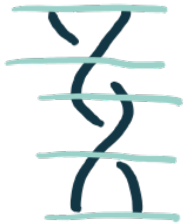
Braids & the FDTC

"how many full twists does β contain?"

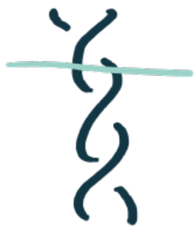
★ read braids from bottom to top

β



FDTC

1



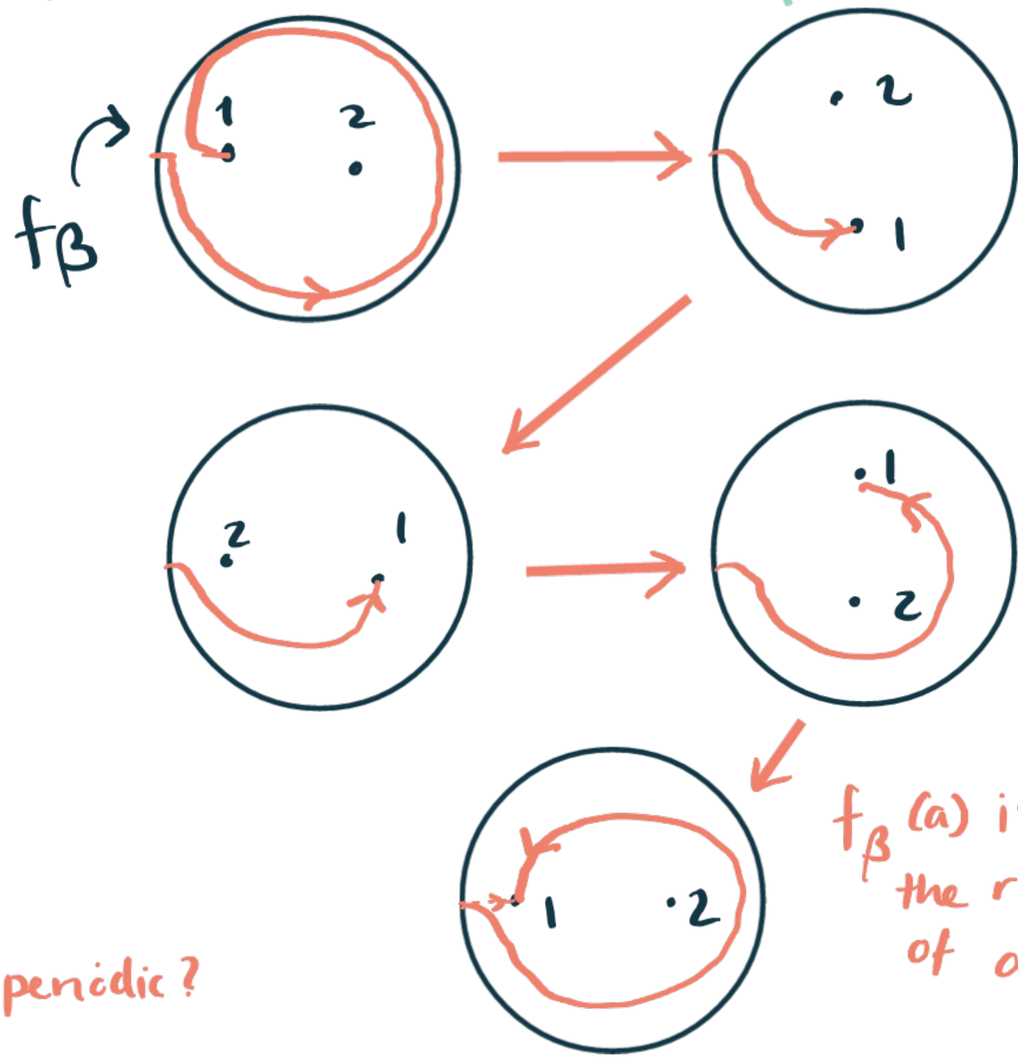
FDTC

3/2

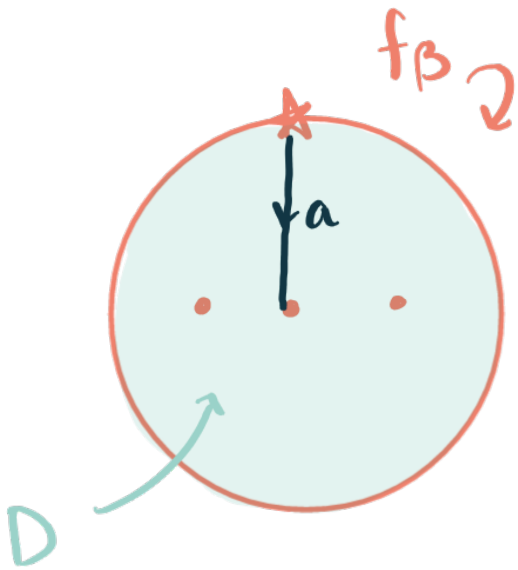
FDTC

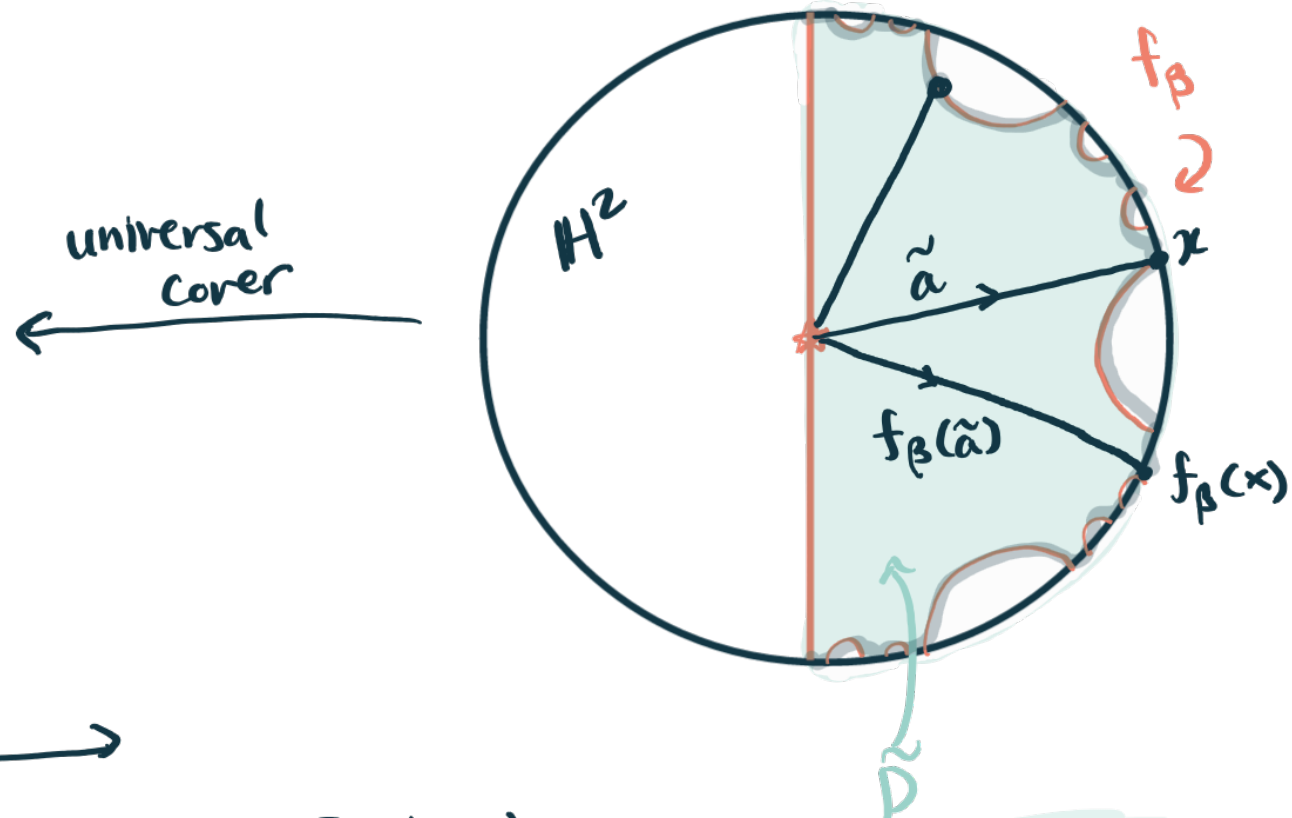
1

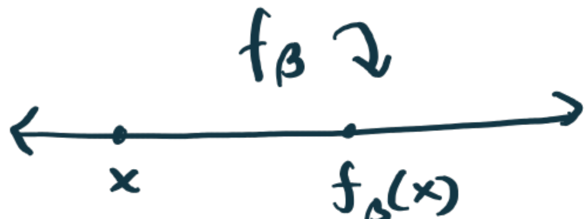
