Spatial Manifestations of Order Reduction, and Remedies via Weak Stage Order

Benjamin Seibold (Temple University)

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Collaborators

Abhijit Biswas (Temple)
David Ketcheson (KAUST)
Rodolfo Ruben Rosales (MIT)
David Shirokoff (NJIT)
Dong Zhou (CalState LA)

Support

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Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Pre-Talk Advertisement: Neuro-VISOR Project

VISOR = Virtual Interactive Simulation Of Reality

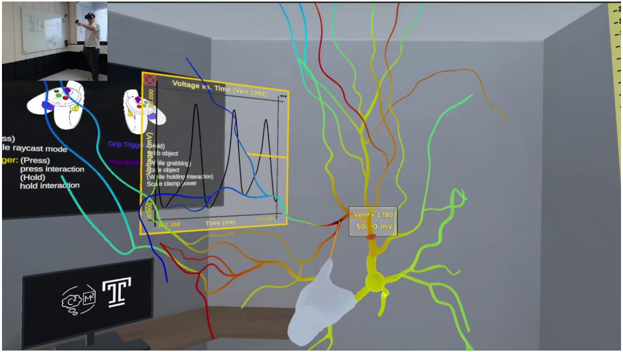
Real-Time HPC

→ Small-scale simulation runs in real time; reacts instantaneously (no restart) to user manipulating state.

→ Rapid prototyping and intuition building.

→ New HPC challenges ($\leq$10ms per step); time-steppers must deal well with discontinuous changes of system (e.g., user places voltage clamp).

PhD student J. Rosado using our VR system



[B. Seibold, G. Queisser, R. Chinomona, many students]

Oculus and desktop version:

`https://github.com/c2m2/Neuro-VISOR`

Testers welcome!

Overview

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Overview

Implicit Runge-Kutta (RK) Methods

Butcher tableau with A invertible: $\begin{array}{c|c} \vec{c} & A \\ \hline & \vec{b}^T \end{array}$, $\quad \vec{c} = A\vec{e}$, $\quad \vec{e} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Def.: If A is lower triangular, scheme is *diagonally implicit* (DIRK).

Thm.: *Stability function* $R(\zeta) = 1 + \zeta \vec{b}^T (I - \zeta A)^{-1} \vec{e}$ measures growth u^{n+1}/u^n per step Δt, when solving $u'(t) = \lambda u$. Here $\zeta = \lambda \Delta t$.

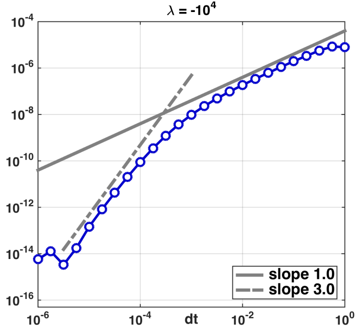
Order Reduction for Stiff ODE

$y' = \lambda(y - \phi(t)) + \phi'(t)$
with i.c. $y(0) = \phi(0)$ and $\mathrm{Re}\,\lambda \leq 0$.
Exact solution: $y(t) = \phi(t)$.

Convergence $\Delta t \to 0$:

- Traditional limit: $\zeta \to 0$. Observe scheme's order.
- Stiff limit: $\zeta \to -\infty$. Reduced order.

Error Convergence (DIRK3)



Understanding Order Reduction for (Linear) Stiff ODE

Model equation [Prothero–Robinson]: $\quad y' = \lambda(y - \phi(t)) + \phi'(t)$

Apply RK scheme. Error at t_{n+1} (with $\zeta = \lambda\Delta t$):

$$\epsilon^{n+1} = R(\zeta)\,\epsilon^n + \zeta\vec{b}^{\,T}(I - \zeta A)^{-1}\vec{\delta}_s^{\,n+1} + \delta^{n+1}\,.$$

Truncation errors at intermediate stages and end of step

$$\vec{\delta}_s^{\,n+1} = \sum_{j\geq 2} \frac{\Delta t^j}{(j-1)!}\,\vec{\tau}^{\,(j)}\phi^{(j)}(t_n)\,,\quad \delta^{n+1} = \sum_{j\geq 1} \frac{\Delta t^j}{(j-1)!}\left(\vec{b}^{\,T}\vec{c}^{\,j-1} - \tfrac{1}{j}\right)\phi^{(j)}(t_n)$$

in terms of derivatives of ϕ, and...

Def.: *Stage order residuals* $\quad \vec{\tau}^{\,(j)} = A\vec{c}^{\,j-1} - \frac{1}{j}\vec{c}^{\,j}\,,\quad j = 1, 2, \ldots$

Def.: Scheme has *stage order* q if $\vec{\tau}^{\,(j)} = 0$ for $1 \leq j \leq q$.

Convergence:

- Limit $\zeta \to 0$: Order conditions $[\vec{b}^{\,T}A^j\,\vec{c}^{\,k} = \frac{1}{(j+k+1)\cdots(k+1)}$ for $0 \leq j+k \leq p-1]$ imply order p (as $\zeta(I-\zeta A)^{-1} = \zeta + \zeta^2 A + \zeta^3 A^2 + \ldots$ and $\vec{b}^{\,T}\vec{\tau}^{\,(j)} = 0$ for $j \leq p-1$).
- Stiff $\zeta \to -\infty$: Order $< p$ (as $\zeta(I-\zeta A)^{-1} = -A^{-1} - \zeta^{-1}A^{-2} - \zeta^{-2}A^{-3} - \ldots$ and $\vec{b}^{\,T}A^{\ell}\vec{\tau}^{\,(j)} \neq 0$ for $\ell < 0$). In fact, can guarantee stage order q.

