Verifying Terao's freeness conjecture for small arrangements in arbitrary characteristic

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Joint work with Mohamed Barakat, Reimer Behrends, Christopher Jefferson, and Martin Leuner

DEFINITIONS

- $\mathcal{A}$ arrangement of hyperplanes in V over $\mathbb{F}$ with equations α_H for $H \in \mathcal{A}$.

- The module of logarithmic derivations $D(\mathcal{A})$ is defined as

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in \langle \alpha_H \rangle_S \ \forall H \in \mathcal{A}\}.$$

where $S=\mathbb{F}[x_1,\dots,x_\ell]$ $(\dim V = \ell)$

$\mathcal{A}$ is called free if $D(\mathcal{A})$ is a free S-module

- The intersection lattice $L(\mathcal{A})$ is

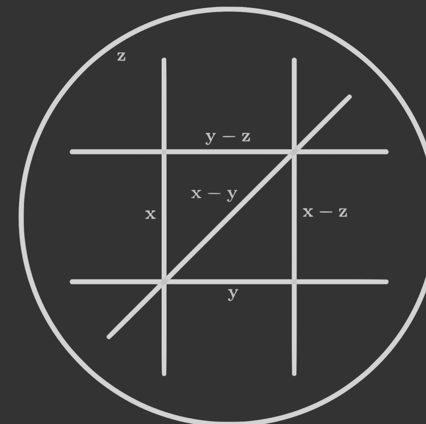
$$L(\mathcal{A}) = \{\bigcap_{H\in\mathcal{B}} H \mid \mathcal{B}\subseteq \mathcal{A}\}.$$

EXAMPLE

The Braid arrangement $\mathcal{A}_3$

$x, y, z, x-y$

$x-z, y-z$



$\mathcal{A}_3$ is free:

$$\theta_1 = x\delta_x + y\delta_y + z\delta_z$$

$$\theta_2 = x^2\delta_x + y^2\delta_y + z\delta_z$$

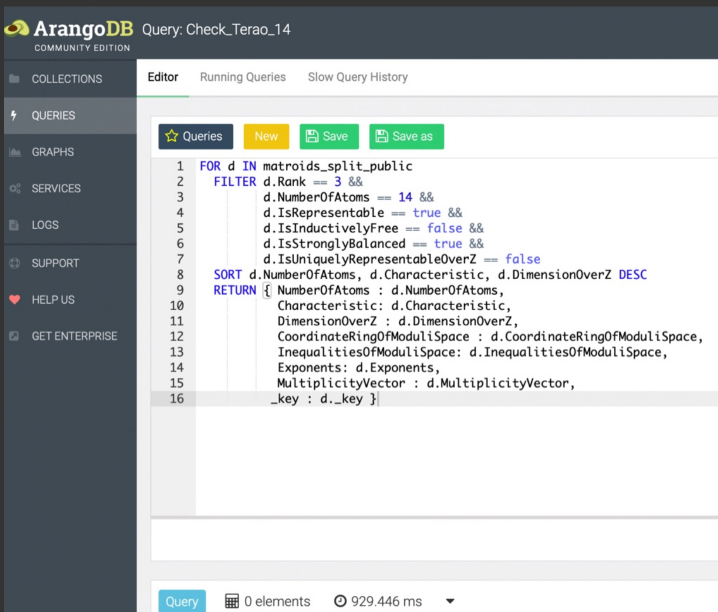
$$\theta_3 = x^3\delta_x + y^3\delta_y + z^3\delta_z$$

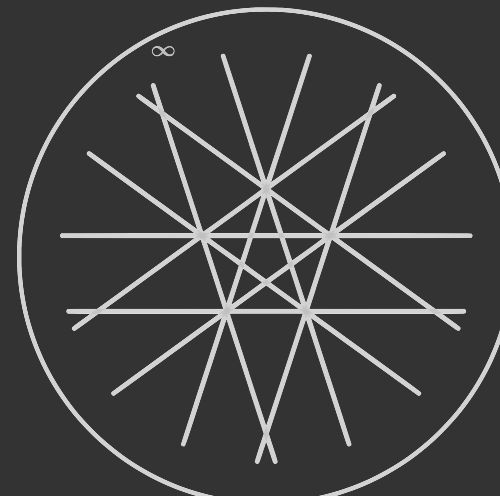
CONJECTURE (Terao)

