \frac{\Delta t}{\varepsilon} R^{hu,n} \right) - \frac{a^n \Delta t}{\varepsilon^2} \left( h^{n+1}_y - \frac{\Delta t}{\varepsilon} h^{n+1}_x \right) \right]
\end{align*}
\]

where

\[ K := 1 + (\Delta t/\varepsilon)^2 \]
We differentiate equations for \((hu)^{n+1}\) and \((hv)^{n+1}\) with respect to \(x\) and \(y\), respectively, and substitute them into equation into the first equation and obtain the following elliptic equation for \(h^{n+1}\):

\[
h^{n+1} - \frac{a^n(1 - \alpha)}{\tilde{K}} \Delta h^{n+1} = h^n + \Delta t R^{h,n} - \frac{\Delta t(1 - \alpha)}{K} \left[ (hu)_x^n + (hv)_y^n \right.
\]
\[
+ \frac{\Delta t}{\varepsilon} \left( (hv)_x^n - (hu)_y^n \right) + \Delta t \left( R_x^{hu,n} + R_y^{hv,n} \right) + \frac{(\Delta t)^2}{\varepsilon} \left( R_x^{hv,n} - R_y^{hu,n} \right) \right]
\]

where

\[
\tilde{K} := 1 + (\varepsilon/\Delta t)^2
\]

Solve for \(h^{n+1}\) and substitute it into the second and third equation to obtain

\[
(hu)^{n+1} = \ldots
\]
\[
(hv)^{n+1} = \ldots
\]
Stability of the Proposed AP Scheme

\[ U^{n+1} = U^n - \Delta t \left[ \tilde{F}(U)_x^n + \tilde{G}(U)_y^n \right] - \Delta t \left[ \hat{F}(U)_{x}^{n+1} + \hat{G}(U)_{y}^{n+1} - S(U)_n^{n+1} \right] \]

\[ \text{R(U)}^n \]

The stability of the proposed AP scheme is controlled by the CFL condition:

\[ \Delta t_{AP} \leq \nu \cdot \min \left( \frac{\Delta x}{\max_{u,h} \left\{ |u| + \sqrt{(1 - \alpha)u^2 + \alpha \frac{h-a(t)}{\varepsilon^2}} \right\}}, \frac{\Delta y}{\max_{v,h} \left\{ |v| + \sqrt{(1 - \alpha)v^2 + \alpha \frac{h-a(t)}{\varepsilon^2}} \right\}} \right). \]

The denominators on the RHS are independent of \( \varepsilon \) (provided \( \alpha \sim \varepsilon^s \)). Therefore, the use of large time steps of size \( \Delta t_{AP} = \mathcal{O}(\Delta_{\text{min}}) \), is sufficient to enforce the stability of the proposed AP scheme.
Proof of Consistency

Recall that for the $\varepsilon \to 0$ limit: equations for $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ terms imply that

$$h_x^{(0)} = 0, \quad h_y^{(0)} = 0 \implies h^{(0)} \equiv \text{Const}, \quad h_x^{(1)} = v^{(0)}, \quad h_y^{(1)} = -u^{(0)},$$

which can be substituted into equations of $O(1)$ terms to obtain the limit equations:

$$\begin{align*}
\left\{ \begin{array}{l}
  h_t^{(0)} + (h^{(0)}u^{(0)})_x + (h^{(0)}v^{(0)})_y = 0 \quad \implies \quad u_x^{(0)} + v_y^{(0)} = 0 \\
  (h^{(0)}u^{(0)})_t + \left[ h^{(0)}(u^{(0)})^2 \right]_x + (h^{(0)}u^{(0)}v^{(0)})_y + h^{(0)}h_x^{(2)} = fh^{(0)}v^{(1)} \\
  (h^{(0)}v^{(0)})_t + \left[ h^{(0)}(v^{(0)})^2 \right]_y + (h^{(0)}u^{(0)}v^{(0)})_x + h^{(0)}h_y^{(2)} = -fh^{(0)}u^{(1)} \\
  h_t^{(1)} - h^{(0)} \left( h_{xx}^{(1)} + h_{yy}^{(1)} \right)_t = \cdots \\
  h_t^{(2)} - \left( h^{(0)}v^{(1)} + h^{(1)}v^{(0)} \right)_{xt} + \left( h^{(0)}u^{(1)} + h^{(1)}u^{(0)} \right)_{yt} = \cdots 
\end{array} \right.
\end{align*}$$

