Existence and Uniqueness in Viscoelasticity of strain-rate type

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Viscoelasticity

\[ \frac{\partial^2 y}{\partial t^2} = \text{div } T \]

Viscoelasticity models feature the dependence on memory effects

- \( T = \hat{T}(F(t, x), \dot{F}(t, x)) \) viscoelasticity of strain-rate type
- \( \dot{T} = \tilde{G}(T(t, x), F(t, x)) \) viscoelasticity of stress-rate type, Maxwell models
- kinetic models

Motion \( y(x, t) \)

Velocity \( \nu = \frac{\partial y}{\partial t} \)

Deformation gradient \( F = \nabla y \)

Stress \( T \)

\[ \Omega \]

\[ \Omega_t = y_t(\Omega) \]
Viscoelasticity of strain-rate type

\[
\frac{\partial^2 y}{\partial t^2} = \text{div} \ T(\nabla y, \nabla y_t)
\]

Kelvin-Voigt model

\[
T(F, \dot{F}) = \frac{\partial W}{\partial F}(F) + \dot{F}
\]

Key features:
- \( W(F) \) elastic stored energy \( \text{nonconvex} \)
- model consistent w. entropy dissipation
- not frame indifferent
- invariant under the extended Galilean group

\[
x^* = Qx + d(t) \quad \text{where } Q \text{ is a time-independent rotation}
\]

Defficiency: no account of the constraint \( \det F > 0 \)
\[ u_t = v_x \]
\[ v_t = \tau(u, v_x) \quad \text{e.g.} \quad v_t = \sigma(u)_x + \varepsilon v_{xx} \]

**Literature**

A lot of results in dimension \( d = 1 \), Greenberg-McCamy-Mizel 68, Dafermos 69, ...

several connected to phase transitions \( \sigma(u) \) non-monotone

Slemrod, Hoff, Pego, Andrews-Ball, Trivisa ...

Strain-rate viscoelasticity: Friedman-Nečas 88, Friesecke-Dolzmann 97, Demoulini 00

nonlinear strain-rate dependence and implicit stress theories: Tvedt 08, Y. Sengül 10, Malek-Bulíček-Rajagopal, 12 ...,

\[ T(F, \dot{F}) = \frac{\partial W}{\partial F}(F) + \dot{F} \quad W(F) \text{ semiconvex} \]
viscoelasticity as a hyperbolic-parabolic system

\[ \partial_t \nu = \text{div} \, S(F) + \Delta \nu \]
\[ \partial_t F = \nabla \nu \]
\[ \text{curl} \, F = 0 \]

\[ F \bigg|_{t=0} = \nabla y_0 \]
\[ \nu \bigg|_{t=0} = \nu_0 \]

periodic boundary conditions

\[ \text{curl} \, F = 0 \quad \text{involution} \quad \text{propagated from the initial data} \]

energy identity
\[ S(F) = \frac{\partial W}{\partial F}(F) \]

\[ \partial_t \left( \frac{1}{2} |\nu|^2 + W(F) \right) = \text{div} \left( \nu \cdot (S(F) + \nabla \nu) \right) - |\nabla \nu|^2 \]
\[
\int \frac{1}{2} |v|^2 + W(F) \, dx + \int_0^t \int |\nabla v|^2 = \int \frac{1}{2} |v_0|^2 + W(F_0)
\]

Hypotheses on $W$

\[
c |F|^p - \tilde{c} \leq W(F) \leq C(1 + |F|^p) \quad \text{for } p \geq 2
\]

\((H)\)

\[
|DW(F)| \leq C(1 + |F|^{p-1})
\]

For a sequence of approximate solutions that are bounded in energy, one concludes

\[
v_n \rightarrow v \quad s-L^2_{t,x}
\]

\[
F_n \rightharpoonup F \quad \text{wk-}^* \text{ in } L^\infty(L^p)
\]
Existence main theorem existence

Andrews-Ball condition

\[(AB) \quad (S(F_1) - S(F_2)) : (F_1 - F_2) \geq 0 \quad \forall |F_1|, |F_2| \geq R\]

Condition imposes monotonicity of $S(F)$, convexity of $W(F)$ at infinity

**Existence theorem** Under Hypotheses (H) and (AB)

$$v_0 \in L^2 \quad F_0 \in L^p \quad p \geq 2$$

(i) There exists a weak solution

$$v \in L^\infty(L^2) \cap L^2(H^1) \quad F \in L^\infty(L^p)$$

$$\int \frac{1}{2}|v|^2 + W(F) \, dx + \int_0^t \int |\nabla v|^2 \leq \int \frac{1}{2}|v_0|^2 + W(F_0)$$
main theorem existence - continues

(ii) If also

\[ F_0 \in H^1 \quad F_0 = \nabla y_0 \]

then

\[ F \in L^\infty(H^1) \]

Solution here is (almost) what we call strong solution (modulo uniqueness) - which we do not know in general