Overcoming Order Reduction for (Linear) Stiff ODE

Need that error term $\zeta\vec{b}^T(I-\zeta A)^{-1}\vec{\delta}_s^{n+1}$ is high order, where

$$\vec{\delta}_s^{n+1} = \sum_{j\geq 2} \tfrac{\Delta t^j}{(j-1)!}\,\vec{\tau}^{(j)}\phi^{(j)}(t_n)$$

and *stage order residuals* $\vec{\tau}^{(j)} = A\vec{c}^{j-1} - \frac{1}{j}\vec{c}^j$.

Fact: If stage order q equals order p, i.e., $\vec{\tau}^{(1)} = \cdots = \vec{\tau}^{(p)} = 0$, then order reduction is avoided.

Problem: DIRK (non-EDIRK) schemes are limited to stage order $q = 1$.

[Kennedy, Carpenter, 2016 review]: "To date, we are not aware of any efforts to remedy boundary order reduction by using Butcher coefficients for DIRK-type methods."

Goal: Concept weak stage order that also remedies order reduction, and that *is* compatible with DIRK structure.

One simple Idea: Require $A\vec{\tau}^{(j)} = \mu_j\vec{\tau}^{(j)}$ for some μ_j for $1 \leq j \leq \tilde{q}$.

Why does it work: $\vec{b}^T(I-\zeta A)^{-1}\vec{\tau}^{(j)} = (1-\zeta\mu_j)^{-1}\vec{b}^T\vec{\tau}^{(j)}$.

Application in which stiff limit arises naturally: PDE IBVPs.

MOL perspective, "PDE IBVP is just a stiff ODE", is insight-limited.

Overview

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Initial-Boundary-Value Problem (IBVP)

$$\begin{cases} u_t = \mathcal{L}u + f & \text{for } x \in \Omega, t \in (0, T) \\ u = g & \text{for } x \in \partial\Omega, t \in [0, T] \\ u = u_0 & \text{for } x \in \Omega, t = 0 \end{cases}$$

where $\mathcal{L}$ differential operator.

Example: 1D Heat Equation

$$\begin{cases} u_t = u_{xx} + f(x,t) & \text{PDE} \\ u = g(x_\mathrm{b}, t) & \text{b.c.} \\ u = u_0(x) & \text{i.c.} \end{cases}$$

where $x \in [0, 1]$.

Implicit Time-Stepping of IBVP

Why? Avoid $\Delta t \leq O(\Delta x^2)$ time-step restriction of explicit schemes.

Semi-discretization in time (Rothe; justified if uncond. stable) yields BVP:

Backward Euler:
$$\begin{cases} \frac{1}{\Delta t}(u^{n+1} - u^n) = \mathcal{L}u^{n+1} + f^{n+1} & \text{in } \Omega \\ u^{n+1} = g^{n+1} & \text{on } \partial\Omega \end{cases}$$

Local (one time step) truncation error: $O(\Delta t^2)$

Global ($O(1/\Delta t)$ time steps) truncation error: $O(\Delta t)$

Order reduction is a temporal error phenomenon.

($\rightsquigarrow$ use super-fine spatial grids in examples)

Initial-Boundary-Value Problem (IBVP)

$$\begin{cases} u_t = \mathcal{L}u + f & \text{for } x \in \Omega, t \in (0, T) \\ u = g & \text{for } x \in \partial\Omega, t \in [0, T] \\ u = u_0 & \text{for } x \in \Omega, t = 0 \end{cases}$$

where $\mathcal{L}$ differential operator.

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where $x \in [0, 1]$.

Diagonally Implicit Runge-Kutta (DIRK) Schemes

Rothe–DIRK paradigm: user designs spatial problem for backward Euler; high-order via only minor modifications of that call in each stage (cf. nontrivial (b.c.) coupled system in full IRK schemes).

$$\text{DIRK2:} \begin{cases} u_1^{n+1} = u^n + \Delta t\gamma(\mathcal{L}u_1^{n+1} + f^{n+\gamma}) & \text{in } \Omega \\ u_1^{n+1} = g^{n+\gamma} & \text{on } \partial\Omega \\ u^{n+1} = u^n + \Delta t(1-\gamma)(\mathcal{L}u_1^{n+1} + f^{n+\gamma}) + \Delta t\gamma(\mathcal{L}u^{n+1} + f^{n+1}) & \text{in } \Omega \\ u^{n+1} = g^{n+1} & \text{on } \partial\Omega \end{cases}$$

Butcher tableau: $\begin{array}{c|cc} \gamma & \gamma & \\ 1 & 1-\gamma & \gamma \\ \hline & 1-\gamma & \gamma \end{array}$ where $\gamma = 1 - \frac{\sqrt{2}}{2}$

