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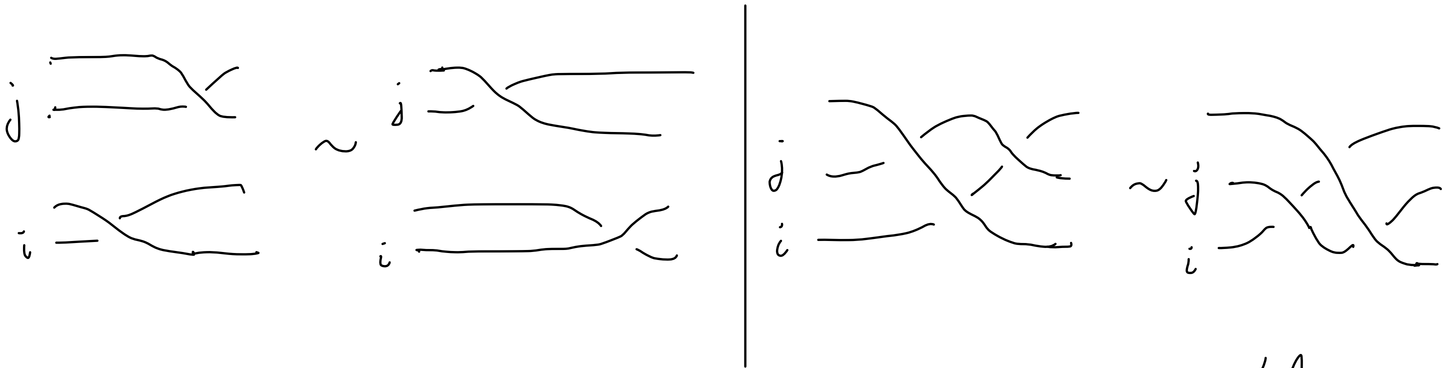
DAY MONTH

Artin groups and normal forms

Braid group: $B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j|=1 \end{array} \right\rangle$



To work with B_n algebraically, one uses the monoid of positive braids

$B_n^+ = \langle \text{same presentation} \rangle^+ \leftarrow$ (presentation as a monoid: no inverses)

Fact: B_n^+ embeds in B_n [Garside '69]

(two positive words which are equivalent in B_n, are also equivalent in B_n^+).

Reason: B_n^+ satisfies Ore's conditions: it is cancellative and every two elements have a least common right multiple.

$\Rightarrow$ B_n^+ embeds in its groups of fractions, which is B_n //

Hence, every braid x can be decomposed as

$\boxed{x = a^{-1}b}$ with $a, b \in B_n^+$. By symetry, also

as $\boxed{x = cd^{-1}}$

$$B_n^+ = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j|=1 \end{array} \right\rangle^+$$

Relations are homogeneous $\Rightarrow$ All words representing a given element have the <u>same length</u>

$\forall a \in B_n^+$, $|a|$ = length of any representative.

Standard generators $\{\sigma_1, \dots, \sigma_{n-1}\} = \{$elts. of length 1$\}$

They are called atoms

Lattice order in B_n^+

Given $a, b \in B_n^+$, we say that a is a prefix of b ($a \preccurlyeq b$) if $\exists\, c \in B_n^+ \;/\; ac = b$.

[Similarly, there is a suffix order $\succcurlyeq$]

Properties of the partial order $\preccurlyeq$

① It is invariant under left-multiplication

$$a \preccurlyeq b \iff xa = xb$$

② It is a lattice order:

$\forall\, a, b \in B_n^+$, $\quad \exists!\; a \wedge b \quad$ (g.c.d. or meet)

$\qquad\qquad\qquad \exists!\; a \vee b \quad$ (l.c.m. or join)

Examples:

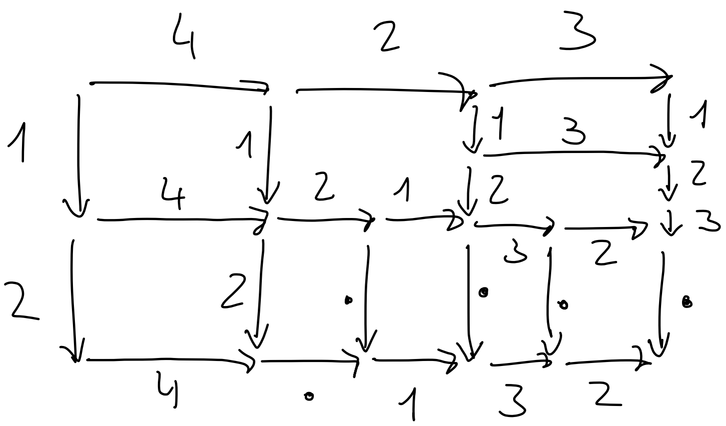
$$\sigma_1 \vee \sigma_3 = \sigma_1\sigma_3 = \sigma_3\sigma_1$$

$$\sigma_1 \vee \sigma_2 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

(Can be easily shown, by length arguments)

