Parallelism in algebraic geometry
Examples with Singular and GPI-Space

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joint work with Janko Böhm SFB/TRR 195
and with Mirko Rahn, Fraunhofer ITWM, Kaiserslautern
(and many others from the SINGULAR and GPI-Space Teams)

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Common knowledge I

Why parallel?

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state of the art hardware:

- multicore computers
- high performance clusters

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- simulation of flows and turbulence

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practical drawback:
management of distribution of resources and data

Common knowledge II

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linear in the number of cores

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- embarrassing
  (nearly no communication between tasks)

Parallelism in Algorithmic Algebraic Geometry

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More recently: inherent natural parallelism of a setting
e.g.:

- gfan
- talk of Lars Kastner using polymake and MPTOPCOM

Separating computation and coordination layer

Tasks in parallel setting (classically):

1. design a mathematical algorithm
2. implement mathematical algorithm

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Success stories along these lines

Fan/Graph-Traversals:

- GIT-fan (Böhm, FK, Rahn, Reinbold):
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Coverings:

- smoothness test (Böhm, Decker FK, Rahn, Ristau, Pfreundt) – see below
- desing. of 2-dim. schemes/$\mathbb{Z}$ (FK, Gaube, Schober, Ristau)
- order zeta functions (FK, Maglione, Schober, Voll)

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- Petri net based workflow engine (automated parallel execution, dependency tracking)

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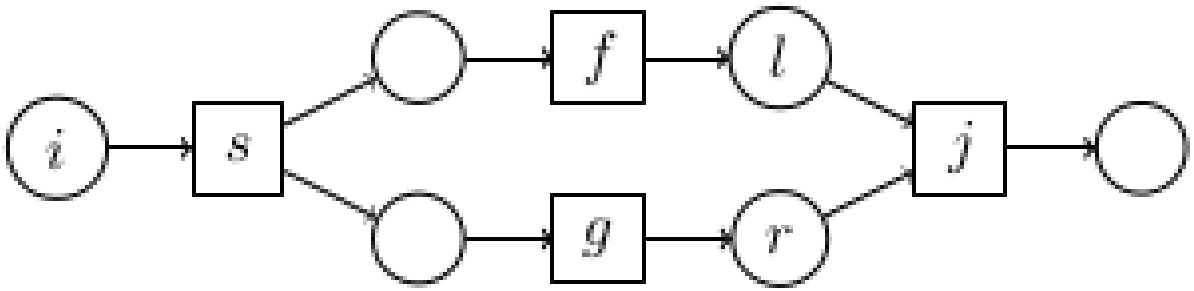
biggest advantage:

Parallelisation without modification to the backend

Petri nets

Petri net:



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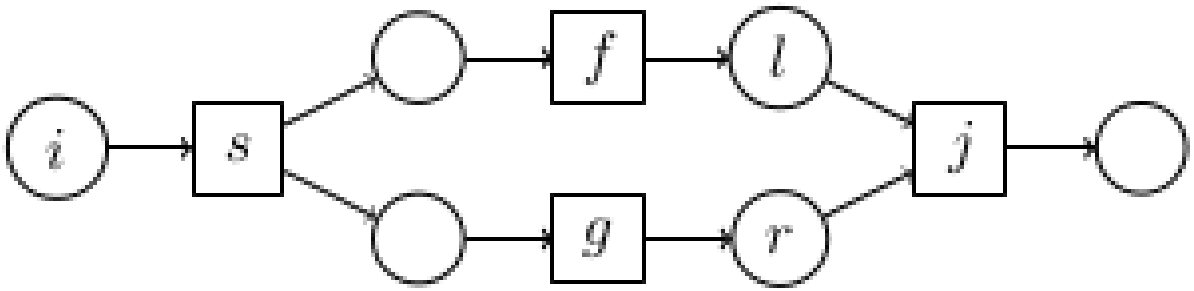

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- 2 kinds of vertices: places (circle) and transitions (box)
- places hold marked tokens
- transitions contain functional units