Example: 1D Heat Equation

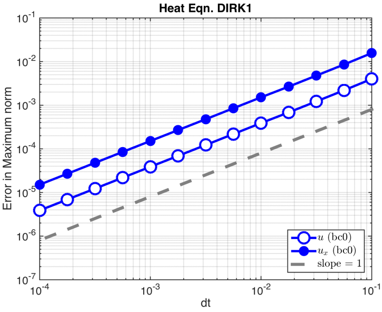
$$\begin{cases} u_t = u_{xx} + f(x,t) & \text{PDE} \\ u = g(x_\text{b}, t) & \text{b.c.} \\ u = u_0(x) & \text{i.c.} \end{cases}$$

Method of Manufactured Solutions

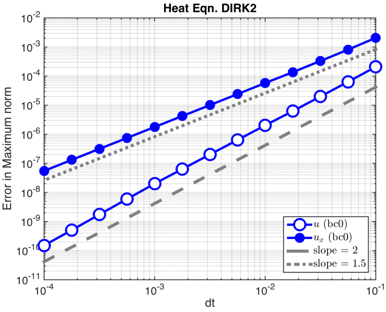
Choose $u(x,t)$. Calculate f, g, and u_0 s.t. IBVP has the chosen solution.

Simplest example: $u(x,t) = \cos(t)$; $x \in [0,1]$, $t \in [0,1]$.

DIRK1 (backward Euler)



Second-Order DIRK2



Expected orders in u. Loss of half an order in u_x for DIRK2.

Example: 1D Heat Equation

$$\begin{cases} u_t = u_{xx} + f(x,t) & \text{PDE} \\ u = g(x_b, t) & \text{b.c.} \\ u = u_0(x) & \text{i.c.} \end{cases}$$

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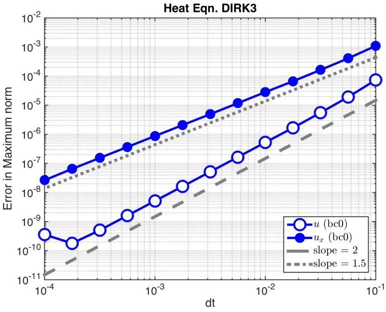
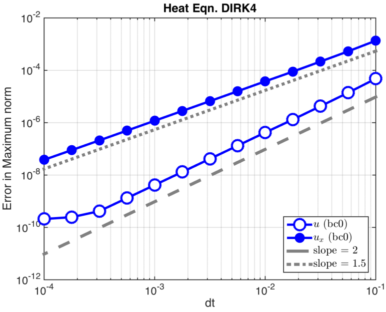
Choose $u(x,t)$. Calculate f, g, and u_0 s.t. IBVP has the chosen solution.

Simplest example: $u(x,t) = \cos(t)$; $x \in [0,1]$, $t \in [0,1]$.

Third-Order DIRK3

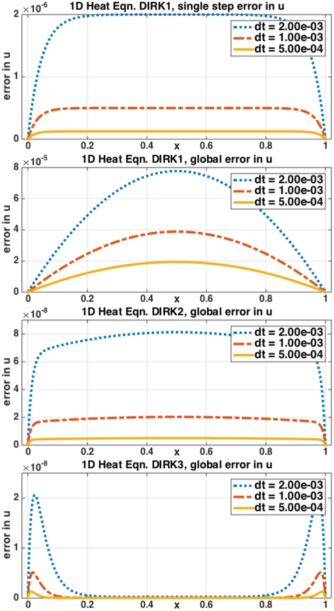


Fourth-Order DIRK4



DIRK3/4 only as accurate as DIRK2. Order-loss in u (and u_x).

Shape of Temporal Errors



Why are there Boundary Layers (BL) at all?

DIRK1 one-step error $\epsilon(x)$ solves BVP

$$\begin{cases} \epsilon - \Delta t\, \epsilon_{xx} = -\Delta t \sin(\Delta t) & \text{for } x \in (0,1) \\ \epsilon = 0 & \text{for } x \in \{0,1\} \end{cases}$$

Singularly perturbed problem:
$\epsilon = O(\Delta t^2)$ outside BL; BL thickness $O(\Delta t^{0.5})$.

Spatial boundary layers, caused by the temporal error.

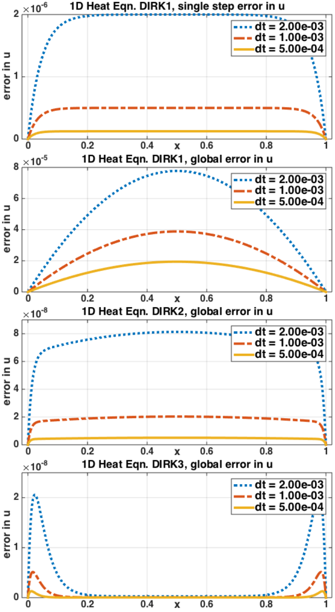
Why loss of 1/2 order in u_x?

Error away from BL: $O(\Delta t^p)$;
error on boundary: 0;
BL thickness: $O(\Delta t^{0.5})$.

Why DIRK 3 and DIRK 4 only second order?

Stages have different BL thickness. No order p Taylor series cancellation inside BLs. Error as accurate as each stage ($O(\Delta t^2)$).