If $\mathcal{A}$, $\mathcal{B}$ are arrangements over $\mathbb{F}$ with $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{B})$ then $\mathcal{A}$ is free $\iff$ $\mathcal{B}$ is free

THEOREM (BBJKL '19, Barakat, K. '21+)

Terao's conjecture holds for arrangements of size up to 14 and rank 3 in arbitrary characteristic.





matroid.mathematik.uni-siegen.de

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Sage experimentation with stable Grothendieck polynomials

arXiv: 1911.08732

The Electronic Journal of Combinatorics 27(2) (2020), #P2.29

Joint with Jennifer Morse, Wencin Poh and Anne Schilling

Sage/Oscar Days for Combinatorial Algebraic Geometry, ICERM

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$$
\begin{tikzcd}
{[X_w]} \arrow[d, "\text{poly rep, } K\text{-theory}"] & & \left\{ \begin{array}{c} \text{decreasing} \\ \text{factorization of } \mathcal{H}_0(n) \end{array} \right\} \arrow[dl, "\text{gen. fun.}"'] \arrow[d, "\textcolor{red}{\text{today!}}"] \\
\mathfrak{G}_w^{(\beta)} \arrow[r, "\text{stable limit}"] \arrow[d, "\beta = 0"] & \textcolor{red}{G_w^{(\beta)}} \arrow[d, "\beta = 0"] & \begin{array}{c} \text{crystals} \\ \text{combinat. model for rep.} \\ \text{directed graphs} \end{array} \\
\mathfrak{S}_w \arrow[r, "\text{stable limit}"] & F_w & \left\{ \begin{array}{c} \text{reduced decreasing} \\ \text{factorization of } \mathcal{S}_n \end{array} \right\} \arrow[l, "\text{gen. fun."']
\end{tikzcd}
$$

- (31)(32)( )
- (31)(1)(2)
- (31)(2)(2)
- (31)(3)(2)
- (1)(31)(2)
- (1)(32)(2)
- (3)(31)(2)
- (31)( )(32)
- (1)(1)(32)
- (1)(3)(32)
- (3)(1)(32)
- ( )(31)(32)

$$G_{132}(\mathbf{x},\beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

$$\begin{tikzcd}
 & (\,)(31)(32) \arrow[d, "2"] & & (1)(1)(32) \arrow[d, "1"] & (1)(3)(3,2) \arrow[d, "1"] \\
 & (3)(1)(32) \arrow[dl, "2"] \arrow[dr, "1"] & & (1)(31)(2) \arrow[d, "2"] & (1)(32)(2) \arrow[d, "2"] \\
(31)(\,)(32) \arrow[dr, "1"] & & (3)(31)(2) \arrow[dl, "2"] & (31)(1)(2) & (31)(2)(2) \\
 & (31)(3)(2) \arrow[d, "1"] & & & \\
 & (31)(32)(\,) & & &
\end{tikzcd}$$

$$\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 3 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 2 & 2 \\ \hline \end{array}$$

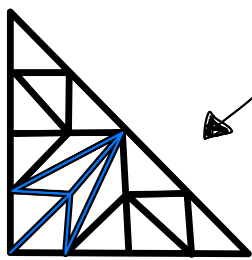
Deformation classes of bitangents to tropical quartic curves