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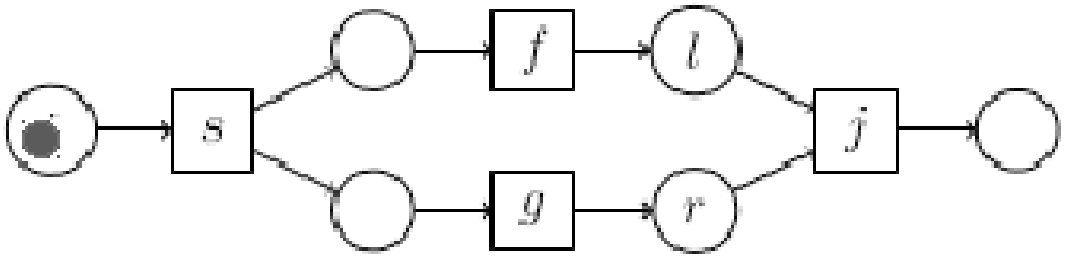

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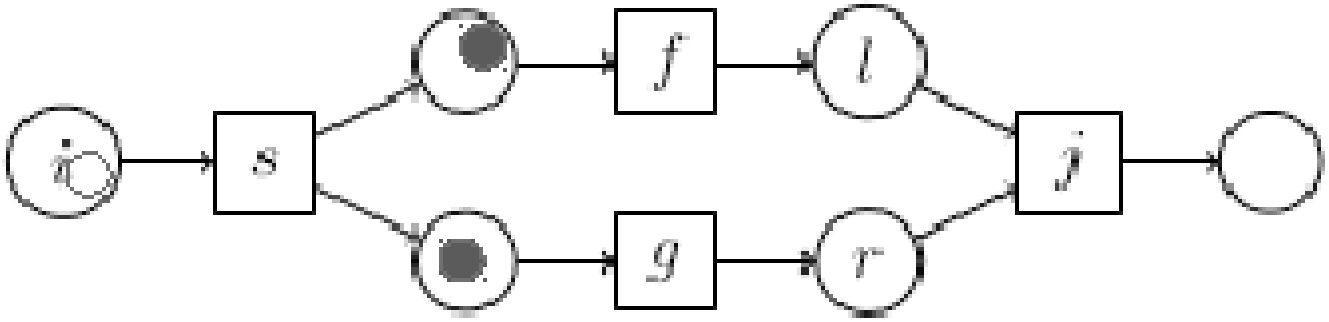

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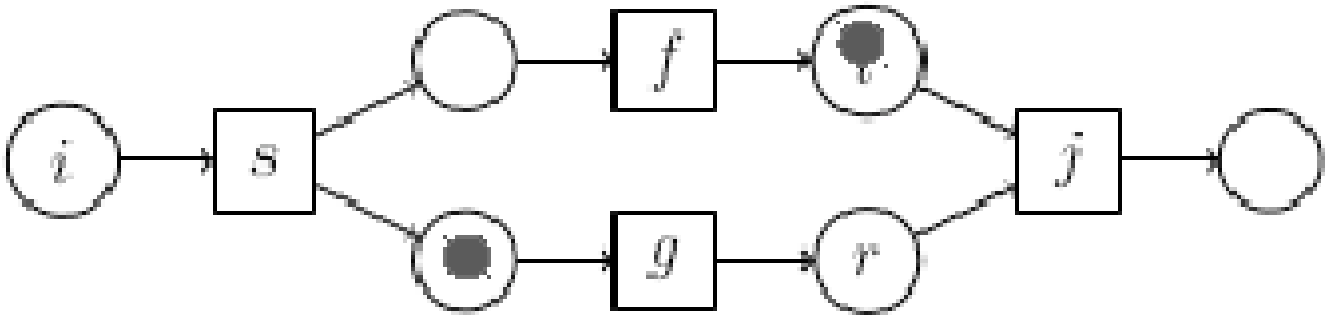

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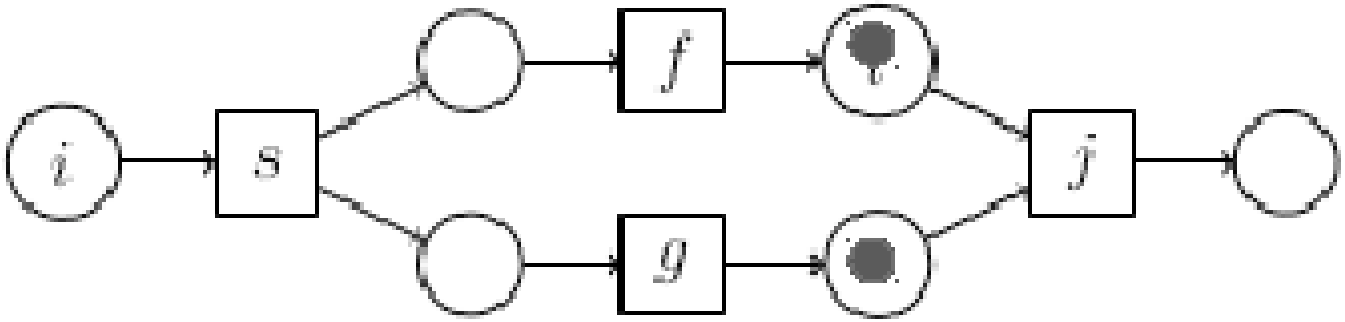