Shape of Temporal Errors



Boundary Layer Error Theory for General Runge-Kutta Schemes

Spatial Manifestations of Order Reduction in Runge-Kutta Methods for Initial Boundary Value Problems, `arxiv.org/abs/1712.00897`

1. Modal analysis of semi-discretized (in space) system: $\vec{u}^n(x) = \vec{v}(x)e^{i\omega\Delta t\, n}$.
2. Yields BVP $\vec{v} = M \cdot \mathcal{L}\vec{v} + M \cdot \vec{\phi}$.
3. Spectrum of M:
$$M = \underbrace{\frac{\Delta t}{e^{i\omega\Delta t} - 1}\vec{e}\vec{b}^T}_{\text{O(1) rank 1 matrix}} + \underbrace{\Delta t A}_{O(\Delta t)\text{ perturbation}}$$
4. One $O(1)$ eigenvalue, others $O(\Delta t)$.
5. Hence: Single-stage methods are devoid of OR. RK methods have BLs.
6. Avoiding OR means: BLs are present but are of the order of the method (or higher).

Overview

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Stage Order Concepts (assuming sufficiently high order p)

Def.: The j-th *stage order residual* is $\vec{\tau}^{(j)} = A\vec{c}^{j-1} - \frac{1}{j}\vec{c}^{j}$ for $j \geq 1$.

Def.: A scheme has *(strong) stage order* q if $\vec{\tau}^{(j)} = 0$ for $1 \leq j \leq q$.

Def.: A scheme has *weak stage order* (WSO) $\tilde{q}$ if there is an A-invariant subspace $\mathcal{V}$ that is orthogonal to $\vec{b}$ and $\vec{\tau}^{(j)} \in \mathcal{V}$ for $1 \leq j \leq \tilde{q}$.

Def.: A scheme satisfies the *weak stage order eigenvector criterion* of order $\tilde{q}_e$ if $A\vec{\tau}^{(j)} = \mu_j \vec{\tau}^{(j)}$ for $1 \leq j \leq \tilde{q}_e$ and $\vec{b}^T\vec{\tau}^{(j)} = 0$.

Properties

- $1 \leq q \leq \tilde{q}_e \leq \tilde{q}$.
- WSO is the most general condition that ensures $\zeta\vec{b}^T(I - \zeta A)^{-1}\vec{\tau}^{(j)} = 0$ for all $\zeta > 0$ and $1 \leq j \leq \tilde{q}$.

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Thm.: For the linear stiff ODE, WSO $\tilde{q} \geq p$ avoids order reduction.

Proof: One-step error:

$$\epsilon^1 = \sum_{j \geq q+1} \frac{\Delta t^j}{(j-1)!} \phi^{(j)}(t_0) \left(\zeta \vec{b}^{\,T} (I - \zeta A)^{-1} \vec{\tau}^{(j)} \right) + O(\Delta t^{p+1}) \,.$$

In the sum, the first $j \leq \tilde{q}$ terms vanish to to weak stage order, because $\vec{\tau}^{(j)} \in \mathcal{V} \overset{\mathcal{V}\ A\text{-inv.}}{\Longrightarrow} (I - \zeta A)^{-1} \vec{\tau}^{(j)} \in \mathcal{V} \overset{\mathcal{V} \perp \vec{b}}{\Longrightarrow} \vec{b}^{\,T} (I - \zeta A)^{-1} \vec{\tau}^{(j)} = 0$.
Hence $\epsilon^1 = O(\Delta t^{\min\{\tilde{q}+1, p+1\}})$. One order is lost for global error. □

Thm.: For a linear IBVP, WSO $\tilde{q} \geq p - 1$ avoids order reduction.
(constant coefficient differential operator)

Mechanism:

Stage order q makes *all* stages $O((\Delta t)^{q+1})$.

In contrast, weak stage order $\tilde{q}$ makes some eigenfunctions of the global error $O((\Delta t)^{\tilde{q}+1})$, and renders the remaining (low order) ones unused.

Fundamental Limitation of DIRK Schemes

Thm.: The stage order q of an irreducible DIRK scheme is at most 2. Moreover, if A is non-singular, then $q = 1$.

Proof: Verify by first two components of $\vec{\tau}^{(2)}$ and $\vec{\tau}^{(3)}$.

DIRK Schemes With High Weak Stage Order

Thm.: (Limitation of eigenvector criterion) DIRK schemes with non-singular A have $\tilde{q}_e \leq 3$.

Proof: Technical (study solutions of nonlinear relations).

Thm: DIRK schemes with $\tilde{q}_e = 3$ and orders 3 and 4 do exist (next slide).

WSO Conditions

Thm.: WSO $\tilde{q}$ is equivalent to $\vec{b}^T A^\ell \vec{\tau}^{(j)} = 0$ for $1 \leq j \leq \tilde{q}$ and $0 \leq \ell \leq s-1$.

Yields, with order conditions, polynomial equations for the entries of A.

Solve via Matlab's optimization toolbox, minimizing (proxy for) LTE.

Resulting schemes likely reasonable, but not guaranteed to be globally optimal.