For longer elements, one can draw a diagram

$$\sigma_1\sigma_2 \vee \sigma_4\sigma_2\sigma_3 = \sigma_1\sigma_2\,\sigma_4\sigma_1\sigma_3\sigma_2 = \sigma_4\sigma_2\sigma_3\,\sigma_1\sigma_2\sigma_3$$

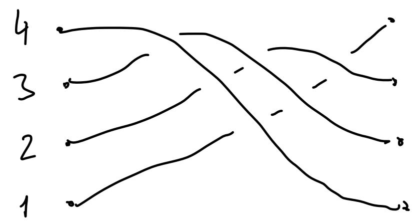


To compute gcd's, we need more properties.

Garside element

Lcm of all standard generators:

$$\boxed{\Delta = \sigma_1 \vee \sigma_2 \vee \dots \vee \sigma_{n-1}}$$



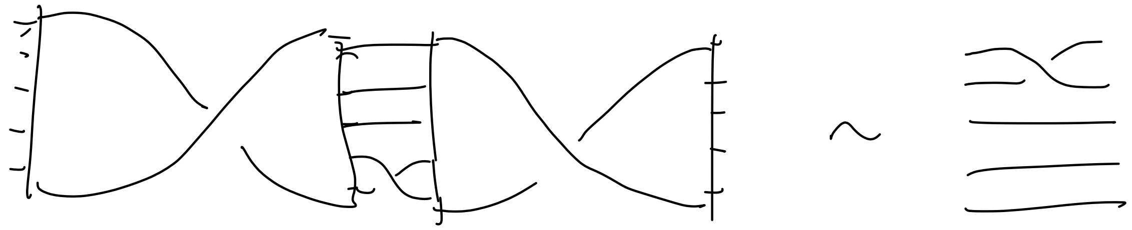
"Half twist"

$$\Delta = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1) = (\sigma_1\cdots\sigma_{n-1})(\sigma_1\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1$$

$$= (\sigma_{n-1}\cdots\sigma_1)(\sigma_{n-1}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})\sigma_{n-1}$$

Properties of Δ:

- By definition: $\forall i \quad \Delta = \sigma_i \cdot \underbrace{\partial(\sigma_i)}_{\in B_n^+}$
- $\forall i \quad \Delta^{-1}\sigma_i\Delta = \sigma_{n+1-i} = \Delta\sigma_i\Delta^{-1}$



- Hence $\Delta^2 \in Z(B_n)$. Actually $Z(B_n) = \langle \Delta^2 \rangle$
  center
  [Chow, 1948]
- Also: $\Delta B_n^+ \Delta^{-1} = B_n^+$ (Conjugation by Δ just permutes the generators)
- Conjugation by Δ preserves the lattice structure.
- Every $x \in B_n$ can be written as $\boxed{x = \Delta^m \cdot P}$, where $m \in \mathbb{Z}$, $P \in B_n^+$.

<u>Proof</u>: - Write x as a word in $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$.

- Replace each σ_i^{-1} with $\partial(\sigma_i)\Delta^{-1}$.
- "Move" all Δ^{-1}'s to the left, using the conjugation relation. //

Using these facts, Garside solved the word problem in B_n.

Given two words w_1, w_2, do they represent the same element in B_n?

[Garside, 1969] Write $[w_1] = \Delta^{m_1} \cdot [v_1]$, $[w_2] = \Delta^{m_2} \cdot [v_2]$

(Can assume $m_1 = m_2$ by inserting $\Delta^{-k}\Delta^{k}$ in one of them if necessary).

$[w_1] = \Delta^{m}[v_1]$, $[w_2] = \Delta^{m}[v_2]$

Then $[w_1] = [w_2] \Leftrightarrow [v_1] = [v_2]$

Positive words!

Apply all possible relations in B_n^+ to v_1, to generate <u>all</u> words representing $[v_1]$ (a finite set of words).