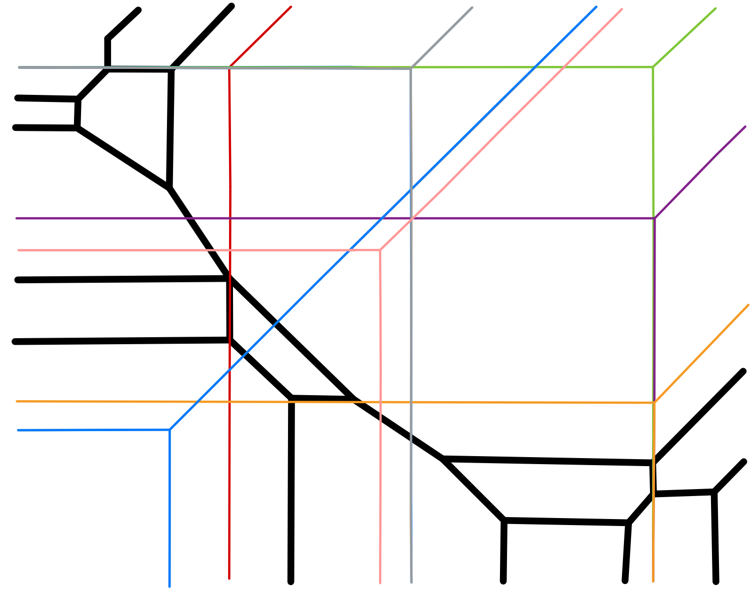
Marta Panizzut (TU Berlin)

jww Alheydis Geiger

Sage/Oscar Days - February 17, 2021

Tropical smooth plane quartic curve Γ dual to the regular unimodular triangulation $\mathcal{T}$ of $4\Delta_2$





Γ has infinitely many bitangent tropical lines grouped into seven bitangent classes.

These classes are encoded by subcomplexes of $\mathcal{T}$ described by Cueto and Markwig '20.

Our starting points:

- regular unimodular triangulations of $4\Delta_2$
  Brodsky, Joswig, Morrison and Sturmfels '15
- bitangent classes and real lifting conditions
  Cueto and Markwig '20.

Current work:

- Enumeration of (deformations of) bitangent classes
- Hyperplane arrangements induced by bitangent classes
- Real lifting conditions

polymake

Goal:

- Computational proof of Plücker's Theorem:

Real smooth quartic curve has either 4, 8, 16 or 28 real bitangents.

Variational GIT for Complete Intersections and a Hyperplane Section via Sage

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15 February 2021

Some Background on (Variational) GIT

- Main goal: describe quotients of (projective) varieties X with (reductive) group actions, with respect to some G-linearization $\mathcal{L}$.
- If $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, the (categorical quotient) $X /\!\!/_{\mathcal{L}} G :=\mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^{\otimes m})^G$ depends on $\mathcal{L} = \mathcal{O}(a, b)$. If $x \in X /\!\!/_{\mathcal{L}} G$ we say x is a semi-stable point. We also have a finite wall-chamber decomposition $[t_i, t_{i+1}]$, where stability conditions are the same for each wall/chamber, $t_i = \frac{b_i}{a_i}$.
- We also have a moduli space $\overline{\mathcal{M}}^{GIT}$ where the points are stable. We find stable/semi-stable points via the Hilbert-Mumford numerical criterion.
- The action of G on X induces an action of $\mathbb{G}_m$ on X via one-parameter subgroups, $\lambda \colon \mathbb{G}_m \to G$, via $x \to \lambda(t) \cdot x$, $t \in \mathbb{G}_m$.
- For all λ, H-M function $\mu_t(X, \lambda) \geq 0$ (> 0 resp.) for (semi)-stable points. If X is a Hilbert scheme parametrising hypersurfaces, μ_t depends on monomials of a specific degree.

