# Existence and Uniqueness in Viscoelasticity of strain-rate type

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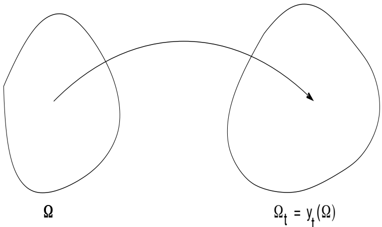
$$\frac{\partial^2 y}{\partial t^2} = \text{div } T$$

Viscoelasticity models feature the dependence on memory effects

- $T = \hat{T}(F(t,x), \dot{F}(t,x))$ viscoelasticity of strain-rate type
- $\dot{T} = \tilde{G}(T(t,x), F(t,x))$ viscoelasticity of stress-rate type, Maxwell models
- kinetic models

| | |
|---|---|
| motion | $y(x,t)$ |
| velocity | $v = \frac{\partial y}{\partial t}$ |
| deformation gradient | $F = \nabla y$ |
| stress | $T$ |



## viscoelasticity of strain-rate type

$$\frac{\partial^2 y}{\partial t^2} = \text{div}\, T(\nabla y, \nabla y_t)$$

Kelvin-Voigt model

$$T(F, \dot{F}) = \frac{\partial W}{\partial F}(F) + \dot{F}$$

Key features : $W(F)$ elastic stored energy nonconvex

- model consistent w. entropy dissipation
- not frame indifferent
  invariant under the extended Galilean group

$$x^\star = Qx + d(t) \qquad \text{where } Q \text{ is a time-independent rotation}$$

Defficiency : no account of the constraint $\det F > 0$

$$u_t = v_x$$
$$v_t = \tau(u, v_x) \qquad \text{e.g.} \qquad v_t = \sigma(u)_x + \varepsilon v_{xx}$$

Literature

A lot of results in dimension $d = 1$, Greenberg-McCamy-Mizel 68, Dafermos 69, ...

several connected to phase transitions $\sigma(u)$ non-monotone

Slemrod, Hoff, Pego, Andrews-Ball, Trivisa ...

Strain-rate viscoelasticity: Friedman-Nečas 88, Friesecke-Dolzmann 97, Demoulini 00

nonlinear strain-rate dependence and implicit stress theories : Tvedt 08, Y. Sengül 10, Malek-Bulíček-Rajagopal, 12 ...,

$$T(F, \dot{F}) = \frac{\partial W}{\partial F}(F) + \dot{F} \qquad W(F) \text{ semiconvex}$$

## viscoelasticity as a hyperbolic-parabolic system

$$
\begin{aligned}
\partial_t v &= \operatorname{div} S(F) + \Delta v \\
\partial_t F &= \nabla v \\
\operatorname{curl} F &= 0
\end{aligned}
$$

$$
\begin{aligned}
F\Big|_{t=0} &= \nabla y_0 \\
v\Big|_{t=0} &= v_0 \\
&\text{periodic boundary conditions}
\end{aligned}
$$

$\operatorname{curl} F = 0$ involution propagated from the initial data

energy identity $S(F) = \frac{\partial W}{\partial F}(F)$

$$
\partial_t \left( \tfrac{1}{2}|v|^2 + W(F) \right) = \operatorname{div} \left( v \cdot \left( S(F) + \nabla v \right) \right) - |\nabla v|^2
$$

$$\int \tfrac{1}{2}|v|^2 + W(F)dx + \int_0^t \int |\nabla v|^2 = \int \tfrac{1}{2}|v_0|^2 + W(F_0)$$

Hypotheses on $W$

(H)
$$c|F|^p - \tilde{c} \leq W(F) \leq C(1 + |F|^p) \qquad \text{for } p \geq 2$$
$$|DW(F)| \leq C(1 + |F|^{p-1})$$

For a sequence of approximate solutions that are bounded in energy, one concludes

$$v_n \to v \quad \text{s-}L^2_{t,x}$$
$$F_n \rightharpoonup F \quad \text{wk-}\star \text{ in } L^\infty(L^p)$$

## main theorem existence

### Andrews-Ball condition

(AB) $$(S(F_1) - S(F_2)) : (F_1 - F_2) \geq 0 \qquad \forall\, |F_1|, |F_2| \geq R$$