what if β is not periodic?



$f_\beta(a)$ is to the right of a







Δ^2 ↻ by translation by 1

$f_\beta \in \mathrm{Homeo}^+(\mathbb{R})$ (actually in $\widetilde{\mathrm{Homeo}^+(S^1)}$)

$\mathrm{FDTC}(\beta) :=$ translation # $f_\beta := \lim_{n\to\infty} \dfrac{f^n(x)-x}{n}$ doesn't depend on x

$$\lim_{n\to\infty} \frac{x+n-x}{n} = 1 \qquad \lim_{n\to\infty} x+n+\frac{n}{2}-x$$

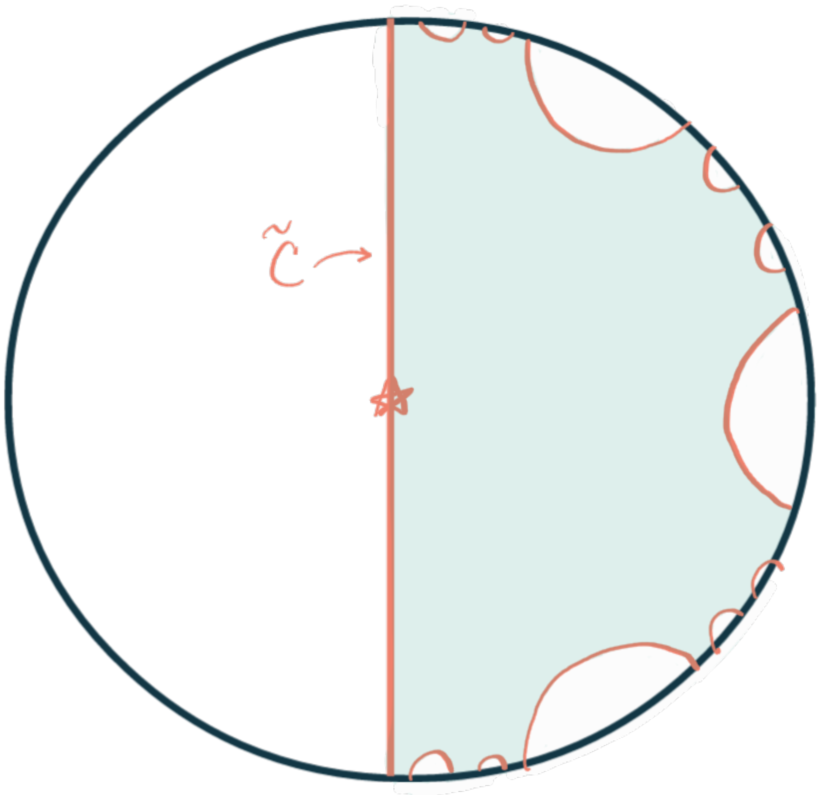
a good reference for FDTC of braids is Malyutin's "The twist number of (closed) braids" - he does not seem to know the definition of FDTC for more general mapping class groups so his work is somewhat disjoint from other literature; see also Section 4 of Ito-Kawamuro's "Essential open book foliations and FDTC" esp. Section 4.3