**Goal:** To have a consistent approximation of the above limiting equations.
Proof of Consistency

We consider the asymptotic expansions for the unknowns

\[
\begin{align*}
\overline{h}_{j,k}^n &= h_{j,k}^{(0),n} + \varepsilon h_{j,k}^{(1),n} + \varepsilon^2 h_{j,k}^{(2),n} + \ldots, \\
u_{j,k}^n &= u_{j,k}^{(0),n} + \varepsilon u_{j,k}^{(1),n} + \varepsilon^2 u_{j,k}^{(2),n} + \ldots, \\
v_{j,k}^n &= v_{j,k}^{(0),n} + \varepsilon v_{j,k}^{(1),n} + \varepsilon^2 v_{j,k}^{(2),n} + \ldots, \\
a^n &= h^{(0),n} + \varepsilon a^{(1),n} + \varepsilon^2 a^{(2),n} + \ldots,
\end{align*}
\]

and assume that the discrete analogs of the first four equations are satisfied at time level \( t = t^n \):

\[
\begin{align*}
h_{j,k}^{(0),n} &= h^{(0),n}, \\
D_x u_{j,k}^{(0),n} &+ D_y v_{j,k}^{(0),n} = 0, \\
v_{j,k}^{(0),n} &= D_x h_{j,k}^{(1),n}, \\
u_{j,k}^{(0),n} &= -D_y h_{j,k}^{(1),n}, \quad \forall j, k.
\end{align*}
\]
Proof of Consistency

• From the elliptic equation for \( h_{j,k}^{n+1} \), we have

\[
\left[ I - \frac{a_n(1 - \alpha)}{K} \Delta \right] (h_{j,k}^{n+1} - h^{(0),n}) = O(\varepsilon),
\]

where \( a_n(1 - \alpha)/K = h^{(0),n} + O(\varepsilon) \).

Matrix \( I - \frac{a_n(1 - \alpha)}{K} \Delta \) is positive definite and non-singular (with eigenvalues bounded away from zero independently of \( \varepsilon \)), therefore

\[
\bar{h}_{j,k}^{n+1} = h^{(0),n} + O(\varepsilon)
\]

• We also have \( (hu)_{j,k}^{n+1} \) = \( O(1) \) and \( (hv)_{j,k}^{n+1} \) = \( O(1) \), which gives

\[
\bar{u}_{j,k}^{n+1} = u^{(0),n+1} + O(\varepsilon) \quad \text{and} \quad \bar{v}_{j,k}^{n+1} = v^{(0),n+1} + O(\varepsilon)
\]
Proof of Consistency

We plug the asymptotic expansions

\[ h^n = h^{(0),n} + \varepsilon h^{(1),n} + \varepsilon^2 h^{(2),n}, \quad h^{n+1} = h^{(0),n+1} + \varepsilon h^{(1),n+1} + \varepsilon^2 h^{(2),n+1} \]
\[ u^n = u^{(0),n} + \varepsilon u^{(1),n} + \varepsilon^2 u^{(2),n}, \quad u^{n+1} = u^{(0),n+1} + \varepsilon u^{(1),n+1} + \varepsilon^2 u^{(2),n+1} \]
\[ v^n = v^{(0),n} + \varepsilon v^{(1),n} + \varepsilon^2 v^{(2),n}, \quad v^{n+1} = v^{(0),n+1} + \varepsilon v^{(1),n+1} + \varepsilon^2 v^{(2),n+1} \]

into the implicit-explicit scheme and equate the like powers of \( \varepsilon \) to obtain the following equations...

\[ O(\varepsilon^{-2}) : \quad h^{(0),n+1} h^{(0),n+1}_x = 0 \]
\[ h^{(0),n+1} h^{(0),n+1}_y = 0 \]

\[ O(\varepsilon^{-1}) : \quad h^{(0),n+1} h^{(1),n+1}_x + h^{(0),n+1} h^{(1),n+1}_y = h^{(0),n+1} v^{(0),n+1} \]
\[ h^{(0),n+1} h^{(1),n+1}_y + h^{(0),n+1} h^{(1),n+1}_x = -h^{(0),n+1} u^{(0),n+1} \]

\[ O(1) : \quad \ldots \]
Proof of Consistency

- The equations of $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ terms imply that

$$D_x h_{j,k}^{(0),n+1} \equiv 0, \quad D_y h_{j,k}^{(0),n+1} \equiv 0 \implies h_{j,k}^{(0),n+1} \equiv h^{(0),n+1} = \text{Const}$$

- For the $O(1)$ terms, we obtain

$$u_{j,k}^{(0),n+1} = -D_y h_{j,k}^{(1),n+1}, \quad v_{j,k}^{(0),n+1} = D_x h_{j,k}^{(1),n+1}, \quad \forall j, k$$

- Taking central differences of the above equations with respect to $y$ and $x$, respectively, we obtain

$$D_y v_{j,k}^{(0),n+1} + D_x u_{j,k}^{(0),n+1} = D_x D_y h_{j,k}^{(1),n+1} - D_y D_x h_{j,k}^{(1),n+1} = 0,$$

which implies that the divergence-free condition for the discrete velocity holds at all time levels.
Summary

**Theorem.** The proposed hyperbolic flux splitting method coupled with the described fully discrete scheme is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number \( \varepsilon \to 0 \).