Condition imposes monotonicity of $S(F)$, convexity of $W(F)$ at infinity

EXISTENCE THEOREM Under Hypotheses (H) and (AB)

$$v_0 \in L^2 \quad F_0 \in L^p \qquad p \geq 2$$

(i) There exists a weak solution

$$v \in L^\infty(L^2) \cap L^2(H^1) \quad F \in L^\infty(L^p)$$

$$\int \tfrac{1}{2}|v|^2 + W(F)dx + \int_0^t \int |\nabla v|^2 \leq \int \tfrac{1}{2}|v_0|^2 + W(F_0)$$

## main theorem existence - continues

(ii) If also

$$F_0 \in H^1 \quad F_0 = \nabla y_0$$

then

$$F \in L^\infty(H^1)$$

Solution here is (almost) what we call strong solution (modulo uniqueness) - which we do not know in general

(iii) For $F_0 \in H^1 \cap L^p$, $p \geq 2$

$$\int \tfrac{1}{2}|v|^2 + W(F)dx + \int_0^t \int |\nabla v|^2 = \int \tfrac{1}{2}|v_0|^2 + W(F_0)$$

$$\text{energy is conserved for} \quad \begin{cases} 2 \leq p & d = 2 \\ 2 \leq p \leq 4 & d = 3 \end{cases}$$

## approach of Friesecke-Dolzmann

(i) Solve the minimization problem: Given $(v^0, F^0)$

$$\min \int \tfrac{1}{2}|v - v^0|^2 + \frac{1}{2h}|F - F^0|^2 + W(F)dx$$

subject to the affine constraint

$$\frac{F - F^0}{h} = \nabla v$$

(ii) Build approximate solutions $y^h(t,x)$, $F^h(t,x)$, $v^h(t,x)$

(iii) Propagation of compactness

$$\limsup_{h\to 0} \int_0^t \int \left|F^h(t,x) - F(t,x)\right|^2 dxdt \le \left(\limsup_{h\to 0} \int |F_0^h - F_0|^2 dx\right) e^{Kt}$$

Friesecke-Dolzmann 96

## present approach

$$(AB) \quad \Longleftrightarrow \quad (S(F) - S(\bar{F})) : (F - \bar{F}) \geq -K|F - \bar{F}|^2 \quad \forall F, \bar{F}$$

$$\Longleftrightarrow \quad W(F) + \tfrac{K}{2}|F|^2 \quad \text{convex}$$

(AB) condition $\Longrightarrow$

$$\frac{d}{dt}\int \left(|v - \tfrac{1}{2}\operatorname{div} F|^2 + |\nabla F|^2 + W(F)\right) + \int D^2\tilde{W} : (\nabla F, \nabla F) + |\nabla v|^2 \, dx$$

$$\leq K \int |\nabla F|^2 dx$$

where $D^2\tilde{W} = D^2 W + K\,\mathbb{I} \geq 0$, in turn

$$\int |\nabla F|^2 \leq \left(E_0 + \int |\nabla F_0|^2 dx\right) e^{Kt}$$

## transfer of dissipation

$$\partial_t v - \operatorname{div} S(F) - \varepsilon \Delta v = 0 \qquad \varepsilon > 0$$
$$\partial_t F - \nabla v = 0$$

$$\frac{d}{dt} \int \tfrac{1}{2}|v|^2 + W(F) dx + \varepsilon \int |\nabla v|^2 = 0$$

$$D^2 W : (\nabla F, \nabla F) - |\nabla v|^2 = ... = \partial_t (v \cdot \operatorname{div} F - \varepsilon |\operatorname{div} F|^2) + \operatorname{div}(...)$$

In $d = 1$ dissipation is transfered from velocity to the strain DiPerna 83, similar in relaxation T. 99 Here, the identity achieving the same objective is

$$\frac{d}{dt} \int \tfrac{1}{2}|v - \tfrac{\varepsilon}{2}\operatorname{div} F|^2 + \tfrac{\varepsilon^2}{4}|\operatorname{div} F|^2 + W(F) dx + \frac{\varepsilon}{2} \int D^2 W : (\nabla F, \nabla F) + |\nabla v|^2 = 0$$