Properties of the FDTC

- $FDTC(\Delta^{2n}) = n$
- $FDTC(\Delta^{2n}\beta) = n + FDTC(\beta)$
- conjugation invariant
- if f_β sends an arc to the right then $FDTC(\beta) \geq 0$

Claim If f_β sends an arc a to the right then $\mathrm{FDTC}(\beta) \geqslant 0$

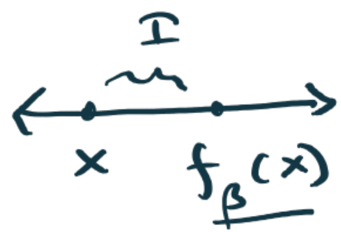
$$\lim_{n\to\infty} \frac{f_\beta^n(x) - x > 0}{n} \geqslant 0$$

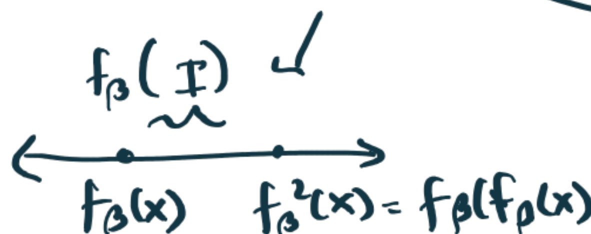


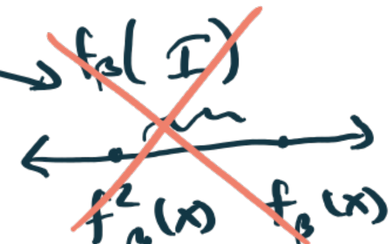
going to show: $f_\beta(a)$ is to the right of a