(iii) For \( F_0 \in H^1 \cap L^p, \ p \geq 2 \)

\[ \int \frac{1}{2} |v|^2 + W(F) \, dx + \int_0^t \int |\nabla v|^2 = \int \frac{1}{2} |v_0|^2 + W(F_0) \]

energy is conserved for \[ \begin{cases} 2 \leq p & d = 2 \\ 2 \leq p \leq 4 & d = 3 \end{cases} \]
(i) Solve the minimization problem: Given \((v^0, F^0)\)

\[
\min \int \frac{1}{2} |v - v^0|^2 + \frac{1}{2h} |F - F^0|^2 + W(F) \, dx
\]

subject to the affine constraint

\[
\frac{F - F^0}{h} = \nabla v
\]

(ii) Build approximate solutions \(y^h(t, x), F^h(t, x), v^h(t, x)\)

(iii) Propagation of compactness

\[
\limsup_{h \to 0} \int_0^t \int |F^h(t, x) - F(t, x)|^2 \, dx \, dt \leq \left( \limsup_{h \to 0} \int |F^h_0 - F_0|^2 \, dx \right) e^{Kt}
\]
**Existence**

**present approach**

\[(AB) \iff (S(F) - S(\bar{F})) : (F - \bar{F}) \geq -K|F - \bar{F}|^2 \quad \forall F, \bar{F}\]

\[\iff W(F) + \frac{K}{2}|F|^2 \quad \text{convex}\]

\[(AB) \text{ condition } \implies \frac{d}{dt} \int (|v - \frac{1}{2} \text{div } F|^2 + |\nabla F|^2 + W(F)) + \int D^2 \tilde{\mathcal{W}} : (\nabla F, \nabla F) + |\nabla v|^2 \, dx\]

\[\leq K \int |\nabla F|^2 \, dx\]

where \(D^2 \tilde{\mathcal{W}} = D^2 \mathcal{W} + K I \geq 0\), in turn

\[\int |\nabla F|^2 \leq \left(E_0 + \int |\nabla F_0|^2 \, dx\right) e^{Kt}\]
transfer of dissipation

\[
\partial_t \nu - \text{div } S(F) - \varepsilon \Delta \nu = 0 \quad \varepsilon > 0 \\
\partial_t F - \nabla \nu = 0 \\
\frac{d}{dt} \int \frac{1}{2} |\nu|^2 + W(F) \, dx + \varepsilon \int |\nabla \nu|^2 = 0 \\
D^2 W : (\nabla F, \nabla F) - |\nabla \nu|^2 = \ldots = \partial_t (\nu \cdot \text{div } F - \varepsilon |\text{div } F|^2) + \text{div } (\ldots)
\]

In \( d = 1 \) dissipation is transfered from velocity to the strain DiPerna 83, similar in relaxation T. 99 Here, the identity achieving the same objective is

\[
\frac{d}{dt} \int \frac{1}{2} |\nu - \varepsilon \text{div } F|^2 + \frac{\varepsilon^2}{4} |\text{div } F|^2 + W(F) \, dx + \frac{\varepsilon}{2} \int D^2 W : (\nabla F, \nabla F) + |\nabla \nu|^2 = 0
\]

Conclusions

\[
D^2 W \geq cI \quad \Rightarrow \quad \varepsilon |\nabla \nu^\varepsilon|^2 \in_b L^1 \quad \varepsilon |\nabla F^\varepsilon|^2 \in_b L^1 \\
\text{Condition (AB)} \quad \Rightarrow \quad \int |\nabla F|^2 \, dx \leq e^{Kt} \int |\nabla F_0|^2
\]
existence theory

\( F_0 \in H^1 \): existence theory based on

\[
\int |\nabla F|^2 dx \leq C
\]

\( F_0 \in L^p \): existence theory based on approximation

\[
F_{0n} \in H_1 \quad F_{0n} \to F_0 \quad \text{in } L^p
\]

and the property that strong-convergence propagates from data to solution

\[
F_n \to F \quad \text{in } L^p
\]

If \( F_{0n} \rightharpoonup F_0 \) weakly in \( L^p \) what happens?
sustained oscillations

\[ u_t = v_x \]
\[ v_t = \sigma(u)_x + v_{xx} \]

Construct a family of oscillatory solutions by mixing

- Uniform shear solutions which are universal, i.e. independent of the constitutive functions

\[ u_s(t) = \kappa t \quad v_s(x) = \kappa x, \quad \kappa \in \mathbb{R} \]

- Solutions with discontinuities for $1 - d$ viscoelasticity Pego 87, Hoff 87

\[ -s[u] = [v] \quad [v] = 0 \quad s = 0 \]
\[ -s[v] = [\sigma(u) + v_x] \quad [\sigma(u) + v_x] = 0 \]
Figure: periodic solution

\[ \exists \sigma(u) \text{ non-monotone} \] such that this construction is achieved for \( 1 \leq t \leq 2 \)

By rescaling this periodic solution

\[
\begin{align*}
\nu_n(t, x) &\to (a\theta + b(1 - \theta))x \quad s - L^2 \\
\chi_n(t, x) &\to (a\theta + b(1 - \theta))t \quad \text{wk} - \star L^\infty
\end{align*}
\]
Figure: continuous motion jumping on a steady interface - twinning