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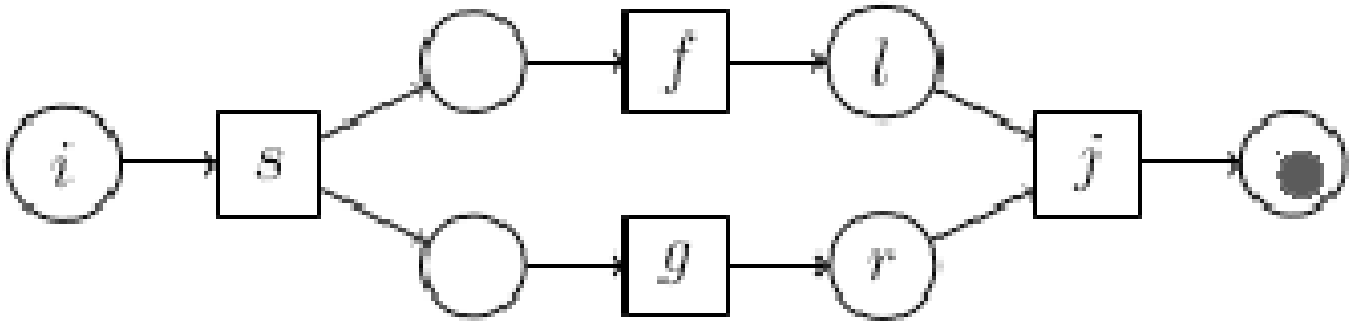

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The singular locus I

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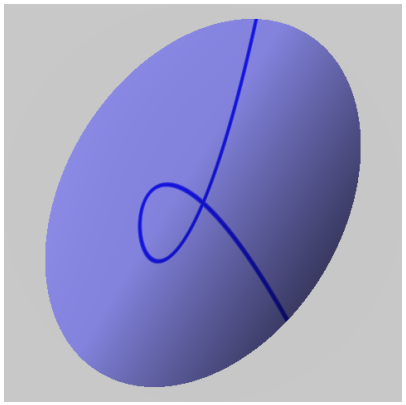

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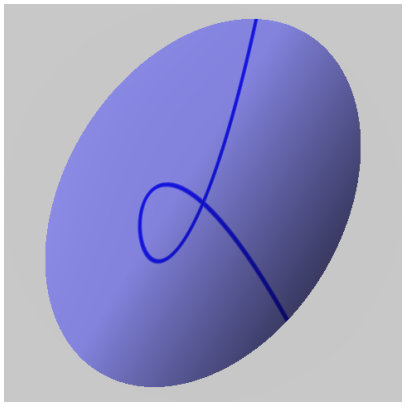

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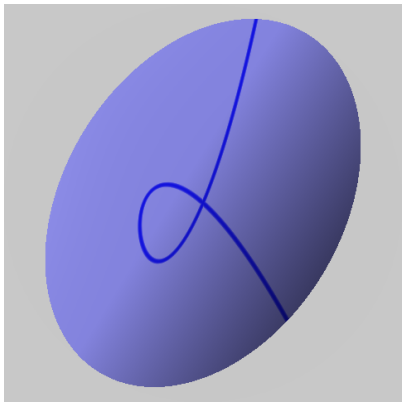

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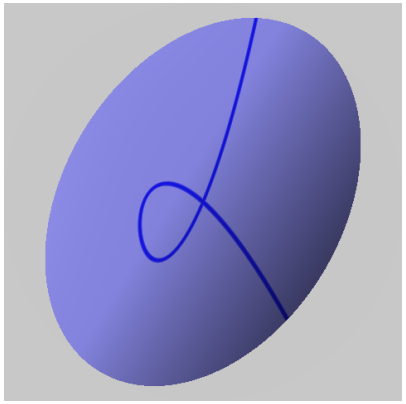