$[v_1] = [v_2] \Leftrightarrow v_2$ is in that set //

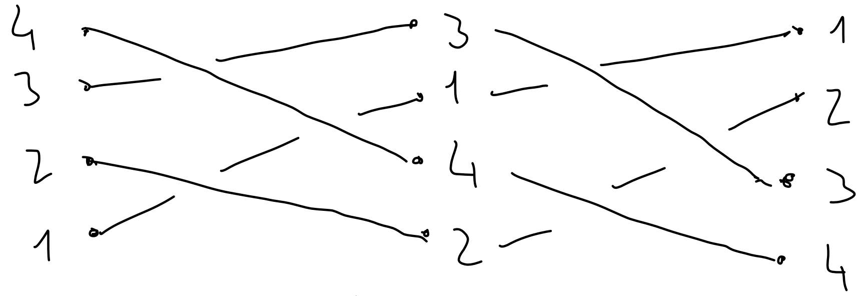
VERY BAD ALGORITHM!

Improvement [Adjan, Deligne, Elrifai-Morton, Thurston]

$x = \Delta^m P$ ← Decompose P in a unique way.

The factors will be the prefixes of Δ, also called simple elements or permutation braids

$$\{\text{Permutations in } S_n\} \xleftrightarrow{\text{bij}} \{\text{prefixes of } \Delta\}$$



$= \Delta$

$\left\{\begin{array}{l}\text{+ve braids in which} \\ \text{every two strands} \\ \text{cross \underline{at most once}}\end{array}\right\} = \{\text{Prefixes of } \Delta\}$

Left greedy decomposition of a positive element P.

$x_1 = \Delta \wedge P$ (biggest simple prefix of P)

Write $P = x_1 P_1$. Define $x_2 = \Delta \wedge P_1$

Write $P = x_1 x_2 P_2$ Define $x_3 = \Delta \wedge P_2$ ... etc

One obtains $P = \underbrace{x_1 \cdots x_r}_{\text{simple factors}}$

All Δ factors, if any, are on the left.

Hence we can write $\forall x \in B_n$:

$$x = \Delta^m x_1 \cdots x_r \quad \text{where } x_i \in Div(\Delta) \setminus \{1, \Delta\}$$

LEFT GREEDY NORMAL FORM

$$x_i = \Delta \wedge x_i \cdots x_r$$

This decomposition is unique.

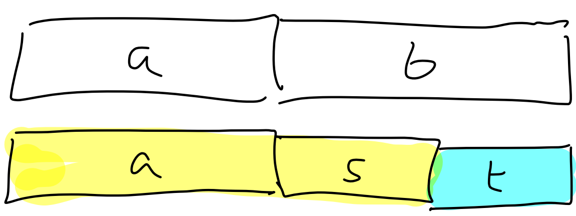
Fact: $\Delta^m x_1 \cdots x_r$ is in normal form $\Longleftrightarrow$

$\forall i$, $\boxed{x_i = \Delta \wedge x_i x_{i+1}}$ (say that $x_i x_{i+1}$ is left-weighted)

"Local property"

HOW TO COMPUTE THE NORMAL FORM

Given $a, b \in Div(\Delta)$, if $a \cdot b$ is not left-weighted,



Let $s = \partial(a) \wedge b$

Write $b = st$.

Then $(as) \cdot t$ is left-weighted.

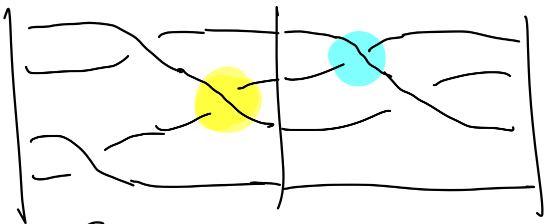
Example: $a = \sigma_1 \sigma_3$, $b = \sigma_3 \sigma_2 \sigma_3$ in B_4

a b

$\sigma_1 \sigma_3 \quad \sigma_3 \sigma_2 \sigma_3$

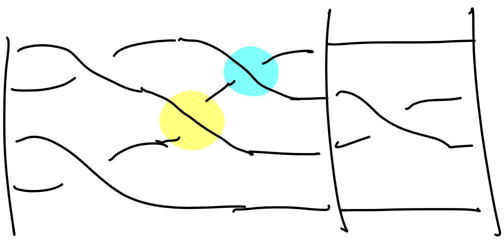
Is there some atom, $\overbrace{\text{prefix of } b}^{\sigma_3 \text{ or } \sigma_2}$, which can be added to a, keeping its simplicity?