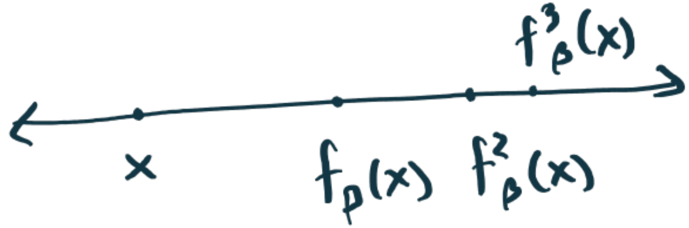
$\Longrightarrow$

$f_\beta^n(a)$ is to the right of a







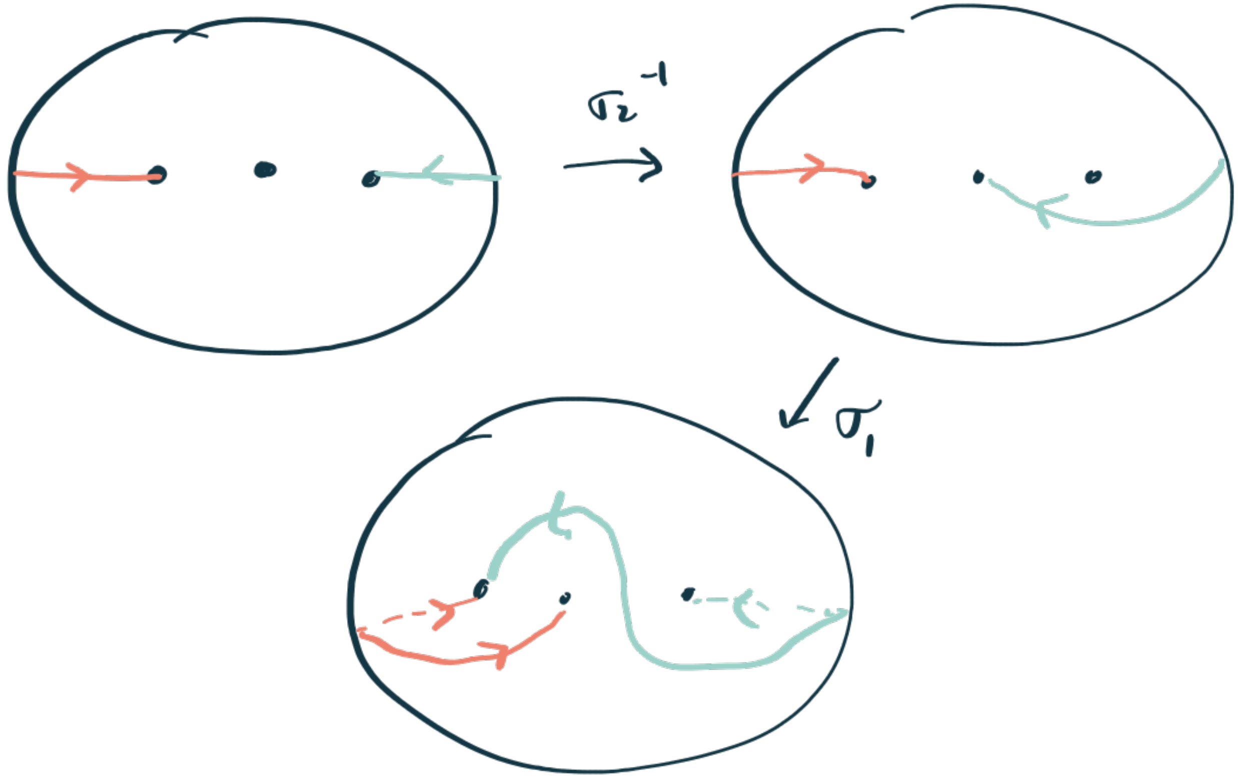


$f_\beta^3(x)$
x
$f_\beta(x)$
$f_\beta^2(x)$

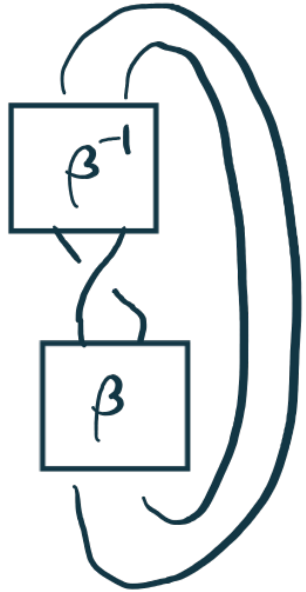
Corollary: If f_β sends an arc a to the right and an arc b to the left then $FDTC(\beta)=0$

p. A

FDTC = 0!



Closures of braids



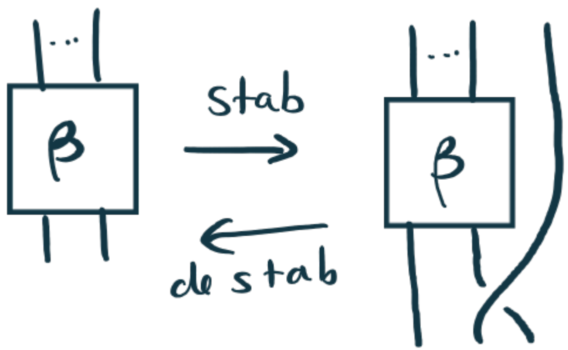
conjugate braids have the same closure

non conjugate braids with same closure

Alexander's Thm Any link $L \hookrightarrow S^3$ can be represented as a braid closure $\hat{\beta}$.

Markov's Thm Any two braids which have the same closure are related by:

- (braid moves)
- conjugation
- (de) stabilization



Goal: Use FDTC (β) to study $\hat{\beta}$

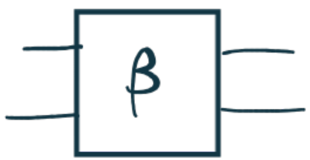
Problem:

FDTC $= 1/3$ FDTC $= -1/2$ FDTC $= 0$

all close to the unknot!