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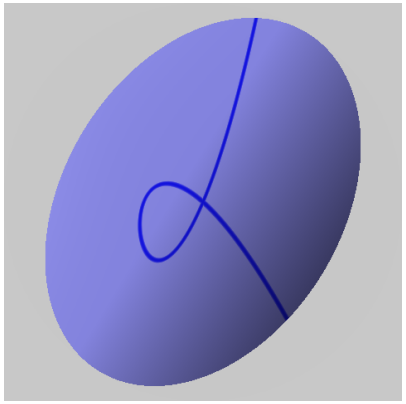

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$\Leftrightarrow$ Jacobian matrix J of f at p of rank < 1

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A real-life example – the problem

numerical Godeaux-Surface (from current research/ surface with prescribed properties):

13 variables, dimension 2, 22 polynomials

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- that is: $1 \in I + \langle (n-d) \text{ minors von } J \rangle$?
- Groebner Basis computations: all minors as input (size of pairset!)
- certificate of smoothness or singularity after **all** computations

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Goal:

Test whether X is singular
without computation of singular locus!

Hironaka's Approach

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- $\mathrm{ord}_p(g_i) = \max\{k\in\mathbb{Z} \mid g_i \in \mathfrak{m}^k_{\mathbb{A}^n_K,p}\}$

Hironaka's Approach II

defined a moment ago:
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Lemma (Hironaka)

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practical problem:

standard basis locally at **each** point impossible

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locus of order ≥ 2 of X:

$$\Delta(I_x) = V(\langle f_1, \ldots, f_s, \frac{\partial f_i}{\partial x_j} \mid 1 \leq i \leq s, 1 \leq j \leq n \rangle)$$

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Transfer to $(\mathbb{K}[x_1, \ldots, x_n]/\langle h \rangle)_g$ possible

Subsequent Entry of ν^*

Sketch of approach:

- Choose appropriate $V(h)$ und $D(g)$ covering X (g suitably chosen derivative of h)
- on each $D(g)$ consider: $X \cap D(g) \subset V(h) \cap D(g)$
- repeat previous construction (slightly modified):
  - $\dim(V(h)) = n - 1$, drop in codimension
  - derivatives w.r.t. local systems of parameters in $V(h) \cap D(g)$
  - provides covering of $X \cap D(g)$ by new $V(h, h_{neu}) \cap D(g_{neu})$
- iterate

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Challenges:

- choose large neighbourhoods for new ambient space
- keep number of charts low/use easily computable charts
- consistent choice of local system of parameters
- derivatives w.r.t. a system of parameters

Parallel, but not embarassingly parallel

Challenges call for parallel approach:

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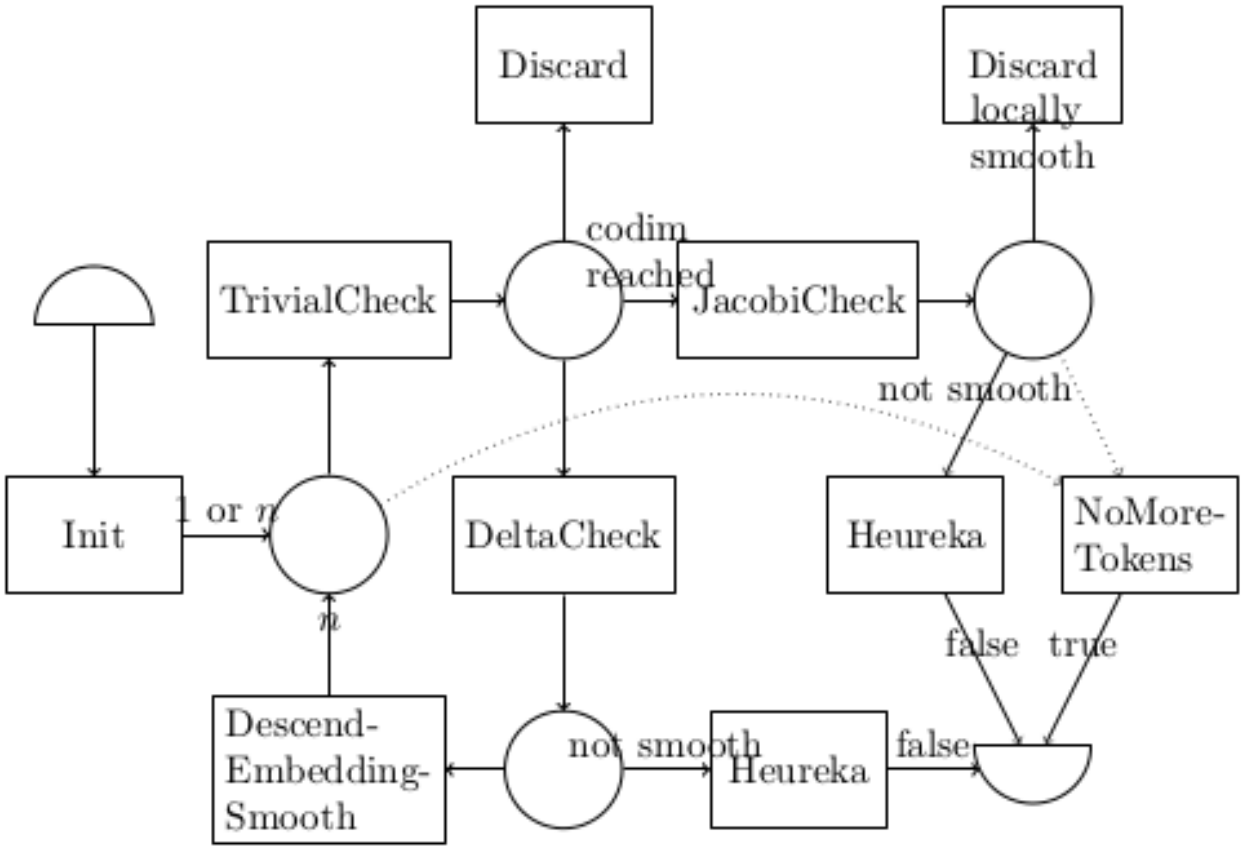