σ_2 does:



$\sigma_1 \sigma_3 \sigma_2 \quad \sigma_3 \sigma_2$

Now we can slide σ_3 too:



$\sigma_1 \sigma_3 \sigma_2 \sigma_3 \cdot \sigma_2$ ← Left-weighted factorization

<u>In general</u>: Take P, written as a product of simple elements (maybe atoms):

$$P = s_1 s_2 \cdots s_t$$

- Take any pair of consecutive factors, and make it left-weighted. (If 2nd factor becomes trivial, remove it)
- Repeat.

This terminates, producing the left greedy normal form of P.

<u>Complexity</u>: Given a word of length ℓ in B_n, can compute its normal form in $O(\ell^2 n \log n)$

<u>VERY FAST!</u> [Epstein et al, 92]

↑
Thurston

Some interesting properties

1. The lattice order extends to B_n

$\forall a, b \in B_n \quad a \leq b$ if $\exists c \in B_n^+ / ac = b$.
$\iff a^{-1}b \in B_n^+$

[$a \wedge b$? Take $N \gg 0$ s.t. $\Delta^N a, \Delta^N b \in B_n^+$

Compute $g = \Delta^N a \wedge \Delta^N b \in B_n^+$

Then $\boxed{\Delta^{-N} g = \Delta^{-N}(\Delta^N a \wedge \Delta^N b) = a \wedge b}$]

2. These properties hold in all Artin groups of spherical type

$$A = \left\langle a_1, \dots, a_n \;\middle|\; \begin{array}{l} \text{For each pair } (i,j) \text{, at most one reln.} \\ \text{of type } \underbrace{a_i a_j a_i \cdots}_{} = \underbrace{a_j a_i a_j \cdots}_{} \end{array} \right\rangle$$

Same length

Spherical type if $A / \langle\langle a_1^2, \dots, a_n^2 \rangle\rangle =: W$ is finite

Coxeter group

$\boxed{3}$ Garside groups are groups satisfying this kind of properties

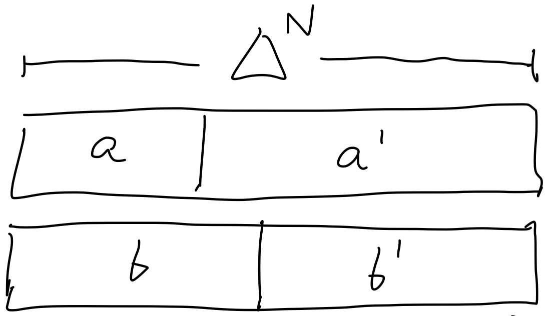
- Monoid $\subset$ Group.
- Lattice order
- Garside element Δ
- finiteness conditions.

$\boxed{4}$ To compute gcd's. Two options:

$a, b \in B_n^+$

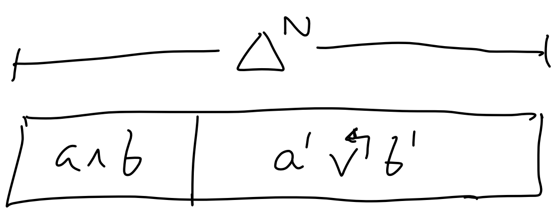
1) Find Δ^N so that $a \preccurlyeq \Delta^N$, $b \preccurlyeq \Delta^N$

Write $\Delta^N = a \cdot a' = b \cdot b'$



Compute $\alpha = a' \vee^{\uparrow} b'$ (lcm for suffix order)

Then $a \wedge b = \Delta^N \alpha^{-1}$



2) Compute the normal forms:

$a = x_1 \cdots x_r \qquad b = y_1 \cdots y_s$

If $x_1 \wedge y_1 = 1 \Rightarrow a \wedge b = 1$.

If $x_1 \wedge y_1 = \alpha_1 \neq 1$, write $a = \alpha_1 a_1$, $b = \alpha_1 b_1$

Repeat with a_1 and b_1.

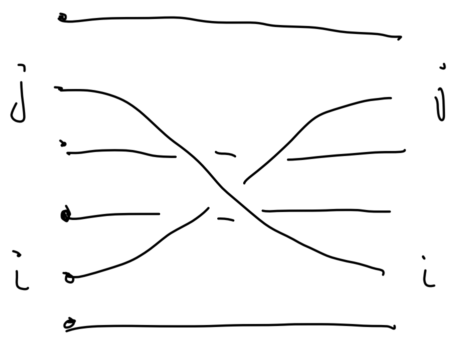
The product $\alpha_1 \alpha_2 \cdots \alpha_t$ when the process ends is $a \wedge b$.