DIRK Schemes with WSO Eigenv. Crit. (all stiffly accurate and L-stable)

Order 3 and WSO 2 $\rightsquigarrow$ full 3rd order for IBVPs, 2nd order for stiff ODEs:

0.01900072890	0.01900072890			
0.78870323114	0.40434605601	0.38435717512		
0.41643499339	0.06487908412	−0.16389640295	0.51545231222	
1	0.02343549374	−0.41207877888	0.96661161281	0.42203167233
	0.02343549374	−0.41207877888	0.96661161281	0.42203167233

Order 3 and WSO 3 $\rightsquigarrow$ full 3rd order for IBVPs and for stiff ODEs:

0.13756543551	0.13756543551			
0.80179011576	0.56695122794	0.23483888782		
2.33179673002	−1.08354072813	2.96618223864	0.44915521951	
1	0.59761291500	−0.43420997584	−0.05305815322	0.88965521406
	0.59761291500	−0.43420997584	−0.05305815322	0.88965521406

Order 4 and WSO 3 $\rightsquigarrow$ full 4th order for IBVPs, 3rd order for stiff ODEs:

0.0796724	0.0796724					
0.4643647	0.3283554	0.1360093				
1.3485592	−0.6507728	1.7428591	0.2564730			
1.3126642	−0.7145806	1.7937458	−0.0782548	0.3117538		
0.9894693	−1.1200928	1.9834523	3.1173939	−3.7619302	0.7706460	
1	0.2148237	0.5363674	0.1544881	−0.2177486	0.0722264	0.2398430
	0.2148237	0.5363674	0.1544881	−0.2177486	0.0722264	0.2398430

Overview

Prothero-Robinson ODE

$$u_t = \lambda(u - \phi) + \phi_t$$

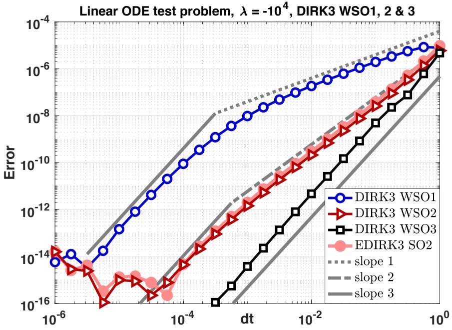
with i.c. $u = \phi(0)$,
true sol. $\phi(t) = \sin(t + \frac{\pi}{4})$,
and $\lambda = -10^4$.

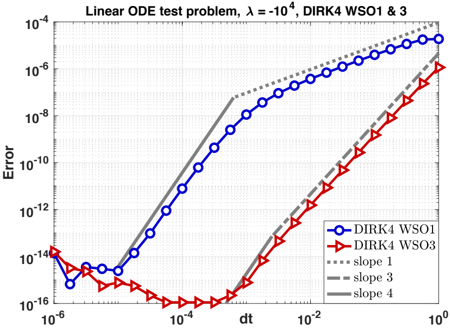
Result

High WSO: As expected, better convergence in stiff $\zeta \gg 1$ regime.

Possibly also better error constants in non-stiff $\zeta \ll 1$ regime.

Error Convergence with Various WSOs (and Stage Orders)





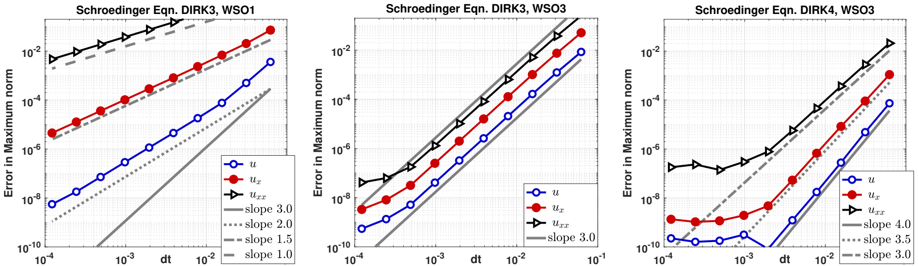
Schrödinger Equation

$$u_t = \frac{i\omega}{\xi^2} u_{xx} + f$$

Manufactured Solution

$$u(x,t) = \exp(i(\xi x - \omega t))$$

with $\xi = 2\pi$, $\omega = 5$, and $T = 1.2$.



WSO2 recovers full 3rd order in u. Loses half order in u_x.

Due to dispersive waves, order-reduction effects away from BLs.

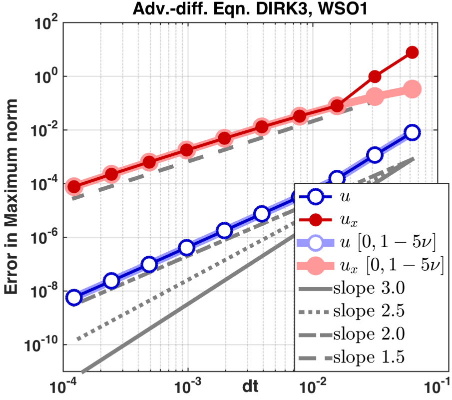
Same error results for heat equation (not shown here).

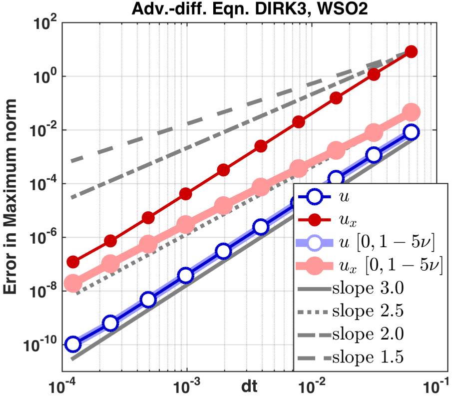
Advection-Diffusion Equation

$$u_t + u_x = \varepsilon u_{xx} + f$$

Manufactured Solution

$u(x,t) = \sin(2\pi(x-t))$
with $\varepsilon = 10^{-3}$.