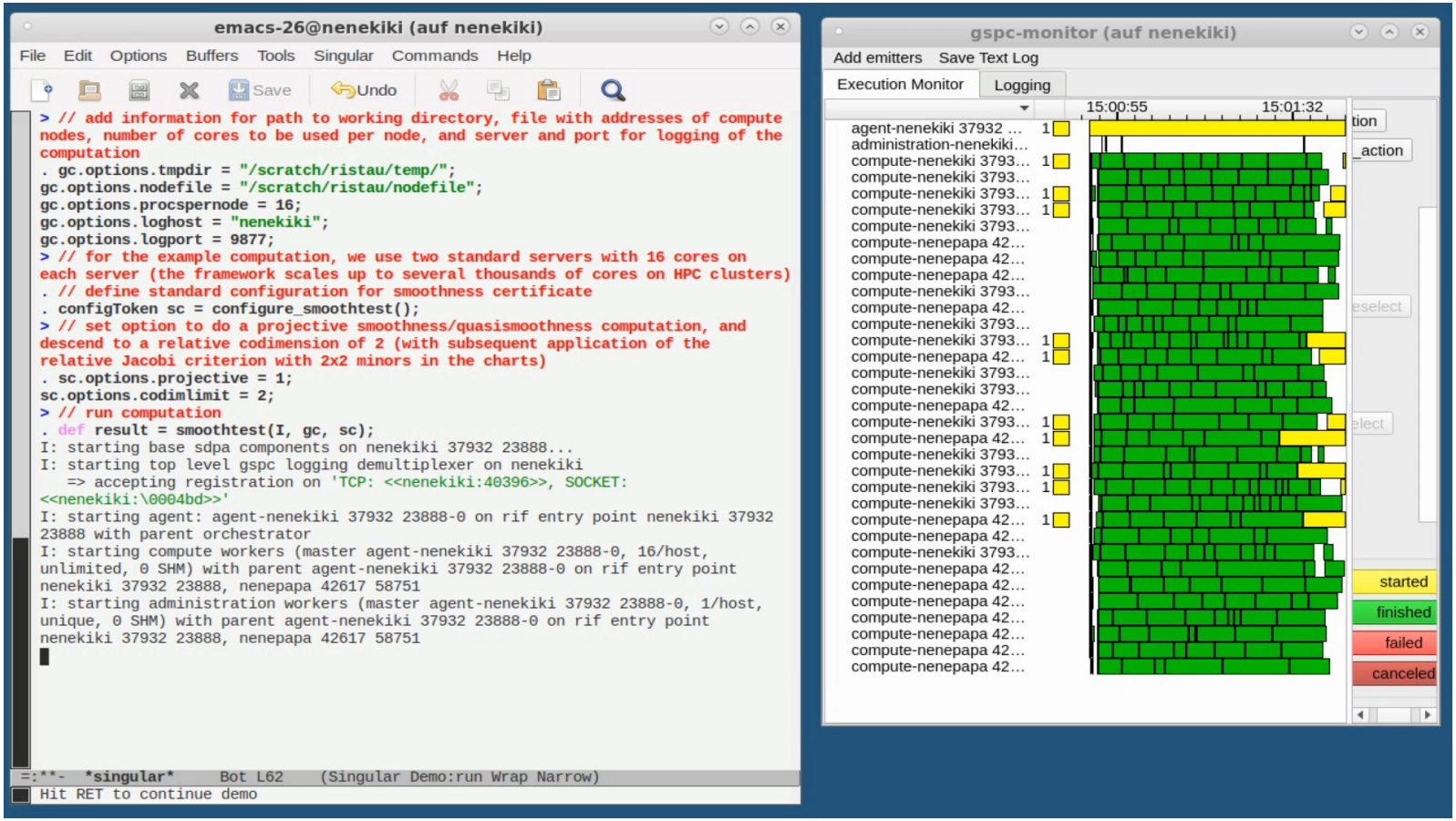
- robust against huge variations in computing time and size