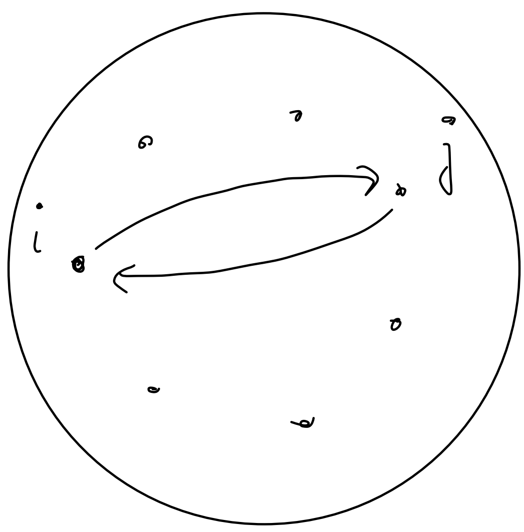
$\boxed{5}$ Other Garside structure in B_n

Birman-Ko-Lee structure (dual structure)

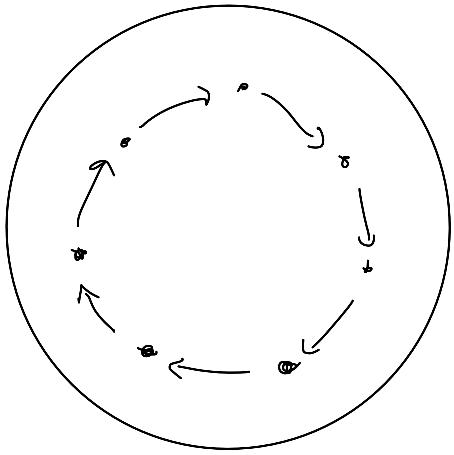
Atoms:

a_{ij}



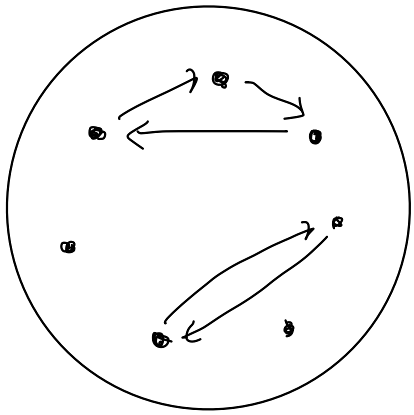


Garside element: δ



Simple elements:

Non-crossing partitions



$$\#(\{\text{simple elements}\}) = C_n \leftarrow \text{Catalan number.}$$

$\boxed{6}$ Using these normal forms, one can solve the conjugacy problem in Garside groups.

7 NP-decomposition. Mixed normal form

From $x = \Delta^m x_1 \cdots x_r$, we have a decomposition

$$x = \alpha^{-1} \cdot \beta \quad , \text{ with } \alpha, \beta \in B_n^+ \quad \left(\begin{array}{c}\text{they could}\\ \text{be trivial}\end{array}\right)$$

Let $d = \alpha \wedge \beta$, and write $\alpha = d\,a$, $\beta = d\,b$

Then $x = a^{-1} \cancel{d^{-1}} \cancel{d}\, b$

$$\boxed{x = a^{-1} b} \quad \text{with } a \wedge b = 1$$

This decomposition is unique. (NP-decomp.)

If we compute the left greedy normal forms of a & b: $\quad a = a_1 \cdots a_r$
$\quad b = b_1 \cdots b_s$

$$\boxed{x = a_r^{-1} \cdots a_1^{-1} b_1 \cdots b_s} \quad \text{Mixed normal form (Thurston)}$$

It is a geodesic in the Cayley graph of B_n with simple elements as generators. [Charney '95]

This normal form provides a bi-automatic structure for B_n [Thurston '92]
[Charney '95]

$\boxed{8}$ Garside groups are torsion-free

Proof: Let G be a Garside group.

Let $x \in G$ be such that $x^m = 1$ for some $m > 0$

Consider $\alpha = 1 \wedge x \wedge x^2 \wedge \cdots \wedge x^{m-1}$

Then $x\alpha = x(1 \wedge x \wedge \cdots \wedge x^{m-1}) =$

$$= x \wedge x^2 \wedge \cdots \wedge x^{m-1} \wedge \underset{\substack{\| \\ 1}}{x^m} = \alpha$$

So $x\alpha = \alpha$ in G. Hence $x = 1$ //