Across the steady interface, balance of momentum \([SN] = 0\)

Assume that \(W(F)\) satisfies

\[
\frac{\partial W}{\partial F}(\theta A + \theta \xi \otimes N) - \frac{\partial W}{\partial F}(\theta A) + \xi \otimes N = 0
\]

\[
1 \leq \theta \leq 2
\]

\[
\xi = b - a \in \mathbb{R}^d
\]

\[
N \in S^{d-1}
\]

The above picture gives a solution of the viscoelasticity system, time-dependent twinning solution.
diffusion-dispersion approximation for elasticity / gradient theory

\[ \partial_t v - \text{div } S(F) = \varepsilon \Delta v - \delta \text{div } \Delta F \quad \varepsilon > 0 \quad \delta > 0 \quad A > 0 \text{ parameter} \]
\[ \partial_t F - \nabla v = 0 \]
\[ \text{curl } F = 0 \]

Set \( \omega = v - \kappa \text{div } F \). Then we may rewrite
\[ \partial_t \omega - \text{div } S(F) = (\varepsilon - \kappa) \Delta \omega \]
\[ \partial_t F - \nabla \omega = \kappa \Delta F \]
\[ \text{curl } F = 0 \]

For this to work we need to restrict to
\[ \delta = \varepsilon^\rho \quad \text{with} \quad \begin{cases} \rho > 2 & \text{any } A \\ \rho = 2 & 0 < A \leq \frac{1}{4} \end{cases} \]

First noted by Slemrod for dimension \( d = 1 \).
\textbf{Uniqueness} Suppose in dimension $d = 2$ that
\[ v_0 \in L^2 \quad F_0 \in H^1 \cap L^p \quad p \geq 2 \]

and that Hypothesis (H) and a strengthened variant of (AB) hold

\[(AB') \quad (S(F_1) - S(F_2)) : (F_1 - F_2) \geq \left[ C(|F_1|^{p-2} + |F_2|^{p-2}) - K \right] |F_1 - F_2|^2 \quad \forall |F_1|, |F_2| \geq R \]

\[ \tilde{W}(F) = W(F) + \frac{K}{2} |F|^2 \quad \text{satisfies} \quad D^2 \tilde{W}(F) \geq c |F|^{p-2} \]

then

there exists a unique weak solution $F \in L^\infty(H^1)$
Regularity Suppose in dimension $d = 2$ that

$$v_0 \in H^3 \quad F_0 \in H^3$$

The solution constructed before enjoys the regularity

$$v \in L^\infty(H^3) \quad F \in L^\infty(H^3)$$

provided $W$ satisfies (AB) and $2 \leq p \leq 5$, or $W$ satisfies (AB’) and $2 \leq p \leq 6$. 
Uniqueness

Uniqueness in the class \( \int |\nabla F|^2 \leq C \)

\[
\begin{align*}
\partial_t \nu - \Delta \nu &= \text{div } S(F) \\
\partial_t F - \nabla \nu &= 0
\end{align*}
\]

Comments on proof

\[
\partial_t (\nu_1 - \nu_2) - \Delta (\nu_1 - \nu_2) = \text{div } (S(F_1) - S(F_2))
\]

Employ the \( L^p \) theory for the heat equation, the a-priori estimate \( \int |\nabla F|^2 \leq C \), Sobolev embedding \( H^1 \subset L^q \) \( \forall q \) in \( d = 2 \) and Brezis-Gallouet, to obtain the inequality

\[
\|F_1 - F_2\|_r^r \leq q \int_0^t \|F_1 - F_2\|_r^{r-\frac{1}{q}} \, ds \quad \forall q >> 1
\]
The differential inequality

\[
\frac{dy}{dt} \leq qy^{1-\frac{1}{q}} \quad q > 1
\]

was employed by Yudovich 63 in his uniqueness proof for the 2d-incompressible Euler with bounded vorticity.

Argument is

\[
\frac{d}{dt}y^\frac{1}{q} \leq 1 \quad y(0) = 0
\]

\[y(t) \leq t^q \quad \forall q >> 1\]

gives uniqueness in interval \((0, \frac{1}{2})\) as \(q \to \infty\) and by bootstrapping for \(t \geq 0\)
A more elaborate argument provides uniqueness for any $p \geq 2$ under a strengthened variant of the Andrews-Ball condition

$$(S(F_1) - S(F_2), F_1 - F_2) \geq (C(|F_1|^{p-2} + |F_2|^{p-2}) - K)|F_1 - F_2|^2.$$ 

The latter provides some additional coercivity and yields a bound

$$\int_0^t \int \nabla |F|^\frac{p}{2} dx ds \leq C$$

This proof is analogous to the 2d Euler uniqueness proof, valid under the assumption of $L^\infty$ vorticity.
Figure: 1994 San Remo, courtesy of Eitan Tadmor
Uniqueness

Oxford1982

Claude Bardos, Stella Dafermos, Andy Majda, Ron DiPerna, Costas Dafermos
Uniqueness

Kwosta

χρονια πολλα με υγεια

many healthy years