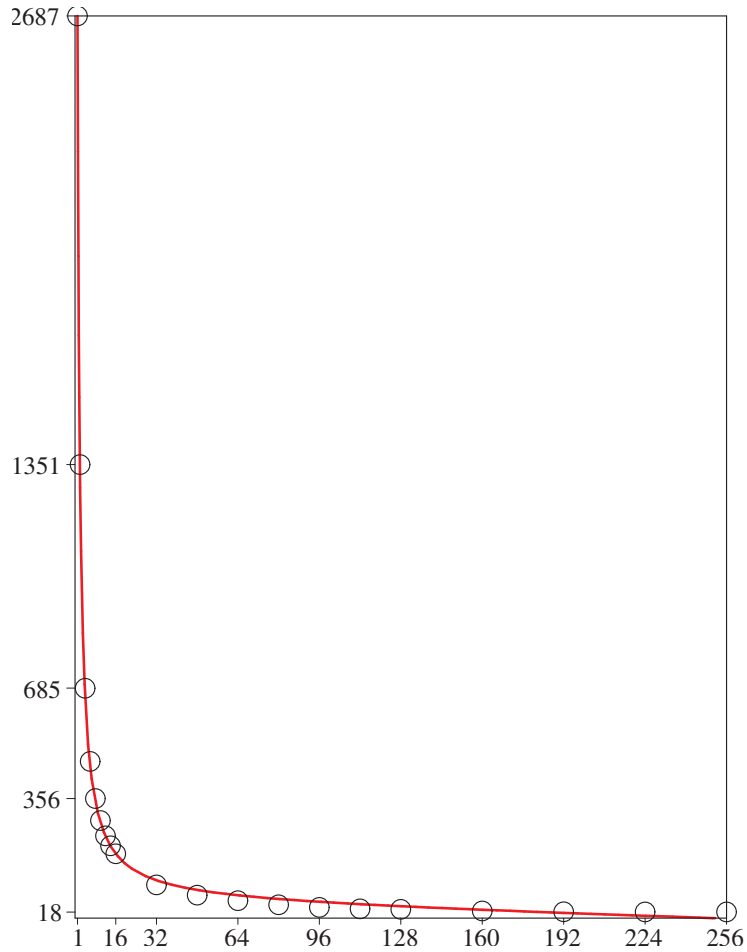
The Algorithm as a Petri-Net



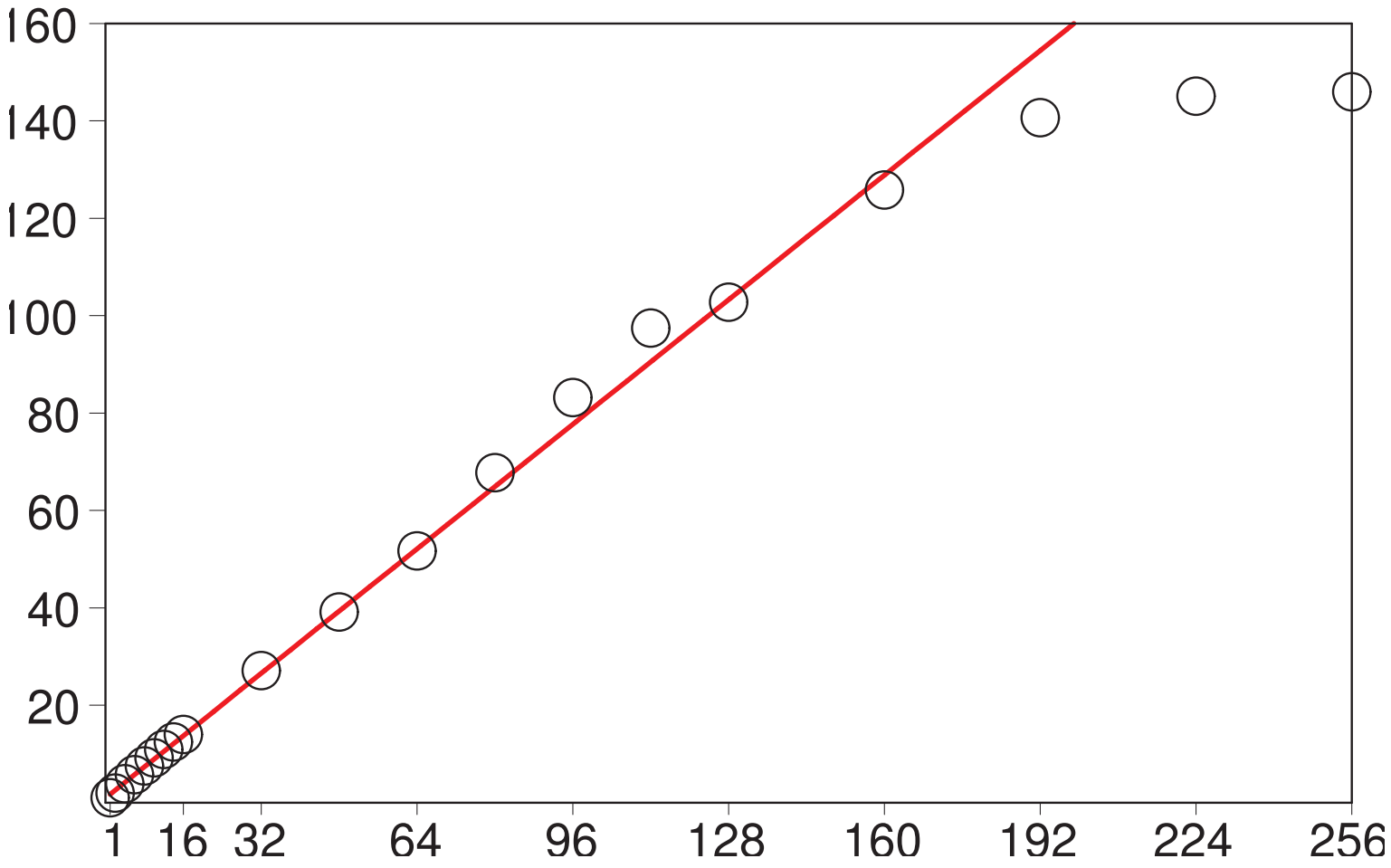
A screenshot



Campedelli-Fläche – Timings



Campedelli-Fläche – Speed-Up versus cores



Links and References

- https://github.com/singular-gpispace/framework
- https://www.mathematik.uni-kl.de/~boehm/singulargpispace
- J. Böhm, W. Decker, A. Frühbis-Krüger, F.-J. Pfreundt, M. Rahn, L. Ristau: *Towards Massively Parallel Computations In Algebraic Geometry*, to appear in Foundations of Computational Mathematics (2021)
- W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann: SINGULAR – A computer algebra system for polynomial computations http://www.singular.uni-kl.de
- Fraunhofer ITWM / F.-J. Pfreundt, M. Rahn, et al.: GPI-space http://www.gpi-space.de

Thank you!