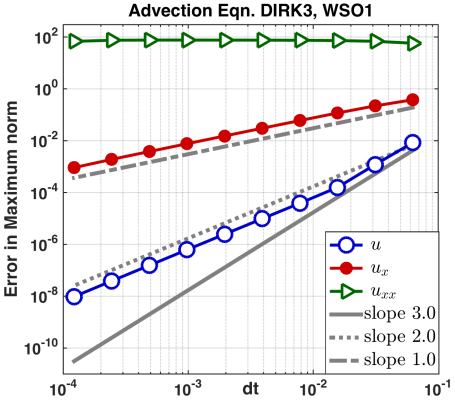


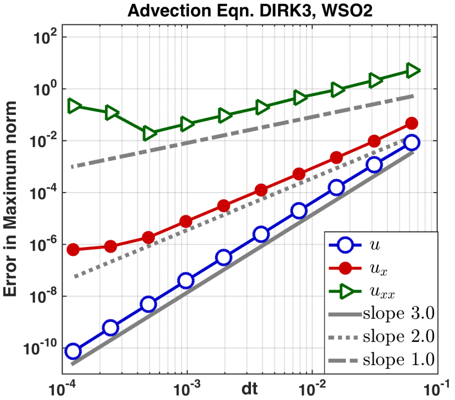


WSO2 recovers full 3rd order in u. Loses half order in u_x.
Order reduction not visible for large Δt (physical BL dominates).

Linear Advection Equation

$$u_t = -u_x + f$$





A full order is lost per spatial derivative.

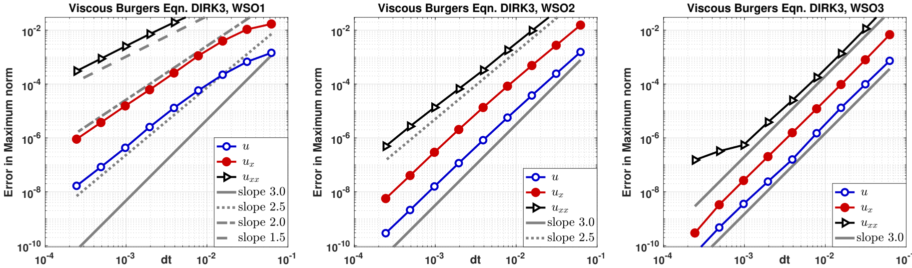
WSO2 recovers full 3rd order in u.

Viscous Burgers' Equation

$$u_t + u u_x = \nu u_{xx} + f \ , \quad \nu = 0.1$$

Manufactured Solution

$$u(x,t) = \cos(2+10t)\sin(0.2+20x)$$



Even for this nonlinear problem, the errors scale in accordance to: WSO2 recovers full 3rd order in u, and WSO3 also recovers full 3rd order in u_{xx}.

However: WSO does not guarantee this, because linear theory. These results are better than the theory.

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Van der Pol Oscillator

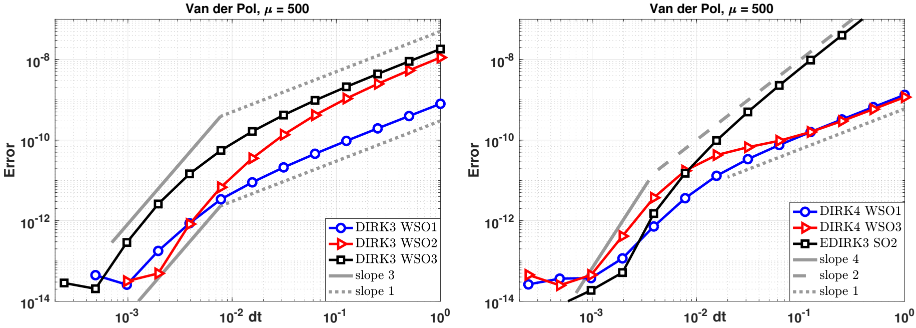
$x' = y \quad \text{and} \quad y' = \mu(1 - x^2)y - x$

$(x(0), y(0)) = (2, 0)$

Parameters

$\mu = 500,$

$T_{\text{final}} = 10.$



For this nonlinear problem, WSO does not remedy order reduction.

Interesting: EDIRK with stage order 2 remedies order reduction, but does so by yielding larger errors for large time steps.

Overview

Breaking the Weak Stage Order 3 Barrier via WSO Theory

- Recall: WSO eigenvector criterion ($A\vec{\tau}^{(j)} = \mu_j \vec{\tau}^{(j)}$) limited to $\tilde{q} \leq 3$.
- But: General WSO ($\vec{b}^T A^\ell \vec{\tau}^{(j)} = 0$ for $\ell, 1 \leq j \leq \tilde{q}$) is not.
- Goal: find L-stable DIRK schemes with $(p, \tilde{q})$: (4,4), (5,4), (5,5).
- Challenge: Brute force optimization does not work well.
- [A. Biswas PhD thesis]: Use WSO theory to construct feasible (starting) schemes with A-invariant subspace $\vec{\tau}^{(j)} \in \mathcal{V}$ having $\dim(\mathcal{V}) = 2$.