★ FDTC(β) is not an invariant of $\hat{\beta}$

Example: Given any $\beta \in B_n$ there is $\beta' \in B_{n+2}$ with $\hat{\beta} = \hat{\beta'}$ but FDTC $(\beta') = 0$.



The FDTC and ... the braid index

braid index for a link $L \hookrightarrow S^3$ is the fewest number of strands needed to represent L as a braid closure.

has braid index 4 by Feller-Hubbard

braid index 1

braid index 2

braid index 2

Feller-Hubbard's Thm If $|\mathrm{FDTC}(\beta)| \geq n$ for $\beta \in B_n$ then β realizes the braid index of $\hat{\beta}$.

...and the L-space conjecture

via Roberts

Boyer-Hu's Thm If β an odd strand ($\beta \in B_{2n+1}$) P.A. braid with $|FDTC(\beta)| \geq 2$ then $\Sigma_{2n}(\hat{\beta})$ admits a taut foliation for all $n \geq 1$

(and • $\pi_1(\Sigma_{2n}(\hat{\beta}))$ is LO
• $\Sigma_{2n}(\hat{\beta})$ is not an L-space)

3-manifold obtained as the index $2n$ cyclic branched cover of S^3 branched along $\hat{\beta}$.

generally, FDTC helps construct taut foliations

...and the geometry of knot complements

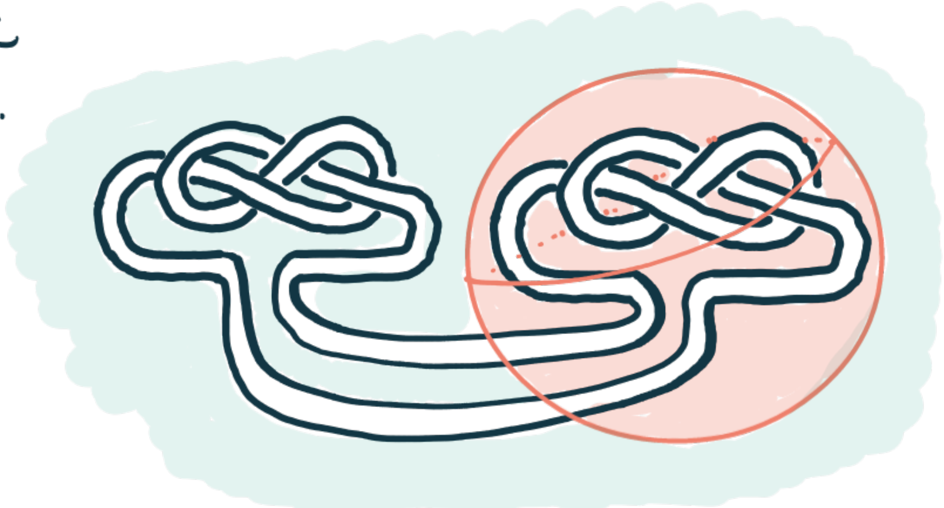
Malyutin's Thm if $|FDTC(\beta)|>1$ then $\hat{\beta}$ is prime

knot

Ito-Kawamuro's Thm if $|FDTC(\beta)|>1$ then $\hat{\beta}$ is a:

$4_1 \# 4_1$ doesn't have any braid rep with large FDTC

1. satellite knot iff f_β is reducible
2. torus knot iff f_β is periodic
3. hyperbolic knot iff f_β is p.A.



periodic

P. A.

reducible

all close to the unknot!

Claim Any link has a braid representative which is reducible.

Other things I wanted to mention

Using FDTC to study . . .

- conjugacy classes with the same closure – see Malyutin-Netsvetaev
- surfaces bounded by $\hat{\beta}$ (Seifert|slice genus) – see Ito, upcoming work of Feller
- contact stuff – transverse braids

FDTC of more general mapping classes

FDTC of braids not in S^3