Complete Intersections and Hyperplane Section

- We study pairs $X = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{P}^n$, H a hyperplane, parametrized by their Hilbert scheme $\mathcal{R}$ where $\mathrm{Pic}(\mathcal{R})^G = \mathbb{Z}^2$, $G = \mathrm{SL}(n+1)$.
- Can find an explicit finite set $P_{n,k,d}$ of 1-PS that is maximal with respect to unstable and non-stable points. This is achieved by solving a number of linear systems dependent on the n, k, d on Sage. Using this, we can find all walls and chambers $[t_i, t_{i+1}]$ by solving a number of equations dependant on the monomials of degree d and 1.
- We also compute finite sets $N_t^{\ominus}(\lambda)$ for $\lambda \in P_{n,k,d}$, $N_t^{-}(\lambda)$, that parametrise non-stable/unstable pairs. This is achieved by testing the H-M function for each t for all λ with some constraints on the monomials.
- Constraints: as n, k, d increase the program becomes computationally overloaded and slows down.

Matrix Schubert varieties and CM regularity

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joint work with Jenna Rajchgot and Anna Weigandt
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Matrix Schubert varieties and CM regularity

Matrix Schubert varieties $\overline{X}_w$, where $w \in S_n$, are generalized determinantal varieties. To study $\overline{X}_w$ we can consider the algebraic invariant of **CM regularity** $\mathrm{reg}(\mathbb{C}[\overline{X}_w])$.

Theorem

$$\mathrm{reg}(\mathbb{C}[\overline{X}_w]) = \deg \mathfrak{G}_w(x_1, \ldots, x_n) - \ell(w).$$

Problem

Give an easily computable formula for $\deg \mathfrak{G}_w(x_1, \ldots, x_n)$.

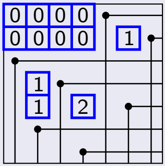
Finding the degree of $\mathfrak{G}_v$ vexillary

Theorem [Rajchgot-R.-Weigandt '21+]

Suppose $v \in S_n$ vexillary. Then

$$\deg(\mathfrak{G}_v) = \ell(v) + \sum_{i=1}^{n} \sum_{\mu \in \mathrm{comp}(\lambda(v)|_{\geq i})} \mathrm{sv}(\mu).$$

Example: $v = 5713624$



$\rightsquigarrow$

0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\deg(\mathfrak{G}_v) = \ell(v) + ((2+1)+(1)) = 12 + 4 = 16.$

$\mathbb{P}(1,2,3) = \mathrm{Proj}\,[a,b,c]$ $w = 1+2+3 = 6$

Goal: study $D^b(\mathbb{P}(1,2,3))$.

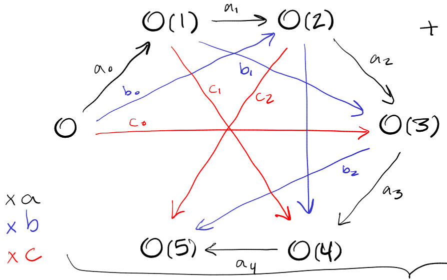
* Exceptional Collection:

$$\{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4), \underset{w-1}{\overset{\parallel}{\mathcal{O}(5)}}\} \Longrightarrow$$

* Tilting Bundle

$$T = \bigoplus_{i=0}^{w-1} \mathcal{O}(i)$$

* Quiver of Sections



+ Relations

$$\begin{pmatrix} a_0 b_1 - b_0 a_2 \\ a_0 c_1 - c_0 a_3 \\ b_0 c_2 - c_0 b_3 \\ a_1 b_2 - b_1 a_3 \\ a_1 b_2 - c_1 a_4 \\ \vdots \end{pmatrix}$$

* Bound Quiver (Q, R)

* Path Algebra

$$A = \mathbb{K}Q/R \cong \mathrm{End}(T)$$

Thm (Bondal '89)

$$D^b(\mathbb{P}(1,2,3)) = D^b(\text{mod-}A)$$

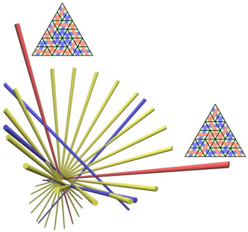
Resolution of the Diagonal, the Hochschild way