### Conclusions

$$D^2 W \geq c\mathbb{I} \quad \Longrightarrow \quad \varepsilon |\nabla v^\varepsilon|^2 \in_b L^1 \qquad \varepsilon |\nabla F^\varepsilon|^2 \in_b L^1$$

$$\text{Condition (AB)} \quad \Longrightarrow \quad \int |\nabla F|^2 dx \leq e^{Kt} \int |\nabla F_0|^2$$

## existence theory

$F_0 \in H^1$ : existence theory based on

$$\int |\nabla F|^2 dx \leq C$$

$F_0 \in L^p$ : existence theory based on approximation

$$F_{0n} \in H_1 \qquad F_{0n} \to F_0 \quad \text{in } L^p$$

and the property that strong-convergence propagates from data to solution

$$F_n \to F \quad \text{in } L^p$$

If $F_{0n} \rightharpoonup F_0$ weakly in $L^p$ what happens?

# sustained oscillations

$$
\begin{aligned}
u_t &= v_x \\
v_t &= \sigma(u)_x + v_{xx}
\end{aligned}
$$

Construct a family of oscillatory solutions by mixing

- Uniform shear solutions which are universal, i.e. independent of the constitutive functions

$$
u_s(t) = \kappa t \quad v_s(x) = \kappa x \,, \quad \kappa \in \mathbb{R}
$$

- Solutions with discontinuities for $1-d$ viscoelasticity Pego 87, Hoff 87

$$
\begin{aligned}
-s[u] &= [v] \\
-s[v] &= [\sigma(u) + v_x]
\end{aligned}
\qquad \to \qquad
\begin{aligned}
[v] = 0 \quad s &= 0 \\
[u] &\neq 0 \\
[\sigma(u) + v_x] &= 0
\end{aligned}
$$

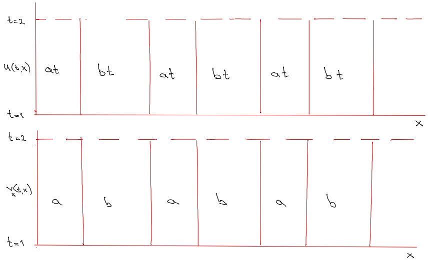



Figure: periodic solution

$\exists\ \sigma(u)$ non-monotone such that this construction is achieved for $1 \leq t \leq 2$

By rescaling this periodic solution

$$\begin{cases} v_n(t,x) \to \big(a\theta + b(1-\theta)\big)x & s - L^2 \\ u_n(t,x) \rightharpoonup \big(a\theta + b(1-\theta)\big)t & wk - \star\ L^\infty \end{cases}$$

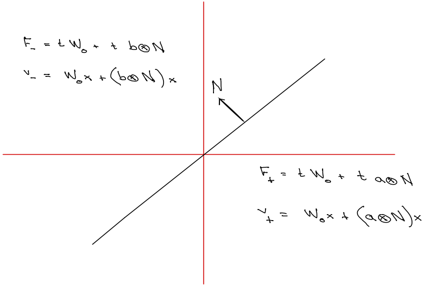



Figure: continuous motion jumping on a steady interface - twinning

Across the steady interface, balance of momentum $[SN] = 0$
Assume that $W(F)$ satisfies

$$\frac{\partial W}{\partial F}(\theta A + \theta \xi \otimes N) - \frac{\partial W}{\partial F}(\theta A) + \xi \otimes N = 0 \qquad \begin{array}{l} 1 \le \theta \le 2 \\ \xi = b - a \in \mathbb{R}^d \\ N \in \mathcal{S}^{d-1} \end{array}$$

The above picture gives a solution of the viscoelasticity system, time-dependent twinning solution

## diffusion-dispersion approximation for elasticity / gradient theory

$$\begin{aligned} \partial_t v - \operatorname{div} S(F) &= \varepsilon \Delta v - \delta A div \Delta F \qquad \varepsilon > 0 \quad \delta > 0 \quad A > 0 \text{ parameter} \\ \partial_t F - \nabla v &= 0 \\ \operatorname{curl} F &= 0 \end{aligned}$$

Set $\omega = v - \kappa \operatorname{div} F$. Then we may rewrite

$$\begin{aligned} \partial_t \omega - \operatorname{div} S(F) &= (\varepsilon - \kappa) \triangle \omega \\ \partial_t F - \nabla \omega &= \kappa \triangle F \\ \operatorname{curl} F &= 0 \end{aligned}$$

For this to work we need to restrict to

$$\delta = \varepsilon^\rho \quad \text{with} \quad \begin{cases} \rho > 2 & \text{any} \quad A \\ \rho = 2 & 0 < A \leq \frac{1}{4} \end{cases}$$

First noted by Slemrod for dimension $d = 1$.