**Remark.** In practice, the fully discrete scheme is both second-order accurate in space and time as we increase a temporal order of accuracy to the second one by implementing a two-stage globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2). The proof holds as well.

**Remark.** The proposed AP scheme is also asymptotically well-balanced in the sense that it preserves geostrophic equilibria in the zero Froude number limit at the discrete level: implies

\[
  u = -\frac{1}{\varepsilon} h_y, \quad v = \frac{1}{\varepsilon} h_x
\]
Example — 2-D Stationary Vortex


\[ h(r, 0) = 1 + \varepsilon^2 \begin{cases} 
\frac{5}{2} (1 + 5\varepsilon^2) r^2 \\
\frac{1}{10} (1 + 5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2} r^2 + \varepsilon^2 (4 \ln(5r) + \frac{7}{2} - 20r + \frac{25}{2} r^2) \\
\frac{1}{5} (1 - 10\varepsilon + 4\varepsilon^2 \ln 2), 
\end{cases} \]

\[ u(x, y, 0) = -\varepsilon y \Upsilon(r), \quad v(x, y, 0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 
5, & r < \frac{1}{5} \\
\frac{2}{r} - 5, & \frac{1}{5} \leq r < \frac{2}{5} \\
0, & r \geq \frac{2}{5}, 
\end{cases} \]

Domain: \([-1, 1] \times [-1, 1]\], \quad r := \sqrt{x^2 + y^2}

Boundary conditions: a zero-order extrapolation in both \(x\)- and \(y\)-directions
$L^\infty$-errors for $h$ computed using the AP scheme on several different grids for $\varepsilon = 0.1$ (left) and $10^{-3}$
Comparison of non-AP and AP methods, $\varepsilon = 1$
Comparison of non-AP and AP methods, $\varepsilon = 0.1$
Comparison of non-AP and AP methods, $\varepsilon = 0.01$
Comparison of non-AP and AP methods, CPU times

<table>
<thead>
<tr>
<th>Grid</th>
<th>$\varepsilon = 1$</th>
<th></th>
<th>$\varepsilon = 0.1$</th>
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<th>$\varepsilon = 0.01$</th>
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<td>Explicit</td>
<td>AP</td>
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<td>0.16 s</td>
<td>0.06 s</td>
<td>1.25 s</td>
<td>0.03 s</td>
<td>10.53 s</td>
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<td>80 × 80</td>
<td>1.57 s</td>
<td>1.32 s</td>
<td>0.29 s</td>
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</tr>
<tr>
<td>200 × 200</td>
<td>24.11 s</td>
<td>21.36 s</td>
<td>5.36 s</td>
<td>163.36 s</td>
<td>3.37 s</td>
<td>804.15 s</td>
</tr>
</tbody>
</table>
Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$

Smaller times: 200 × 200, larger times: 500 × 500
Example — 2-D Traveling Vortex

We take $\varepsilon = 10^{-2}$ and simulate a traveling vortex with the same initial water depth profile as in Example 1 but the initial velocities are now modified by adding a constant velocity vector $(15, 15)^\top$:

$$u(x, y, 0) = 15 - \varepsilon y \Upsilon(r), \quad v(x, y, 0) = 15 + \varepsilon x \Upsilon(r)$$

$$\Upsilon(r) := \begin{cases} 
5, & r \leq \frac{1}{5}, \\
2 - r, & \frac{1}{5} < r \leq \frac{2}{5}, \\
0, & r \geq \frac{2}{5},
\end{cases}$$

where $r := \sqrt{x^2 + y^2}$.

Domain: $[-1, 1] \times [-1, 1]$, $r := \sqrt{x^2 + y^2}$

Boundary conditions: a zero-order extrapolation in both $x$- and $y$-directions

These initial data correspond to a rotating vortex traveling along a circular path.
Comparison of non-AP and AP methods, $\varepsilon = 0.01$

$100 \times 100$
THANK YOU!