Braids & the FDTC
"how many full twists does \( \beta \) contain?"

FDTC 1

FDTC \( \frac{3}{2} \)

FDTC 1

what if \( \beta \) is not periodic?

\( f_\beta \) (a) is to the right of a

\( f_\beta \) read braids from bottom to top
The diagram illustrates a map \( f_\beta \) from \( \mathbb{R} \) to \( \mathbb{R} \) with \( f_\beta(x) \). The map is part of the homeomorphism group \( \text{Homeo}^+(\mathbb{R}) \) (actually in \( \text{Homeo}^+(S^1) \)).

FDTC (\( \beta \)):= translation \( #f_\beta := \lim_{n \to \infty} \frac{f^n(x) - x}{n} \) doesn't depend on \( x \).
\[
\lim_{n \to \infty} \frac{x + n - x}{n} = 1 \\
\lim_{n \to \infty} x + h + \frac{n}{2} - x
\]

A good reference for FDTC of braids is Malyutin's "The twist number of (closed) braids". He does not seem to know the definition of FDTC for more general mapping class groups, so his work is somewhat disjoint from other literature; see also Section 4 of Ho-Kawamura's "Essential open book foliations and FDTC" esp. Section 4.3.
Properties of the FDTC

- \( \text{FDTC}(\Delta^{2n}) = n \)
- \( \text{FDTC}(\Delta^{2n}\beta) = n + \text{FDTC}(\beta) \)
- Conjugation invariant
- If \( f_\beta \) sends an arc to the right then \( \text{FDTC}(\beta) \geq 0 \)
Claim: If $f_\beta$ sends an arc $a$ to the right then $\text{FDTC}(\beta) \geq 0$

$$\lim_{n \to \infty} \frac{f_\beta^n(x) - x}{n} \geq 0$$

Going to show: $f_\beta(a)$ is to the right of $a$

$\Rightarrow$

$f_\beta^n(a)$ is to the right of $a$

$f_\beta(x)$

$f_\beta(I)$

$f_\beta^2(x) = f_\beta(f_\beta(x))$
Corollary: if $f_\beta$ sends an arc $a$ to the right and an arc $b$ to the left then $FDTC(\beta) = 0$.
Closures of braids

conjugate braids have the same closure

non-conjugate braids with same closure
Alexander's Thm

Any link \( L \to S^3 \) can be represented as a braid closure \( \hat{\beta} \).

Markov's Thm

Any two braids which have the same closure are related by:

- (braid moves)
- conjugation
- (cl.stabilization)

\[ \begin{array}{c}
\beta \\
\text{stab}
\end{array} \quad \begin{array}{c}
\beta \\
\text{cl.stab}
\end{array} \]
Goal: Use $\text{FDTC}(\beta)$ to study $\hat{\beta}$

Problem:

$\text{FDTC}=\frac{1}{3}$  $\text{FDTC}=-\frac{1}{2}$  $\text{FDTC}=0$

all close to the unknot!

* $\text{FDTC}(\beta)$ is not an invariant of $\hat{\beta}$

Example: Given any $\beta \in B_n$, there is $\beta' \in B_{n+2}$ with $\hat{\beta} = \hat{\beta'}$ but $\text{FDTC}(\beta') = 0$. 

\[ \text{Diagram of knots with different FDTC values} \]
The FDTC and... the braid index

Braid index for a link \( L \subset S^3 \) is the fewest number of strands needed to represent \( L \) as a braid closure.

Feller-Hubbard's Thm: If \( |FDTC(\beta)| \geq n \) for \( \beta \in B_n \) then \( \beta \) realizes the braid index of \( \hat{\beta} \).
...and the L-space conjecture

via Roberts

Boyer-Hu's Thm

If $\beta$ an odd strand ($\beta \in B_{2n+1}$) P.A. braid with

1. $\text{FDTC}(\beta) \nmid 2$

then $\Sigma_{2n}(\hat{\beta})$ admits a taut foliation for all $n \geq 1$

$\Rightarrow$

3-manifold obtained as the

index 2n cyclic branched cover of

$S^3$ branched along $\hat{\beta}$.

$\lceil \pi_1(\Sigma_{2n}(\hat{\beta})) \text{ is L0} \rceil$

$\lceil \Sigma_{2n}(\hat{\beta}) \text{ is not an L-space} \rceil$

generally, $\text{FDTC}$ helps construct taut foliations
...and the geometry of knot complements

**Malyutin's Thm** if $|\text{FDTC}(\beta)| > 1$ then $\widehat{\beta}$ is prime

**Ito-Kawamuro's Thm** if $|\text{FDTC}(\beta)| > 1$ then $\widehat{\beta}$ is a:

1. satellite knot iff $f_\beta$ is reducible
2. torus knot iff $f_\beta$ is periodic
3. hyperbolic knot iff $f_\beta$ is p.A.

4. #41 doesn't braid any knot with large FDTC

[Diagram of knot complements]
periodic

P. A.

reducible

all close to the unknot!

Claim Any link has a braid representative which is reducible.
Using FDTC to study...

- conjugacy classes with the same closure - see Malyutin-Netsvetaev
- surfaces bounded by \( \hat{\beta} \) (Seifert/slice genus) - see Ito, upcoming work of Feller
- contact stuff - transverse braids

FDTC of more general mapping classes
FDTC of braids not in \( S^3 \)