## Uniqueness

UNIQUENESS Suppose in dimension $d = 2$ that

$$v_0 \in L^2 \quad F_0 \in H^1 \cap L^p \qquad p \geq 2$$

and that Hypothesis (H) and a strengthened variant of (AB) hold

$$(\mathrm{AB}') \qquad (S(F_1) - S(F_2)) : (F_1 - F_2) \geq \Big[ C(|F_1|^{p-2} + |F_2|^{p-2}) - K \Big] |F_1 - F_2|^2$$
$$\forall\, |F_1|, |F_2| \geq R$$

$$\Updownarrow$$

$$\tilde{W}(F) = W(F) + \tfrac{K}{2}|F|^2 \quad \text{satisfies} \quad D^2 \tilde{W}(F) \geq c|F|^{p-2}\mathbb{I}$$

then

there exists a unique weak solution $F \in L^\infty(H^1)$

## Global Regularity

REGULARITY Suppose in dimension $d = 2$ that

$$v_0 \in H^3 \quad F_0 \in H^3$$

The solution constructed before enjoys the regularity

$$v \in L^\infty(H^3) \quad F \in L^\infty(H^3)$$

provided $W$ satisfies (AB) and $2 \le p \le 5$, or $W$ satisfies (AB$'$) and $2 \le p \le 6$.

## uniqueness in the class $\int |\nabla F|^2 \leq C$

$$\partial_t v - \Delta v = \operatorname{div} S(F)$$
$$\partial_t F - \nabla v = 0$$

### Comments on proof

$$\partial_t (v_1 - v_2) - \Delta (v_1 - v_2) = \operatorname{div} \big( S(F_1) - S(F_2) \big)$$

Employ the $L^p$ theory for the heat equation, the a-priori estimate $\int |\nabla F|^2 \leq C$, Sobolev embedding $H^1 \subset L^q \quad \forall q$ in $d = 2$ and Brezis-Gallouet, to obtain the inequality

$$\| F_1 - F_2 \|_r^r \leq q \int_0^t \| F_1 - F_2 \|_r^{r - \frac{1}{q}} \, ds \qquad \forall q >> 1$$

The differential inequality

$$\frac{dy}{dt} \leq q y^{1-\frac{1}{q}} \qquad q > 1$$

was employed by Yudovich 63 in his uniqueness proof for the 2d-incompressible Euler with bounded vorticity

Argument is

$$\frac{d}{dt} y^{\frac{1}{q}} \leq 1 \qquad y(0) = 0$$

$$y(t) \leq t^q \qquad \forall q >> 1$$

gives uniqueness in interval $(0, \frac{1}{2})$ as $q \to \infty$ and by bootstrapping for $t \geq 0$

A more elaborate argument provides uniqueness for any $p \geq 2$ under a strengthened variant of the Andrews-Ball condition

$$(S(F_1) - S(F_2), F_1 - F_2) \geq \big( C(|F_1|^{p-2} + |F_2|^{p-2}) - K \big) |F_1 - F_2|^2 .$$

The latter provides some additional coercivity and yields a bound

$$\int_0^t \int \nabla |F|^{\frac{p}{2}} dxds \leq C$$

This proof is analogous to the 2d Euler uniqueness proof, valid under the assumption of $L^\infty$ vorticity.

Figure: 1994 San Remo, courtesy of Eitan Tadmor

Oxford1982

Claude Bardos, Stella Dafermos, Andy Majda,
Ron DiPerna, Costas Dafermos

$K\omega\sigma\tau\alpha$

$\chi\rho o\nu\iota\alpha\ \pi o\lambda\lambda\alpha\ \mu\varepsilon\ \upsilon\gamma\varepsilon\iota\alpha$

many healthy years