$$S^6 \xleftarrow{\begin{array}{c} \{5,-5\} \\ \{4,-4\} \\ \{3,-3\} \\ \{2,-2\} \\ \{1,-1\} \\ \{0,0\} \end{array} \begin{pmatrix} a & b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & a & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & 0 & -x & 0 & 0 & a & b & c & 0 & 0 & 0 \\ 0 & 0 & -z & 0 & -y & 0 & -x & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & 0 & -y & 0 & -x & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & -y & -x \end{pmatrix}} S^{12}$$

$$S^{12} \xleftarrow{\begin{array}{c} \{5,-4\} \\ \{5,-3\} \\ \{5,-2\} \\ \{4,-3\} \\ \{4,-2\} \\ \{4,-1\} \\ \{3,-2\} \\ \{3,-1\} \\ \{3,0\} \\ \{2,-1\} \\ \{2,0\} \\ \{1,0\} \end{array} \begin{pmatrix} -b & -c & 0 & 0 & 0 & 0 \\ a & 0 & -c & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & -c & 0 \\ -x & 0 & 0 & a & 0 & 0 \\ 0 & -x & 0 & 0 & a & 0 \\ y & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & -x & 0 & a \\ 0 & 0 & -y & 0 & -x & 0 \\ 0 & z & 0 & y & 0 & 0 \\ 0 & 0 & z & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & z & y \end{pmatrix}} S^6$$

A 'QUANTUM EQUALS CLASSICAL' THEOREM FOR N-POINTED GROMOV-WITTEN INVARIANTS OF DEGREE ONE

Weihong Xu
(Joint with Linda Chen, Angela Gibney, Lauren Heller, Elana Kalashnikov, and Hannah Larson)



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A Good Old Example

$$\begin{tikzcd} Fl(1,2;4) \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{P}^3 \\ Gr(2,4) & \end{tikzcd} \quad = \quad \begin{tikzcd} \overline{M}_{0,1}(\mathbb{P}^3,1) \arrow[r, "ev"] \arrow[d] & \mathbb{P}^3 \\ \overline{M}_{0,0}(\mathbb{P}^3,1) & \end{tikzcd}$$

Line $\Gamma \subset \mathbb{P}^3$
$q(p^{-1}(\Gamma))$ Schubert divisor $\square$
$\square^4 = 2 \cdot \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} = 2 \cdot [\text{point}]$

$4 \cdot codim(\Gamma) = dim(\overline{M}_{0,4}(\mathbb{P}^3, 1))$
The Gromov-Witten invariant
$\int_{\overline{M}_{0,4}(\mathbb{P}^3,1)} ev_1^*[\Gamma] \cdots ev_4^*[\Gamma] = 2$

More Generally

$X = G/P$ flag variety
P maximal parabolic corresponding to a long simple root
$ev_i : \overline{M}_{0,n}(X, 1) \to X$
Schubert varieties $\Gamma_1, \cdots, \Gamma_n \subset X$, $\sum_{i=1}^{n} codim(\Gamma_i) = dim(\overline{M}_{0,n}(X, 1))$
Theorem The (n-pointed, genus 0, degree 1) Gromov-Witten Invariant

$$\int_{\overline{M}_{0,n}(X,1)} ev_1^*[\Gamma_1] \cdots ev_n^*[\Gamma_n]$$

$= \#\{\text{line in } X \text{ meeting } g_1\Gamma_1, \cdots, g_n\Gamma_n\}$ for $g_1, \cdots, g_n \in G$ general
$= \int_{G/Q} [q(p^{-1}(\Gamma_1))] \cdots [q(p^{-1}(\Gamma_n))]$

$$\begin{array}{ccccccc} G/(P \cap Q) & \xrightarrow{p} & G/P & & \overline{M}_{0,1}(X,1) & \xrightarrow{ev} & X \\ {\scriptstyle q}\downarrow & & & = & \downarrow & & \\ G/Q & & & & \overline{M}_{0,0}(X,1) & & \end{array}$$

Remarks: 1) $n = 3$ case is known; 2) proof is independent of Lie type