$(s, p, \tilde{q}) = (7, 4, 4)$

1.290066345260422e-01	1.290066345260422e-01						
4.492833135308985e-01	3.315354455306989e-01	1.177478680001996e-01					
9.919659086525534e-03	-8.009819642882672e-02	-2.408450965101765e-03	9.242630648045402e-02				
1.230475897454758e+00	-1.730636616639455e+00	1.513225984674677e+00	1.221258626309848e+00	2.266279031096887e-01			
2.978701803613543e+00	1.475353790517696e-01	3.618481772236499e-01	-5.603544220240282e-01	2.455453653222619e+00	5.742190161395324e-01		
1.247908693583052e+00	2.099717815888321e-01	7.120237463672882e-01	-2.012023940726332e-02	-1.913828539529156e-02	-5.556044541810300e-03	3.707277349712966e-01	
1.000000000000000e+00	2.387938238483883e-01	4.762495400483653e-01	1.233935151213300e-02	6.011995982693821e-02	6.553618225489034e-05	-1.270730910442124e-01	3.395048796261326e-01
	2.387938238483883e-01	4.762495400483653e-01	1.233935151213300e-02	6.011995982693821e-02	6.553618225489034e-05	-1.270730910442124e-01	3.395048796261326e-01

$(s, p, \tilde{q}) = (12, 5, 4)$

2.345371908646273e-01	2.345371908646273e-01											
7.425871511958302e-01	6.874344413888787e-01	5.515270980695153e-02										
3.296674204078279e-02	-1.183552669539587e-01	5.463563002913454e-03	1.458584459918280e-01									
7.379564717201322e-01	-1.832235204042292e-01	5.269029412008775e-02	8.203685085133529e-01	4.812118949092085e-02								
2.376643917109970e-01	9.941572060659400e-02	4.977904930055774e-03	5.414758174284321e-02	-1.666571741820749e-03	8.078975617332473e-02							
1.750238160341377e+00	-9.896614721582678e-01	2.860682690577833e+00	-1.236119341063179e+00	2.130219523351530e+00	-1.260655031676537e+00	2.457717913099987e-01						
2.990308150015702e+00	-5.656238413439102e-02	1.661985685769353e-01	6.464600922362508e-01	6.608854962269927e-01	3.736054198873429e-01	6.294456964407685e-01	5.702752607818027e-01					
2.882138003112822e+00	8.048962104724392e-01	-6.232034990249100e-02	5.737234603323347e-01	-9.613723511489970e-02	5.524106361737929e-01	5.961002486833255e-01	1.978411600659203e-01	3.156238724024008e-01				
2.914399924907188e+00	-1.606381759216300e-01	6.833397073337708e-01	4.734578665308685e-01	8.037708984872738e-01	-1.094498069459834e-02	6.151263362711297e-01	3.908946848682723e-01	8.966103265353116e-02	2.973255537857041e-02			
2.573507348677332e+00	7.074283235644631e-01	4.392037300952482e-01	-3.623592480237268e-02	7.189990308645932e-04	5.820968279166545e-01	3.302003177175218e-01	-2.394564021215881e-01	-7.540283547997615e-03	1.702137469523672e-01	6.268780138721711e-01		
3.567266961364713e+00	1.361197981133694e-01	-7.486549901902831e-01	1.893908350024949e+00	3.940485196730028e-01	6.240233526545023e-02	7.511983862200027e-01	-5.283465265730526e-01	-1.661625677872943e+00	9.998723833190827e-01	1.377776742457387e+00	8.905676409277480e-01	
1.000000000000000e+00	-7.433675378768276e-01	1.490594423766965e-01	-2.042884056742363e-02	8.565329438087443e-04	1.357261590983184e+00	2.067512027776675e-03	9.836884265759428e-02	-1.357936974507222e-02	-5.428992174996300e-02	-3.803299038293005e-02	-9.150525836295019e-03	2.712352651694511e-01
	-7.433675378768276e-01	1.490594423766965e-01	-2.042884056742363e-02	8.565329438087443e-04	1.357261590983184e+00	2.067512027776675e-03	9.836884265759428e-02	-1.357936974507222e-02	-5.428992174996300e-02	-3.803299038293005e-02	-9.150525836295019e-03	2.712352651694511e-01

