Since this is a longer talk, I can expand a bit:

**Def.** Two $n$-component links $L_1, L_2 \subset S^3$ are concordant if

$$S^3 \times I \leftarrow S^3 \times S^3 \times \{0\} \leftarrow S^3 \times S^3 \times \{1\}$$

**Note:** homotopic $\iff$ link homotopic $\iff$ concordant $\iff$ isotopic

**Why you should care:** Every 3-manifold can be obtained by surgery on some link, and concordant links have homology cobordant complements (rel boundary)
Fact: \( \text{Links } \subset S^3 \) form a group, but not in an easy way.

Aside: \( \text{Knots } \subset S^3 \) is a group, and a mysterious one at that (despite being abelian).

One way to form a link concordance group:
- Cut open
- Isotopic in \( S^3 \)

Thm: (Le Dimet ‘88) These knotted and linked arcs in \( D^2 \times I \) (called string links) form a group \( \mathcal{L}_n \) up to (string link) concordance with stacking as the operation, and

\[
\text{pure braid group on } n \text{ strands } \xrightarrow{} \mathcal{P}B_n \leq \mathcal{L}_n \xleftarrow{} \text{string link concordance group on } n \text{ strands}
\]
Thm: (K. 20) If \( n > 1 \), \( \mathcal{L}_n \) /Normal closure (\( PB_n \)) is non-abelian.

Not much is known beyond this!

Q: • How “close” to being abelian is this? Is it solvable? Nilpotent?
• \( PB_n \) is torsion-free. What are the torsion elements of this quotient?
• Does \( 1 \rightarrow \text{Ncl}(PB_n) \rightarrow \mathcal{L}_n \rightarrow \mathcal{L}_n / \text{Ncl}(PB_n) \rightarrow 1 \) split for \( n > 2 \)?
  (De Campos ’05: Splits for \( n = 2 \))

Note: \( \sigma = 1 \in \mathcal{L}_n \) if and only if \( \sigma \) is concordant to the unlink, but usually string link representatives are not unique.

Big Q: • Is there a link concordance group \( G \) where each link has a unique representative?
• How about where the set map \( \text{links} (S^3) \rightarrow G \) is a bijection?
Quickly on to something else I like:

Recall that concordant links lead to homology cobordant 3-manifolds in a few different ways, and that every 3-manifold can be split into two handlebodies glued together by a diffeomorphism (called a Heegaard splitting).

\[ S^3 = \begin{array}{c}
\begin{array}{c}
\text{glue}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{on by}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{c}
\end{array}
\end{array}\]

Fact: (Morita '91) Every $ZHS^3$ can be obtained by gluing by a diffeomorphism that acts trivially on $H_1$ (gluing surface).

Thm: (WIP: Afton-K.-Lidman) Let $Y$ be a $ZHS^3$ with $Y = H_g \cup_c H_g$ where $\varphi \in \Gamma_{g,1} = \Xi$ of $\Sigma_g (ZH_1(\Sigma_g)$ trivially) $\leq \text{Mod}_{g,1}$. Then

\[ |d(Y)| \leq C \| \varphi \| \text{ word length in } \Gamma_{g,1} \]
Thanks for listening and please come chat!

Since some of you are used to seeing my cats on zoom when I speak, here they are.