$(s, p, \tilde{q}) = (12, 5, 5)$

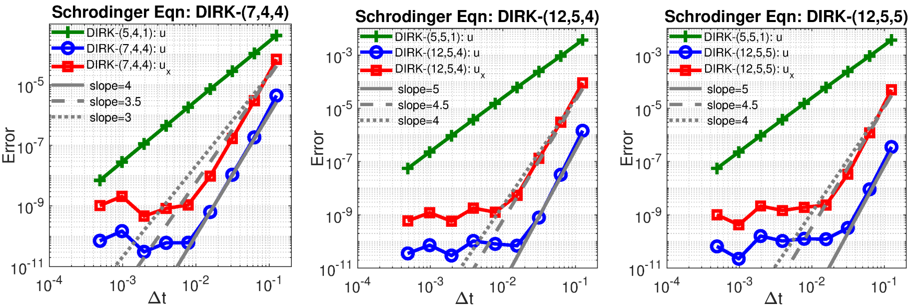
4.113473525867655e-02	4.113473525867655e-02											
2.269850660400232e-01	1.603459327727949e-01	6.663913326722831e-02										
6.222969192243949e-01	-3.424389044264752e-01	8.658006324816373e-01	9.893519116923277e-02									
1.377989449231234e+00	9.437182028870806e+00	-1.088783359642350e+01	2.644025436733866e+00	1.846155800500574e-01								
1.259841986970257e+00	-3.425409029430815e-01	5.172239272544332e-01	9.163589909678043e-01	5.225142808845742e-02	1.165485436026433e-01							
1.228350442796143e+00	-2.094441177460360e+00	2.577655753533404e+00	5.704405293326313e-01	1.213637180023516e-01	-4.752289775376601e-01	5.285605969257756e-01						
1.269855051265635e+00	3.391631788320480e-01	-2.797427027028997e-01	1.039483063369094e+00	5.978770926212172e-02	-2.132900327070380e-01	8.344318363436753e-02	2.410106515779412e-01					
2.496200652601413e+00	5.904282488642163e+00	3.171195765985073e+00	-1.236822836316587e+01	-4.989519066913001e-01	2.160529620826442e+00	1.916104322021480e+00	1.988059486291180e+00	2.232092386922440e-01				
2.783820705331141e+00	4.616443509508975e-01	-1.933433560549238e-01	-1.212541486279519e-01	6.662362039716674e-02	4.254912950625259e-01	7.856131647013712e-01	8.369551389357689e-01	1.604780447895926e-01	3.616125951766939e-01			
3.337101417632813e+00	-7.087669749878204e-01	6.466527094491541e-01	4.758821526542215e-01	-2.570518451375722e-01	1.123185062554392e+00	5.546921612875290e-01	3.192424333237050e-01	3.612077612576969e-01	5.866779836068974e-01	2.353799736246102e-01		
4.173423133876636e+00	4.264162484855930e-01	1.322816663477840e+00	4.245673729758231e-01	-2.530402764527700e+00	-7.822016897497742e-02	1.054463080605071e+00	4.645590541391895e-01	1.145097379521439e+00	4.301337846893282e-01	1.499513057076809e+00	1.447942640822165e-02	
1.000000000000000e+00	1.207394392845339e-02	5.187080074649261e-01	1.121304244847239e-01	-4.959806334780896e-03	-1.345031364651444e+00	3.398828703760807e-01	8.159251531671077e-01	-2.640104266439604e-03	1.439060901763520e-02	-6.556567796749947e-03	6.548135446843367e-04	5.454220210658036e-01
	1.207394392845339e-02	5.187080074649261e-01	1.121304244847239e-01	-4.959806334780896e-03	-1.345031364651444e+00	3.398828703760807e-01	8.159251531671077e-01	-2.640104266439604e-03	1.439060901763520e-02	-6.556567796749947e-03	6.548135446843367e-04	5.454220210658036e-01

Schrödinger Equation

$$u_t = \frac{i\omega}{\xi^2} u_{xx} + f$$

Manufactured Solution

$u(x,t) = \exp(-(x-t)^2)\cos(10x)\sin(t)$
with $\omega = 2\pi$, $\xi = 20$, and $T = 1.2$.



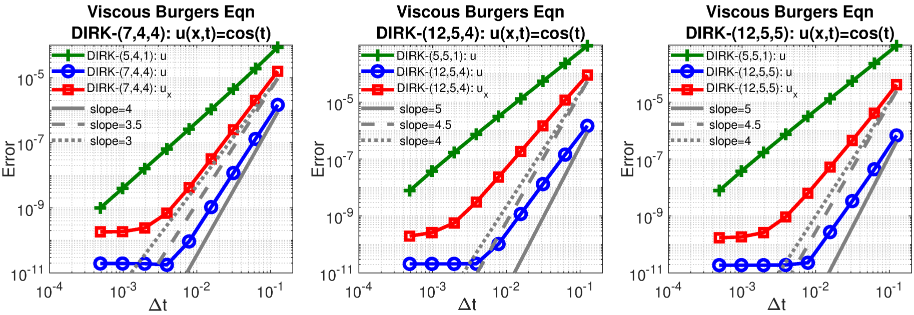
Same outcomes as before: WSO recovers full order.

Viscous Burgers' Equation

$$u_t + uu_x = \nu u_{xx} + f\ , \quad \nu = 0.1$$

Manufactured Solution

$$u(x,t) = \cos(t)$$



For this nonlinear problem,

- WSO does *not* recover the full order;
- but: WSO is beneficial (observed order 3).

Overview

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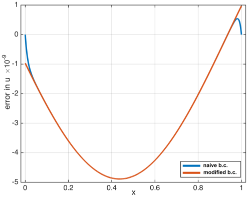
Insights

- Order reduction (OR) is a fundamental problem in high-order time stepping.
- In PDE problems, OR manifests in numerical boundary layers.
- Weak stage order (WSO) is a mechanism to remedy OR.
- Linear concept. But also benefits in some nonlinear problems.
- WSO compatible with DIRK structure. New L-stable stiffly accurate DIRKs.
- Multistep methods do *not* suffer from order reduction (not even in u_x).

Further Thoughts

- General WSO theory in upcoming manuscript.
- WSO conditions not dissimilar to other (ROW) conditions from [Scholz 1989], [Ostermann, Roche 1993].
- If willing to be spatial-specific: modified b.c.
- Here fully implicit. Splitting methods offer more flexibility to remedy OR [S. Roberts PhD thesis].

Modified b.c.



- Rothe vs. MOL: insight on error's spatial boundary layers. If errors peaked near boundaries, why care? lift/drag on airplane wing.

arxiv.org/abs/1712.00897 arxiv.org/abs/1811.01285